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On Stabilizing with PID Controllers

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***Abstract*— In this paper we give an algorithm that determines the set of all stabilizing proportional-integral-derivative (PID) controllers that places the poles of the closed loop system in a desired stability region S. The algorithm is applicable to linear, time invariant, single-input single-output plants. The solution is based on a generalization of the Hermite-Biehler theorem applicable to polynomials with complex coefficients and the the application of a stabilizing gain algorithm to three auxiliary plants.**

I. Introduction

In many applications, stability of the closed loop system is not enough, and it is usually required that the poles of the closed loop system lie in more restrictive stability regions. It is known that time domain specifications for a closed loop system can be translated into desired closed loop pole locations in the frequency domain. These are specified in terms of the damping ratio and damped natural frequency of the closed loop poles. A desired stability region S in the complex plane is shown in Figure 1.

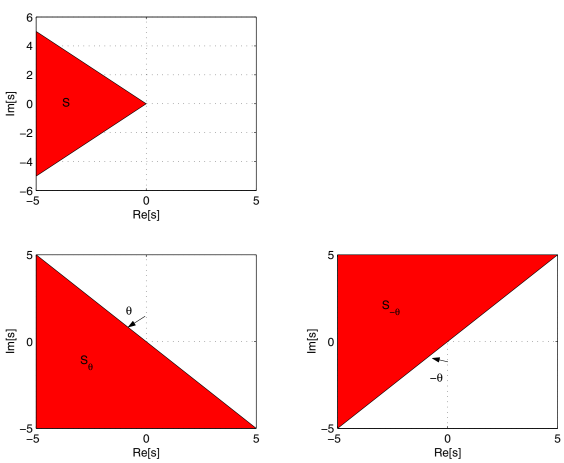

Fig. 1. Stability region S.

The region S is the intersection of two regions S_θ and $S_{-\theta}$ where

- $S_\theta := \{s : s \in \mathbf{C},\ Re[se^{-j\theta}] < 0\}$.
- $S_{-\theta} := \{s : s \in \mathbf{C},\ Re[se^{j\theta}] < 0\}$.

S_θ and $S_{-\theta}$ are rotated Hurwitz stability regions. In [1], it is stated that if all the poles of the closed-loop system lie in the region S, then the step response of the compensated system exhibits a maximum overshoot corresponding to the angle θ. In [2], the region S is approximated by a circular region and a design procedure that combines linear-quadratic optimal control with regional pole placement is given. Recently, a method for determining the set of all proportional controllers that places the closed-loop poles in the region S was given in [3]. In this paper, we give a method to determine the set of all PID controllers that places the poles of the closed-loop system in the region S.

The paper is organized as follows. In section 2, a generalization of the Hermite-Biehler theorem applicable to polynomials with complex coefficients is stated. In section 3, the problem of stabilizing complex polynomials with proportional controllers is considered. In section 4, we give an algorithm that solves the problem of determining all stabilizing PID controllers that places the poles of the closed loop system in the stability region S.

II. A Generalization of the Hermite-Biehler Theorem

In this section, a generalization of the Hermite-Biehler theorem to polynomials with complex coefficients [4] is presented. Before proceeding any further, let us fix the notation used in this paper. Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{C}$ denote the set of complex numbers and let $\mathbf{C}_-$, $\mathbf{C}_0$, $\mathbf{C}_+$ denote the points in the open left half, $j\omega$-axis, and the open right half of the complex plane, respectively. Given a set of polynomials $\psi_1, ..., \psi_k \in \mathbf{R}[s]$ not all zero and $k > 1$, their *greatest common divisor* (with highest coefficient 1) is unique and it is denoted by $gcd\{\psi_1, ..., \psi_k\}$. If $gcd\{\psi_1, ..., \psi_k\} = 1$, then we say $(\psi_1, ..., \psi_k)$ is *coprime*. The derivative of ψ is denoted by ψ'. The set $\mathcal{H}$ of Hurwitz stable polynomials are

$$\mathcal{H} = \{\psi(s) \in \mathbf{C}[s] : \psi(s) = 0 \Rightarrow s \in \mathbf{C}_-\}.$$

The *signature* $\sigma(\psi)$ *of a polynomial* $\psi \in \mathbf{C}[s]$ is the difference between the number of its $\mathbf{C}_-$ roots and $\mathbf{C}_+$ roots. Given $\psi \in \mathbf{C}[s]$, the real and imaginary parts (a, b) of $\psi(s)$ are the unique polynomials $a,\ b \in \mathbf{R}[\omega]$ such that $\psi(j\omega) = a(\omega) + jb(\omega)$. Finally, let us define the *signum function* $\mathcal{S} : \mathbf{R} \to \{-1, 0, 1\}$ by

$$\mathcal{S}r = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ 1 & \text{if } r > 0. \end{cases}$$

Theorem 1. [4] Let a non-zero polynomial $\psi \in \mathbf{C}[s]$ of *degree* n have the real-imaginary parts (a, b). Suppose $b \not\equiv 0$ and (a, b) is coprime. Let $\omega_1 < \omega_2 < ... < \omega_k$ be

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the real, distinct finite roots of b with odd multiplicities. Also let $\omega_0 = -\infty$, $\omega_{k+1} = \infty$, and ξ_n be the leading coefficient of $\psi(s)$. Then

$$\sigma(\psi) = \begin{cases} \frac{1}{2}\big\{\mathcal{S}a(\omega_0)(-1)^k + 2\sum_{i=1}^{k}\mathcal{S}a(\omega_i)(-1)^{k-i} \\ \quad - \mathcal{S}a(\omega_{k+1})\big\}\mathcal{S}b(\infty) \\ \text{if } n \text{ is even and } \xi_n \text{ is purely real,} \\ \text{or } n \text{ is odd and } \xi_n \text{ is purely imaginary.} \\ \\ \frac{1}{2}\big\{2\sum_{i=1}^{k}\mathcal{S}a(\omega_i)(-1)^{k-i}\big\}\mathcal{S}b(\infty) \\ \text{if } n \text{ is even and } \xi_n \text{ is not purely real,} \\ \text{or } n \text{ is odd and } \xi_n \text{ is not purely imaginary.} \end{cases} \tag{1}$$

Proof. See [4], [5]. ■

The following result transforms the problem of determining the number of real roots of a real polynomial to an equivalent problem of finding the signature of a complex polynomial.

Lemma 1. A non-zero polynomial $\psi \in \mathbf{R}[u]$, has r real roots without counting the multiplicities if and only if the signature of the complex polynomial $\bar{\psi}(j\omega) = \psi(w) + j\psi'(w)$ is $-r$.

Proof. See [6]. ■

III. Proportional Controllers

We now describe an alternative method to the constant stabilizing gain method of [3] for complex polynomials. Our method avoids a search in an exponentially growing set. Given a plant $g(s) = \frac{p(s)}{q(s)}$, where $p, q \in \mathbf{C}[s]$ are coprime with $m = \deg\, p$ less than $n = \deg\, q$, the set

$$A_r(p,q) := \{\alpha \in \mathbf{R} :\ \sigma\big[\phi(s,\alpha)\big] = \sigma[q(s)+\alpha p(s)] = r\}$$

is the set of all real α such that $\phi(s,\alpha)$ has signature equal to r.

Let (h,g) and (f,e) be the real-imaginary parts of q and p, respectively, so that $q(j\omega) = h(\omega) + jg(\omega)$, $p(j\omega) = f(\omega) + je(\omega)$. Let $d := gcd\{f,e\}$ so that $f = d\bar{f}$, $e = d\bar{e}$, for coprime polynomials $\bar{f}, \bar{e} \in \mathbf{R}[\omega]$. Then, the polynomial $\bar{p}(s)$ such that $\bar{p}(j\omega) := \bar{f}(\omega) + j\bar{e}(\omega)$ is free of $\mathbf{C}_0$ roots. Let (H,G) be the real-imaginary parts of $q(s)\bar{p}^*(s)$ where $\bar{p}^*(j\omega) := \bar{f}(\omega) - j\bar{e}(j\omega)$. Also let $F(\omega) := p(j\omega)\bar{p}^*(j\omega)$. By a simple computation, it follows that

$$\begin{aligned} H(\omega) &= h(\omega)\bar{f}(\omega) + g(\omega)\bar{e}(\omega), \\ G(\omega) &= g(\omega)\bar{f}(\omega) - h(\omega)\bar{e}(\omega), \\ F(\omega) &= f(\omega)\bar{f}(\omega) + e(\omega)\bar{e}(\omega). \end{aligned} \tag{2}$$

If $G \not\equiv 0$ and if they exist, let the *real zeros with odd multiplicities of* $G(\omega)$ be $\{\omega_1, ..., \omega_k\}$ with the ordering $\omega_1 < \omega_2 < \cdots < \omega_k$, with $\omega_0 := -\infty$ and $\omega_{k+1} := \infty$ for notational convenience and let ξ be the leading coefficient of $q(s)\bar{p}^*(s)$. The following algorithm determines whether $A_r(p,q)$ is empty or not and outputs its elements when it is not empty:

Algorithm 1.

1) *Calculate*

$$\alpha_j = \begin{cases} -\frac{H}{F}(\omega_j), j = 1,\ldots,k \;\;\&\;\; F(\omega_j) \neq 0, \\ \textit{if } n+m \textit{ is even and } \xi \textit{ is not purely real,} \\ \textit{or } n+m \textit{ is odd and } \xi \textit{ is not purely} \\ \textit{imaginary.} \\ \\ -\frac{H}{F}(\omega_j), j = 0,\ldots,k+1 \;\;\&\;\; F(\omega_j) \neq 0, \\ \textit{if } n+m \textit{ is even and } \xi \textit{ is purely real,} \\ \textit{or } n+m \textit{ is odd and } \xi \textit{ is purely imaginary.} \end{cases}$$

and sort the distinct α_j's in ascending order

$$\bar{\alpha}_0 < \bar{\alpha}_1 < \ldots < \bar{\alpha}_{k+2} < \bar{\alpha}_{k+3}$$

where $\bar{\alpha}_0 = -\infty$ and $\bar{\alpha}_{k+3} = \infty$.

2) *Identify all the sequences of signums*

$$\mathcal{I} = \begin{cases} \{i_1,\ldots,i_k\} \\ \textit{if } n+m \textit{ is even and } \xi \textit{ is not purely real,} \\ \textit{or } n+m \textit{ is odd and } \xi \textit{ is not purely imaginary.} \\ \\ \{i_0, i_1,\ldots,i_{k+1}\} \\ \textit{if } n+m \textit{ is even and } \xi \textit{ is purely real,} \\ \textit{or } n+m \textit{ is odd and } \xi \textit{ is purely imaginary.} \end{cases}$$

where $i_j \in \{-1,1\}$ for $j = 0,1,\ldots,k+1$, that correspond to the intervals $(\bar{\alpha}_j, \bar{\alpha}_{j+1})$ for $j = 0,\ldots,k+2$.

3) *For each signum sequence $\mathcal{I}_j$ from step 2, if*

$$r+\sigma(p^*) = \begin{cases} \{(-1)^{k-1}i_0 + \cdots + i_{k-2} - i_{k-1} \\ \quad + i_k\}\mathcal{S}G(\infty) \\ \textit{if } n+m \textit{ is even and } \xi \textit{ is not purely real,} \\ \textit{or } n+m \textit{ is odd and } \xi \textit{ is not purely imaginary.} \\ \\ \frac{1}{2}\{(-1)^k i_0 + \cdots - 2i_{k-1} + 2i_k \\ \quad - i_{k+1}\}\mathcal{S}G(\infty) \\ \textit{if } n+m \textit{ is even and } \xi \textit{ is purely real,} \\ \textit{or } n+m \textit{ is odd and } \xi \textit{ is purely imaginary.} \end{cases}$$

holds, then $(\bar{\alpha}_j, \bar{\alpha}_{j+1}) \in A_r(p,q)$

Remark 1. By Step 3 of Algorithm 1, a necessary condition for the existence of an $\alpha \in A_r(p,q)$ is that the imaginary part of $[q(s) + \alpha p(s)]\bar{p}^*(s)$ has at least $\bar{r} = |r + \sigma(p^*)|$ real roots with odd multiplicities if $n+m$ is even and ξ is not purely real, or $n+m$ is odd and ξ is not purely imaginary, and $\bar{r} = |r + \sigma(p^*) - 1|$ real roots with odd multiplicities if $n+m$ is even and ξ is purely real, or $n+m$ is odd and ξ is purely imaginary. △

IV. PID Controllers

Given a plant $g(s) = \frac{p(s)}{q(s)}$ and a PID controller $c(s) = \frac{k_d s^2 + k_p s + k_i}{s}$, our objective is to find all values of (k_p, k_i, k_d) such that the closed loop characteristic polynomial

$$\phi(s, k_p, k_i, k_d) = sq(s) + (k_d s^2 + k_p s + k_i)p(s)$$

has all its roots in the region S given in Figure 1. This is equivalent to solving two subproblems using the stability regions S_θ and $S_{-\theta}$ and finding the intersection of the solution sets.

Let us first solve the problem for the stability region S_θ. Let $s = s_1 e^{j\theta}$, then

$$\begin{aligned}\phi(s,k_p,k_i,k_d) &= sq(s) + (k_d s^2 + k_p s + k_i)p(s),\\ &= s_1 e^{j\theta} q(s_1 e^{j\theta}) + (k_d s_1^2 e^{j2\theta} + k_p s_1 e^{j\theta}\\ &\quad + k_i)p(s_1 e^{j\theta}).\end{aligned}$$

Since θ is constant, we have $e^{j\theta} = c + jd$, $q(s_1 e^{j\theta}) = \tilde{q}(s_1)$, and $p(s_1 e^{j\theta}) = \tilde{p}(s_1)$ where $\tilde{q}(s_1)$ and $\tilde{p}(s_1)$ are polynomials with complex coefficients. The new characteristic polynomial is given by

$$\begin{aligned}\phi_\theta^0(s_1,k_p,k_i,k_d) &= (c+jd)s_1\tilde{q}(s_1) + [k_d(c^2-d^2\\ &\quad +j2cd)s_1^2 + k_p(c+jd)s_1 + k_i]\tilde{p}(s_1)\\ &= q_0(s_1) + k_i p_0(s_1).\end{aligned}$$

where

$$\begin{aligned}q_0(s_1) = \quad &(c+jd)s_1\tilde{q}(s_1) + [k_d(c^2-d^2+j2cd)s_1^2\\ &+ k_p s_1(c+jd)]\tilde{p}(s_1),\end{aligned}$$

$p_0(s_1) = \tilde{p}(s_1)$. Roots of $\phi(s,k_p,k_i,k_d)$ in stability region S_θ is equivalent to roots of $\phi_\theta^0(s_1,k_p,k_i,k_d)$ in the open left half complex plane. Using the generalized Hermite-Biehler theorem applicable to complex polynomials and Lemma 1, we describe in what follows a method to compute all values of (k_p,k_i,k_d) such that $\phi_\theta^0(s_1,k_p,k_i,k_d)$ is Hurwitz stable. Let

$$\begin{aligned}\tilde{q}(j\omega) &= h(\omega) + jg(\omega),\\ \tilde{p}(j\omega) &= f(\omega) + je(\omega),\\ \bar{p}^*(j\omega) &= \bar{f}(\omega) - j\bar{e}(\omega),\end{aligned}$$

recall that $\bar{f}$ and $\bar{e}$ are coprime polynomials, then

$$\begin{aligned}\tilde{q}(j\omega)\bar{p}^*(j\omega) &= H(\omega) + jG(\omega),\\ \tilde{p}(j\omega)\bar{p}^*(j\omega) &= F(\omega),\end{aligned}$$

where H, G, and F are given by (2). Multiplying $\phi_\theta^0(j\omega,k_p,k_i,k_d)$ by $\bar{p}^*(j\omega)$ we obtain

$$\begin{aligned}\psi_\theta^1(j\omega,k_p,k_i,k_d) &= \phi_\theta^0(j\omega,k_p,k_i,k_d)\bar{p}^*(j\omega)\\ &= [-\omega(dH(\omega)+cG(\omega)) - k_d(c^2\\ &\quad -d^2)\omega^2F(\omega) - k_p d\omega F(\omega) + k_i F(\omega)]\\ &\quad +j[H_1(\omega) + k_pG_1(\omega) + k_dF_1(\omega)]\end{aligned}$$

where

$$\begin{aligned}H_1(\omega) &= \omega(cH(\omega) - dG(\omega))\\ G_1(\omega) &= c\omega F(\omega)\\ F_1(\omega) &= -2cd\omega^2F(\omega)\end{aligned}$$

The reasoning behind the algorithm which determines the set of parameters $k_p,\ k_i,\ k_d$ of a stabilizing PID controller can be explained as follows. Suppose $\phi_\theta^0(s)$ is Hurwitz stable for some $k_p,\ k_i,\ k_d \in \mathbf{R}$. By Remark 1, it follows that the imaginary part $H_1(\omega) + k_pG_1(\omega) + k_dF_1(\omega)$ of $\psi_\theta^1(s)$ has at least $r_1 = |n + 1 + \sigma(p^*)|$ real roots with odd multiplicities. Suppose the imaginary part of $\psi_\theta^1(s)$ has r_1 real roots with odd multiplicities. By Lemma 1, $\sigma[\phi_\theta^1(s_1)] = -r_1$, where

$$\phi_\theta^1(j\omega,k_p,k_d) \quad = \quad q_1(j\omega) + k_dp_1(j\omega) \qquad (3)$$

and

$$\begin{aligned}q_1(j\omega) &= [H_1(\omega) + jH_1^{'}(\omega)] + k_p[G_1(\omega) + jG_1^{'}(\omega)],\\ p_1(j\omega) &= F_1(\omega) + jF_1^{'}(\omega).\end{aligned}$$

Repeating the same process once more we get

$$\begin{aligned}\psi_\theta^2(j\omega,k_p,k_d) &= \phi_\theta^1(j\omega,k_p,k_d)\bar{p}_1^*(j\omega)\\ &= [H_{2r}(\omega) + k_pG_{2r}(\omega) + k_dF_{2r}(\omega)]\\ &\quad +j[H_2(\omega) + k_pG_2(\omega)]\end{aligned}$$

where

$$\begin{aligned}H_{2r}(\omega) &= H_1(\omega)F_1(\omega) - H_1^{'}(\omega)F_1^{'}(\omega)\\ G_{2r}(\omega) &= G_1(\omega)F_1(\omega) - G_1^{'}(\omega)F_1^{'}(\omega)\\ F_{2r}(\omega) &= F_1(\omega)F_1(\omega) - F_1^{'}(\omega)F_1^{'}(\omega)\\ H_2(\omega) &= H_1^{'}(\omega)F_1(\omega) - H_1(\omega)F_1^{'}(\omega)\\ G_2(\omega) &= G_1^{'}(\omega)F_1(\omega) - G_1(\omega)F_1^{'}(\omega)\end{aligned}$$

Once more, by Remark 1, it follows that the imaginary part $H_2(\omega) + k_pG_2(\omega)$ of $\psi_\theta^2(s)$ has at least $r_2 = |r_1 + \sigma(p^*)|$ real roots with odd multiplicities. Now the set of $k_p \in \mathbf{R}$ which achieves r_2 real roots with odd multiplicities in $H_2(\omega) + k_pG_2(\omega)$ can be determined by applying Algorithm 1 to $q_2(s)$ and $p_2(s)$ where

$$\begin{aligned}q_2(j\omega) &= H_2(\omega) + jH_2^{'}(\omega),\\ p_2(j\omega) &= G_2(\omega) + jG_2^{'}(\omega).\end{aligned}$$

The algorithm below traces the above steps backwards by repetition of the steps (i)-(iii) below:

(i) Pick a value of k_p such that the number of real roots with odd multiplicities of $H_2(\omega) + k_pF_2(\omega)$ is r_2 or greater.

(ii) For every k_p determined, find using Algorithm 1, all k_d such that the imaginary part of $\psi_\theta^1(s)$ has at least r_1 real roots.

(iii) For every k_d determined, find using Algorithm 1, all k_i such that $\phi_\theta(s)$ is Hurwitz stable.

Algorithm 2.

1) *Using Algorithm 1, partition the real axis into intervals (or union of intervals) such that the number of real roots with odd multiplicities of $H_2(\omega) + k_pG_2(\omega)$ is constant in each interval.*
2) *Fix $r_1 = |n + \sigma(p_0^*) + 1|$.*
   a) *Find admissible range of k_p from the intervals found in the first step. (This corresponds to values of k_p such that $H_2(\omega) + k_pG_2(\omega)$ has at least $r_2 = |r_1 + \sigma(p_1^*)|$ real roots with odd multiplicities).*
      i) *Fix a k_p in the admissible range.*
      ii) *Apply Algorithm 1 to $q_1(s)$ and $p_1(s)$. (This calculates admissible values of k_d such that $H_1(\omega)+k_pG_1(\omega)+k_dF_1(\omega)$ has r_1 real roots).*
         A) *Fix a k_d in the admissible range.*
         B) *Apply Algorithm 1 to $q_0(s)$ and $p_0(s)$. (This calculates admissible values of k_i such that $\phi_\theta^0(s)$ is Hurwitz stable).*
         C) *Increment k_d and go to step 2.a.ii.B.*
      iii) *Increment k_p and go to step 2.a.ii.*
   b) *If $r_1 < deg(H_1)$, then increment r_1 by one and go to step 2.a.*

For the stability region $S_{-\theta}$, it can be easily shown that the set of stabilizing PID controllers is exactly the same as the set of stabilizing PID controllers for

S_θ. To see this, suppose that for $(\bar{k}_p, \bar{k}_i, \bar{k}_d)$, s_0 is a root of $\phi(s, \bar{k}_p, \bar{k}_i, \bar{k}_d)$, then $s_0 e^{j\theta} q(s_0 e^{j\theta}) + (\bar{k}_d s_0^2 e^{j2\theta} + \bar{k}_p s_0 e^{j\theta} + \bar{k}_i) p(s_0 e^{j\theta}) = 0$. As $q(s)$ and $p(s)$ are real polynomials, it follows that $s_0^* e^{-j\theta} q(s_0^* e^{-j\theta}) + (\bar{k}_d s_0^{*2} e^{-j2\theta} + \bar{k}_p s_0^* e^{-j\theta} + \bar{k}_i) p(s_0^* e^{-j\theta}) = 0$ where s_0^* is the complex conjugate of s_0. Since s_0^* and s_0 have the same real part, it follows that $(\bar{k}_p, \bar{k}_i, \bar{k}_d)$ is stabilizing triplet for the stability region $S_{-\theta}$ if and only if it is stabilizing triplet for the stability region S_θ.

Example 1. Consider a PID controller

$$c(s) = \frac{k_d s^2 + k_p s + k_i}{s}$$

to stabilize the plant $g(s) = \frac{p(s)}{q(s)}$ given in [5], where

$$\begin{aligned} q(s) &= s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17, \\ p(s) &= s^3 - 4s^2 + s + 2. \end{aligned}$$

The stability region S is given in Figure 1 and specified by the parameter $\theta = \frac{\pi}{4}$. For the rotated Hurwitz stability regions S_θ and $S_{-\theta}$, let $s = s_1 e^{j\frac{\pi}{4}}$, then

$$\begin{aligned} \tilde{q}_1(s_1) = \quad & (-0.7071 + 0.7071j)s_1^5 - 8s_1^4 - (22.6274- \\ & 22.6274j)s_1^3 + 46js_1^2 + (32.5269+ \\ & 32.5269j)s_1 + 17, \\ \tilde{p}_1(s_1) = \quad & (-0.7071 + 0.7071j)s_1^3 - 4js_1^2 \\ & +(0.7071 + 0.7071j)s_1 + 2. \end{aligned}$$

Using the new polynomials $\tilde{q}_1(s_1)$, $\tilde{p}_1(s_1)$, and the method described in this section, we obtain the stabilizing values of (k_p, k_i, k_d) as shown in Figure 2. From these

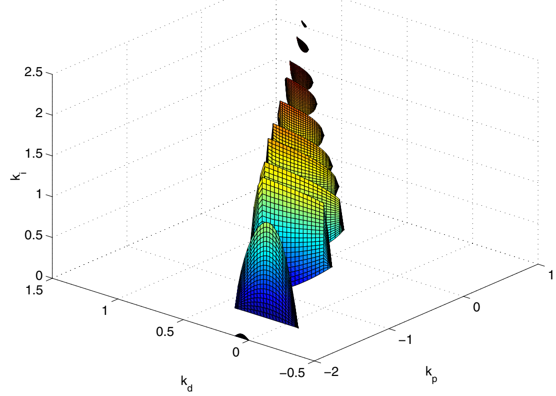

Fig. 2. Stabilizing values (k_p, k_i, k_d).

results , for $k_p = -1$ and $k_i = 0.4049$ we obtain $(0.0255, 0.5612)$ as the stabilizing interval for k_d. The root-locus for the values of k_d in this interval is shown in Figure 3. With $k_p = 0.5$ and $k_i = 1.7$, we obtain $(0.9355, 1.0986)$ as the stabilizing interval for k_d. The root-locus for the values of k_d in this interval is shown in Figure 4. •

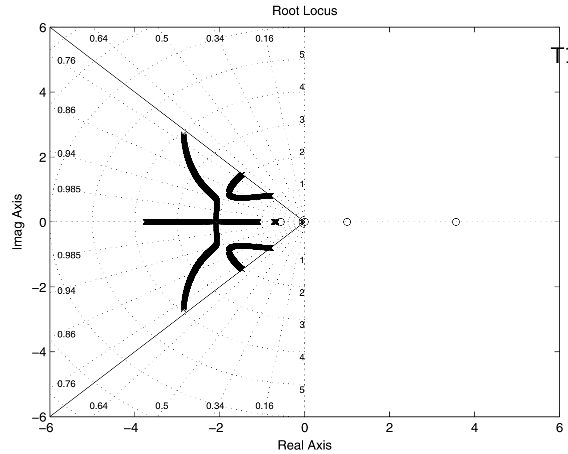

Fig. 3. Attainable roots with respect to regions S_θ and $S_{-\theta}$.

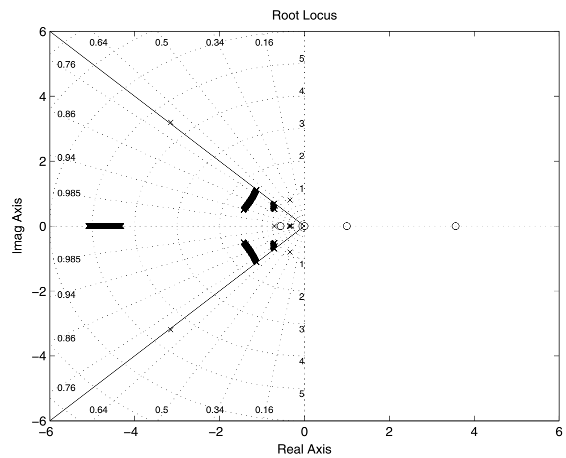

Fig. 4. Attainable roots with respect to region S.

V. Conclusion

In this paper, an algorithm is given for computing the set of all PID controllers that places the poles of a closed loop system in a desired stability region. The method is based on a generalization of the Hermite-Biehler theorem applicable to complex polynomials. We can also consider other stability regions. For example, by simply replacing s by $s + \sigma$ stabilizing with respect to shifted Hurwitz stability region is possible. We can combine both regions, shifted and rotated Hurwitz stability regions, and use the algorithm given in this paper to consider stabilizability with respect to other sectors of the left half-plane.

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