

# ON THE BOUNDARY CONTROL OF KIRCHHOFF'S NONLINEAR STRING <sup>1</sup>

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Abstract: In this paper we propose two new classes of controllers which stabilize Kirchhoff's nonlinear string by using boundary control techniques. We assume that the boundary displacement is the only available measurement. The classes of controllers proposed in this paper are related to the positive real controllers. One of the classes generalizes a special class of such stabilizing controllers which is already proposed in the literature and the other one is new. *Copyright © 2007 IFAC*

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## 1. INTRODUCTION

There are several nonlinear models that represent the dynamics of elastic strings. A model which has been studied the most is a nonlinear partial differential equation known as Kirchhoff's nonlinear string model. This model was originally derived by Kirchhoff, and later by other researchers, see e.g. Kirchhoff (1877), and later by other researchers Carrier(1945), Narasimha (1968). In the past few decades, Kirchhoff's string and its generalizations have been studied from the mathematical point view; see, e.g. Arosio (1993) and references therein. Also, from the practical point of view, the stability and stabilization of Kirchhoff's string have been studied; see, e.g., Shahruz (1998, 1999), and references therein.

In this paper we consider the stabilization of Kirchhoff's nonlinear string by using the boundary displacement feedback. This problem was considered in Kobayashi (2004) where a special class of stabilizing controllers has been proposed and the well-posedness

and stability of the corresponding closed-loop system have been established. In the present paper we first extend the class of controllers given in Kobayashi (2004) to a more general class of stabilizing controllers. The proposed class of controllers is a special class of positive real controllers and includes that proposed in Kobayashi (2004) as a special case. Then we give yet another class of positive real controllers which is completely different from that proposed in Kobayashi (2004).

This paper is organized as follows. In the Section 2, we present the problem statement along with the stabilizing controllers proposed in Kobayashi (2004). We show that these controllers belong to a special class of one dimensional positive real controllers, and relate them to some existing stabilizing controllers for linear strings. In Section 3, we propose a general class of stabilizing controllers and show that these controllers have positive real transfer functions. In Sections 4 and 5, we prove the well-posedness and the stability of the resulting closed-loop system. In Section 6, we propose yet another class of positive real controllers and show that the members of this class result in well-posed and

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asymptotically stable closed-loop systems. Finally, we make some concluding remarks.

## 2. PROBLEM STATEMENT AND PROPOSED CONTROLLER

Following Kobayashi (2004), we consider the following system which represents a nonlinear Kirchhoff string with unit length and unit mass density for  $x \in (0, 1)$ ,  $t \geq 0$ :

$$z_{tt}(x, t) = M(\|z_x(x, t)\|^2)z_{xx}(x, t) \quad (1)$$

where  $z(x, t)$  denotes the transversal displacement of the string at a point  $x \in (0, 1)$  and a time instant  $t \geq 0$ ; a notation such as  $z_x$  or  $z_t$  denotes the partial derivative of a function with respect to the variable in the subscript; the function  $M(\cdot) \in C^1(\mathbf{R}_+)$  is assumed to be nonnegative and  $\|\cdot\|$  represents the standard  $L_2$  norm, i.e.

$$\|z_x(x, t)\|^2 = \int_0^1 z_x^2(x, t) dx. \quad (2)$$

Note that for the sake of brevity, we may omit the arguments and use  $z_x$  or  $z_t$  instead of  $z_x(x, t)$  or  $z_t(x, t)$ . The boundary conditions associated with (1) are given as for  $t \geq 0$ :

$$z(0, t) = 0, \quad M(\|z_x(x, t)\|^2)z_x(1, t) = u(t), \quad (3)$$

where  $u(t)$  is the boundary control force applied at the free end of the string. The problem is to choose appropriate control law for  $u(t)$  so that the closed-loop given by (1), (3) is stable. This problem is investigated in Shahruz (1999) for the case  $M(s) = \alpha + \beta s$  where  $\alpha > 0$  and  $\beta > 0$ , and recently in Kobayashi (2004) for a general  $M(\cdot)$  satisfying  $M(s) \geq c > 0$ . The controller proposed in the latter is as follows:

$$\dot{w}(t) = -aw(t) + bu(t), \quad w(0) = 0, \quad (4)$$

$$u(t) = -k(y(t) + w(t)), \quad y(t) = z(1, t), \quad k > 0, \quad (5)$$

where  $a \geq 0$ ,  $b > 0$  and  $k > 0$ . Note that here the measurement is the end point displacement. Some solutions for the case where the measurement is  $y(t) = z_t(1, t)$ , i.e. the end point velocity, have been given in the literature, see e.g. Shahruz (1999) and the references therein. The usage of displacement measurement instead of velocity measurement at the end point has some merit: It is easier to measure the displacement and such measurements are usually less noisy as opposed to the velocity measurements. The same problem for the case  $M(\cdot) = c > 0$  has been considered in many references, see e.g. Morgül (1994), where the measurement includes both displacement and velocity at the end point. The main result of Morgül (1994) is closely related to the positive realness of the transfer

function of the stabilizing controller. To comply with the notation of Morgül (1994), let us define a force term  $f(t)$  as  $f(t) = -u(t)$ , and apply the Laplace transform to (4)-(5). By using (5) in (4), we obtain:

$$\dot{w}(t) = -(a + bk)w(t) - bky(t). \quad (6)$$

By applying the Laplace transform to (5)-(6), using zero initial conditions, and  $f(t) = -u(t)$ , we obtain:

$$\hat{f}(s) = g(s)\hat{y}(s), \quad (7)$$

where a hat denotes the Laplace transform of the corresponding variable,  $s$  is the Laplace variable and  $g(s)$  is the transfer function of the controller which is given as

$$g(s) = k \frac{s + a}{s + a + bk}. \quad (8)$$

Note that when  $a = 0$ ,  $g(s)$  is a positive real transfer function, and when  $a > 0$ ,  $g(s)$  is strictly positive real. For the definition of positive real transfer functions, see e.g. Morgül (1994), Slotine and Li (1991). Note that here the measurement is the displacement, whereas in Morgül (1994) the transfer function is given with respect to the velocity measurement. Formally, using  $\hat{y}_t(s) = s\hat{y}(s)$  in (7)-(8), we obtain:

$$\hat{f}(s) = \frac{g(s)}{s}\hat{y}_t(s), \quad (9)$$

By using (8), we can easily obtain:

$$\frac{g(s)}{s} = \frac{k_1}{s} + \frac{k_2}{s + a + bk}, \quad (10)$$

where  $k_1$  and  $k_2$  are given as:

$$k_1 = \frac{ka}{a + bk}, \quad k_2 = \frac{bk^2}{a + bk}. \quad (11)$$

Note that for the case  $a \geq 0$ , we have  $k_1 \geq 0$  and  $k_2 > 0$ , and  $g(s)/s$  is a positive real transfer function when  $a > 0$ , and is strictly positive real when  $a = 0$ . By using the results of Morgül (1994), we expect the asymptotic stability of the closed-loop system when  $a \geq 0$  for the case  $M(\cdot) = c > 0$ . For Kirchhoff's string,  $M(\cdot)$  is a nonlinear function, and hence the results of Morgül (1994) are not directly applicable. But by using the idea of positive realness, we can, however, extend the class of controllers proposed in Kobayashi (2004) to a larger class of stabilizing controllers.

In the sequel, we generalize the results of the Kobayashi (2004) in three respects: **i**) We propose a larger class of stabilizing controllers which includes that proposed in Kobayashi (2004) as a special case; **ii**) We relax the unrealistic assumption made in (4) that  $w(0) = 0$ ; **iii**) We propose yet another class of stabilizing controllers the members of which are different from those given in Kobayashi (2004).

Let us define (see e.g. Morgüil (1994))

$$y(t) = z(1, t), \quad f(t) = -u(t). \quad (12)$$

As for the controller, we propose the following system

$$\left. \begin{aligned} \dot{w}(t) &= -Aw(t) - bf(t), \\ f(t) &= \alpha_1 b^T w(t) + \alpha_2 y(t), \end{aligned} \right\} \quad (13)$$

where  $\alpha_1 > 0$  and  $\alpha_2 > 0$  are arbitrary positive constant numbers,  $w \in \mathbf{R}^n$  is the state of the controller,  $A \in \mathbf{R}^{n \times n}$  is a symmetric and positive semi-definite matrix,  $b \in \mathbf{R}^n$ , and a superscript  $T$  denotes the transpose. Note that the controller in Kobayashi (2004) is a special case of that in (12)-(13) where  $n = 1$ ,  $k = \alpha_1 b = \alpha_2$ . For this controller we make the following assumptions

**Assumption 1 :**  $A + bb^T$  is a positive definite matrix.

**Assumption 2 :** The pair  $(A, b)$  is controllable.

Note that in Kobayashi (2004), since  $n = 1$  and  $b > 0$ , these assumptions are automatically satisfied. The transfer function of the controller given in (13) can be easily computed as :

$$\hat{f}(s) = g(s)\hat{y}(s) \quad (14)$$

$$= [\alpha_2 - \alpha_1 \alpha_2 b^T (sI + A + \alpha_1 bb^T)^{-1} b] \hat{y}(s).$$

We show in the sequel that with the stated assumptions,  $g(s)$  given by (14) is a positive real transfer function. By using (13), we obtain :

$$\left. \begin{aligned} \dot{w}(t) &= -(A + \alpha_1 bb^T)w(t) - \alpha_2 by(t) \\ &= Fw(t) + Gy(t), \end{aligned} \right\} \quad (15)$$

$$\left. \begin{aligned} f(t) &= \alpha_1 b^T w(t) + \alpha_2 y(t) \\ &= C^T w(t) + Dy(t), \end{aligned} \right\} \quad (16)$$

where we have

$$\left. \begin{aligned} F &= -(A + \alpha_1 bb^T), \quad G = -\alpha_2 b, \\ C &= \alpha_1 b, \quad D = \alpha_2. \end{aligned} \right\} \quad (17)$$

**Lemma 1 :** Consider the system given by (15)-(17) and let Assumptions 1 and 2 be satisfied. Then the transfer function given by (14) is a positive real transfer function. Moreover, if  $A$  is a positive definite matrix, then  $g(s)$  is a strictly positive real transfer function.

**Proof :** The proof depends on the well-known Kalman-Yakubovitch Lemma, see e.g. Slotine and Li (1991), Lefchetz (1965). According to this Lemma, given a symmetric and positive definite matrix  $Q$ , a controllable pair  $(F, G)$  and an observable pair  $(C, F)$ , the transfer function given by (14), which can be written as  $g(s) = C^T (sI - F)^{-1} B + D$  is strictly positive real transfer function if and only if there exists an  $\varepsilon > 0$ , a symmetric and positive definite matrix  $P \in \mathbf{R}^{n \times n}$ , and

a vector  $q \in \mathbf{R}^n$  such that the following equations are satisfied :

$$F^T P + PF = -\varepsilon Q - qq^T, \quad (18)$$

$$PG - C = \sqrt{2D}q. \quad (19)$$

It easily follows from Assumption 2 that the observability and controllability conditions are satisfied. It can be easily shown that when the matrix  $A$  is positive definite, selecting  $\beta = \alpha_1/\alpha_2$ ,  $P = \beta I$ ,  $\varepsilon = 2\beta$ ,  $Q = A$ , and  $q = -\sqrt{2\alpha_1\beta}b$ , (18) and (19) are satisfied. Hence if  $A$  is positive definite, then  $g(s)$  is a strictly positive real transfer function. On the other hand, if  $A$  is only positive definite, then from the necessity part of the proof of Kalman-Yakubovitch Lemma given in Lefchetz (1965), p. 115, it follows that  $g(s)$  is a positive real transfer function.  $\square$

Note that the transfer function  $g(s)$  in (14) corresponding to the controller proposed in this paper reduces to that proposed in Kobayashi (2004) when  $n = 1$ ,  $k = \alpha_1 b = \alpha_2$ .

Next, we investigate the well-posedness of the closed-loop system given by (1)-(3), (12)-(13). For the sake of simplicity, we call this system as  $\mathcal{S}_1$ . We define various function spaces as follows :

$$H = L_2 = \{f : [0, 1] \rightarrow \mathbf{R} \mid \int_0^1 f^2(x) dx < \infty\},$$

$$H^i = \{f \in H \mid f, f', \dots, f^{(i)} \in H\},$$

$$V = \{f \in H^1 \mid f(0) = 0\}, \quad W = \{f \in H^2 \mid f(0) = 0\},$$

$$D = \{(f, g, w) \in H \times H \times \mathbf{R}^n \mid f \in W, g \in V, \\ M(\|f'\|^2)f'(1) = -\alpha_1 b^T w - \alpha_2 f(1)\}.$$

**Theorem 1 :** Let  $M \in C^1([0, \infty))$  be a positive function such that  $M(s) \geq c > 0$ ,  $\forall s \geq 0$ . Then, for any  $(z_0, z_1, w_0) \in D$ , there exists a unique solution of  $\mathcal{S}_1$  such that for any  $T > 0$  we have.

$$\left. \begin{aligned} z &\in C([0, T]; V) \cap L_2([0, \infty); W), \\ z_t &\in C([0, T]; H), \quad w \in C([0, T]; \mathbf{R}^n), \end{aligned} \right\} \quad (20)$$

**Proof :** For the proof, we use the technique used in Kobayashi (2004). First, let us define the following Lyapunov-like function for the system  $\mathcal{S}_1$  :

$$E(z(t), w(t)) = \frac{1}{2} \|z_t\|^2 + \frac{1}{2} \hat{M}(\|z_x\|^2) \quad (21)$$

$$+ \frac{1}{2} \alpha w(t)^T A w(t)$$

$$+ \frac{1}{2} \alpha_3 [\alpha_1 b^T w(t) + \alpha_2 z(1, t)]^2,$$

where  $\hat{M}(s) = \int_0^s M(z) dz$ , and  $\alpha$  and  $\alpha_3$  are positive constant numbers yet to be determined. Now, for a given  $\alpha_1 > 0$  and  $\alpha_2 > 0$ , we choose the remaining constant numbers  $\alpha$  and  $\alpha_3$  so that  $\alpha = \alpha_1/\alpha_2$ ,  $\alpha_3 = 1/\alpha_2$ . By taking the formal derivative of (21) along

the solutions of  $\mathcal{S}_1$ , and by omitting the spatial and time variables in the argument for simplicity, after straightforward calculations we obtain:

$$\begin{aligned} \dot{E} &= \alpha \dot{w}^T [Aw + \frac{\alpha_1^2 \alpha_3}{\alpha} bb^T w + \frac{\alpha_1}{\alpha} bz(1,t)] \\ &= -\alpha \|\dot{w}\|^2, \end{aligned} \quad (22)$$

where in the last equality, the norm is the standard Euclidean norm in  $\mathbf{R}^n$ . The well-posedness results can be shown by using Theorem 1 of Kobayashi (2004).  $\square$

**Theorem 2 :** Let the assumptions in Theorem 1 hold. Under these conditions, the closed-loop system given by  $\mathcal{S}_1$  is globally asymptotically stable .

**Proof :** Note that by (22) we have :

$$\dot{E} = -\alpha \|\dot{w}\|^2, \quad (23)$$

along the solutions of  $\mathcal{S}_1$ . It can be shown that La Salle's Invariance argument can be applicable, see e.g. see e.g. Luo, Guo and Morgül (1999). Let us define the following set :

$$\mathcal{S} = \{ (z, z_t, w) \in V \times H \times \mathbf{R}^n \mid \dot{E} = 0 \} , \quad (24)$$

By using the techniques similar to the ones used in Kobayashi (2004), it can be shown that the only possible solution of  $\mathcal{S}_1$  which is invariant in  $\mathcal{S}$  is the zero solution. Therefore, by LaSalle's invariance theorem, the system  $\mathcal{S}_1$  is globally asymptotically stable.  $\square$

### 3. A NEW CONTROLLER

In this section, we propose yet a different class of stabilizing controllers for the system given by (1)-(3). Let us define (see Morgül (1994))

$$y(t) = z(1,t), \quad f(t) = -u(t). \quad (25)$$

As for the controller, we propose the following system

$$\left. \begin{aligned} \dot{w}(t) &= Aw(t) + by(t) = Aw(t) + bz(1,t), \\ o(t) &= c^T w, \end{aligned} \right\} \quad (26)$$

$$f(t) = \dot{o}(t) = c^T \dot{w}(t) = c^T Aw(t) + c^T bz(1,t), \quad (27)$$

where  $w \in \mathbf{R}^n$ ,  $A \in \mathbf{R}^{n \times n}$  and  $b, c \in \mathbf{R}^n$ , and a superscript  $T$  denotes the transpose. For the controller given by (25)-(27) we make the following assumptions :

**Assumption 3 :** The transfer function of the triple  $(c, A, b)$  i.e  $g(s) = c^T (sI - A)^{-1} b$  is a strictly positive real transfer function.

**Assumption 4 :** The pair  $(c, A)$  is observable and the pair  $(A, b)$  is controllable.

From these assumptions, and the Kalman-Yakubovitch Lemma stated in Lemma 1, it follows from that for any

given symmetric and positive definite matrix  $Q$ , there exists a symmetric and positive definite matrix  $P$  such that the following hold :

$$A^T P + PA = -Q, \quad Pb - c = 0. \quad (28)$$

Now, with these controllers, we will call the resulting closed-loop system as  $\mathcal{S}_2$ .

**Theorem 3 :** Let  $M \in C^1([0, \infty))$  be a positive function such that  $M(s) \geq c > 0, \forall s \geq 0$ . Then for any  $(z_0, z_1, w_0) \in D$ , there exists a unique solution of  $\mathcal{S}_2$  such that for any  $T > 0$  we have

$$\left. \begin{aligned} z &\in C([0, T]; V) \cap L_2([0, \infty); W), \\ z_t &\in C([0, T]; H), \quad w \in C([0, T]; \mathbf{R}^n), \end{aligned} \right\} \quad (29)$$

**Proof :** Proof of this result is quite similar to the proof of Theorem 1. First, we define an appropriate Lyapunov-like function as follows :

$$\begin{aligned} E(z(t), w(t)) &= \frac{1}{2} \|z_t\|^2 + \frac{1}{2} \hat{M}(\|z_x\|^2) \\ &\quad + \frac{1}{2} \dot{w}(t)^T P \dot{w}(t), \end{aligned} \quad (30)$$

where  $\hat{M}(s) = \int_0^s M(z) dz$ . By formally taking the derivative of (30) along the solutions of  $\mathcal{S}_2$ , and by omitting the spatial and time variables in the arguments for simplicity, after straightforward calculations we obtain :

$$\dot{E} = -\frac{1}{2} \dot{w}(t)^T Q \dot{w}(t), \quad (31)$$

The rest of the proof is exactly the same as the proof of Theorem 1 and is omitted here to avoid repetition.  $\square$

**Theorem 4 :** Let the assumptions in Theorem 3 hold. Under these conditions, the closed-loop system given by  $\mathcal{S}_2$  is globally asymptotically stable .

**Proof :** Similar to Theorem 2, this result can be shown by using LaSalle's invariance theorem.  $\square$

### 4. CONCLUSION

In this paper we considered the stabilization of Kirchhoff's nonlinear string by using boundary control techniques. We assumed that only the displacement measurement is available at the boundary and proposed two different controller structures which asymptotically stabilize the corresponding closed-loop systems. The first type of controllers proposed in this paper is a generalization of the one dimensional controller proposed in Kobayashi (2004) to higher dimensional case. We note that even in one dimensional case, the controller proposed in this paper is still more general than that proposed in Kobayashi (2004). We also proposed a second type of stabilizing controllers the

members of which are quite different from those proposed in Kobayashi (2004). This new class of stabilizing controllers is related to positive real controllers as well.

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