Lower hedging of American contingent claims with minimal surplus risk in finite-state financial markets by mixed-integer linear programming

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A R T I C L E   I N F O

A B S T R A C T

The lower hedging problem with a minimal expected surplus risk criterion in incomplete markets is studied for American claims in finite state financial markets. It is shown that the lower hedging problem with linear expected surplus criterion for American contingent claims in finite state markets gives rise to a non-convex bilinear programming formulation which admits an exact linearization. The resulting mixed-integer linear program can be readily processed by available software.

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1. Introduction

The main purpose of the present paper is to address a problem of crucial importance in mathematical finance using mixed-integer linear programming as a computational tool. While integer programming has been used in financial optimization in the context of portfolio optimization and structuring collateralized mortgage obligations (see [20] for various models), its use in other branches of mathematical finance (e.g., option pricing) has been so far limited to a few papers, namely [2,3,6,14]. The goal is to contribute to this stream of literature by introducing yet another challenging application to the discrete optimization/integer programming community.

A fundamental problem of financial economics is the pricing of uncertain future cash streams generated by financial instruments called contingent claims. As the name "contingent claim" implies, the uncertain cash stream is contingent upon realized values of other financial instruments or economic variables. A common approach to pricing contingent claims is to value their uncertain income streams with respect to other traded instruments in the market, which consists in exactly replicating the income stream by a portfolio of traded instruments in all states of the world. When such perfect replication is possible we say that the financial market is complete and, the present value of the replicating portfolio should be equal to the present value of the uncertain cash stream by the principle of no-arbitrage. When a perfect replication is not possible with existing traded instruments in the market, one faces an incomplete market and the impossibility to compute a unique price using the no-arbitrage principle. In this case, one can compute the so-called lower and upper hedging prices (also referred to as sub-hedging and super-hedging prices). The upper hedging price is obtained by computing the present value of the least costly portfolio of existing instruments whose pay-off dominates the uncertain cash stream. By the same token, the lower hedging price is the present value of the most precious portfolio of existing instruments whose pay-off is dominated by the income stream in question. These two values provide an interval of possible prices where no arbitrage exists for

the buyer or the seller (writer) of a contingent claim. However, in practice the lower and upper hedging values may not be useful for the potential buyer and seller of a contingent claim. While it offers full protection against all states of the world, the upper hedging price is sometimes too high to be interesting for any buyer; see [9] for an example. Therefore, the potential seller may be willing to settle for a smaller price while taking a calculated risk of not being able to fully hedge the pay-out to the buyer. A symmetric argument can also be made for a potential buyer of a contingent claim when the lower hedging price may be too low to be interesting for any potential seller of contingent claim. It even occurs that the lower hedging price is computed to be zero! In this case, the buyer may be prepared to offer a higher price while running the risk of forming a hedge portfolio that may result in a surplus in some future state(s) of the world. This is the setting we consider in this paper. We will be interested in computing the lower hedging portfolio process for American contingent claims, which are instruments that can be exercised at any time until a certain maturity date, using an *expected surplus* criterion which is the reciprocal of an expected shortfall criterion widely studied in the literature; see [4,8,9,12,13,15,17–19]. These references sometimes deal with more general risk measures, e.g., coherent and convex measures of risk of which expected shortfall is a special case, and usually work in infinite-dimensional spaces and continuous time markets. However, with the exception of [15], they address mainly claims of the European type and do not give practical optimization formulations ready to be processed by available software. Since almost all previous work on the expected shortfall criterion for pricing in incomplete markets takes the viewpoint of a writer, we shall concentrate on the problem of the buyer. Furthermore, the lower hedging problem for the American contingent claims breaks the full symmetry with the upper hedging problem, and allows interesting optimization models in finite state markets as we shall demonstrate using a previous characterization of the lower hedging no-arbitrage price for American claims established in [2]. We demonstrate the computational usefulness of the optimization model on a numerical example. To the best of our knowledge, our formulations of the present are the first attempts in the literature to give a practical computing tool for the minimal expected surplus hedging of American contingent claims.

In a closely related, companion paper [16] we treat the problem of computing lower hedging portfolios for European and American claims in infinite-state and finite-state markets in discrete time using the risk measure of *quantile hedging* [7]. The quantile hedging criterion aims to minimize the probability of the event that a replicating portfolio falls short of the target pay-off (or exceeds it). It does not consider the magnitude of the shortfall (or surplus), and thus has been criticized for overlooking this aspect of the risk, which is the reason for preparing the present paper.

The rest of this paper is organized as follows. In Section 2 we briefly introduce the lower hedging problem for European claims under minimal surplus risk. Section 3 is devoted to the study of the lower hedging with minimal surplus risk for American claims, and the derivation of our formulations in finite-dimensional spaces is given in Section 4 along with a numerical example. We conclude the paper in Section 5.

2. Introduction to lower hedging with minimal surplus risk

We work in a financial market $\mathcal{M} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, S, \{\mathcal{F}_t\}_{t \in \mathbb{T}})$ with discrete time trading over the time set $\mathbb{T} = \{0, 1, \ldots, T\}$ and where $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in \mathbb{T}})$ is a complete filtered probability space, and $S = \{S_t\}_{t \in \mathbb{T}}$ is an $\mathbb{R}_+^2$ asset price process over the time set $\mathbb{T}$ adapted to the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{T}}$. We assume without loss of generality that the first component of S is the numéraire security, i.e., $S_t^0 = 1$ for all $t \in \mathbb{T}$. Let $\mathcal{Q}$ be the set of equivalent martingale measures in the arbitrage-free (not necessarily complete) market $\mathcal{M}$. For the rest of the paper we make the following blanket assumption.

Assumption 1. The market $\mathcal{M}$ is arbitrage free, i.e. the set $\mathcal{Q}$ is non-empty.

Let a European contingent claim H maturing at time T be a given non-negative and $\mathbb{P}$-integrable random variable, and let $\Pi^\downarrow(H)$ denote its lower hedging (sub-hedging) price, i,e,.

$$\Pi^\downarrow(H) \equiv \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^\mathbb{Q}[H].$$

For $\ell : \mathbb{R} \to \mathbb{R}$, an increasing function with the property

$$\ell(x) = 0 \quad \text{for } x \leq 0,$$

we consider the problem

$$\min_Y \mathbb{E}^\mathbb{P}[\ell(Y - H)]$$

over all $\mathcal{F}_T$-measurable non-negative random variables Y such that

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^\mathbb{Q}[Y] \geq v$$

where $v \geq \Pi^\downarrow(H)$. The stochastic quantity $Y - H$ measures the "surplus", i.e. the amount by which the variable Y overshoots the target H, and ℓ serves as a "disutility" or a risk function that we wish to minimize in expectation. Observe that if Y^* solves this problem, then so does $\tilde{Y} = H \vee Y^*$. Let $\mathcal{R}$ denote the set of $\mathcal{F}_T$-measurable $[1, \infty]$-valued random variables ψ, and $\mathcal{R}_0$

denote the set of random variables $\psi \in \mathcal{R}$ satisfying

$$\mathbb{E}^{\mathbb{Q}}[H\psi] \geq v \quad \forall \mathbb{Q} \in \mathcal{Q}.$$

For a given portfolio strategy $\xi = (\xi_0, \xi_1, \ldots, \xi_T)$, its value process V is given by $V_0 = \xi_0 \cdot S_0$ (we use $\xi_0 \cdot S_0$ to denote the inner product between S_0 and ξ_0) and $V_t = \xi_t \cdot S_t$ for $t = 1, \ldots, T$. A portfolio strategy ξ is said to be self-financing if it satisfies

$$S_t \cdot (\xi_t - \xi_{t-1}) = 0 \quad \forall t = 1, \ldots, T.$$

Now, for a given portfolio strategy ξ and its value process V, we define the *failure ratio*

$$\psi_V = \mathbb{1}_{\{V_T \leq H\}} + \frac{V_T}{H} \mathbb{1}_{\{V_T > H\}}.$$

Under the light of the above definitions, the lower hedging problem of interest to the buyer over variables ξ that are admissible adapted portfolio strategies (a self-financing trading strategy is called an admissible portfolio strategy if its value process V satisfies $V_T \geq 0$), and their value processes V, is problem [LECP]

$$\begin{aligned} \inf \quad & \mathbb{E}^{\mathbb{P}}[\ell(H\psi_V - H)] \\ \text{s.t.} \quad & V_0 \geq v. \end{aligned}$$

In other words, for an amount $v > \Pi^{\downarrow}(H)$, the lower hedging problem under minimal surplus risk consists in searching among all admissible strategies with initial endowment equal to at least v one that minimizes expected surplus.

In the following section we extend the above concept to the case of American contingent claims.

3. Minimum surplus lower hedging of American claims

An American contingent claim (ACC) is a financial instrument or contract that promises future pay-offs contingent on the evolution of a stochastic quantity such as a stock, an index, or interest rates. The owner of the ACC can choose to exercise the ACC at any time t between the present and the maturity date T, and acquire a pay-off C_t after which the contract is terminated. The owner may also choose not to exercise the ACC at all until the maturity date T. Therefore, the computation of a price for the ACC requires the calculation of an optimal exercise strategy on the part of the potential owner.

Let an American contingent claim $C = \{C_t\}_{t \in \mathbb{T}}$ be a given non-negative adapted and $\mathbb{P}$-integrable process. One common way to describe exercise strategies of ACCs is by stopping times. These are functions $\tau : \Omega \rightarrow \{0, \ldots, T\} \cup \{+\infty\}$ such that $\{\omega \in \Omega | \tau(\omega) = t\} \in \mathcal{F}_t$, for each $t = 0, \ldots, T$. Let $\mathcal{T}$ denote the set of stopping times. It is well-known (see [3]) that the lower hedging price for an American claim can be expressed as

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau].$$

We refer to $\Pi^{\downarrow}(C)$ as the "sub-hedging price" for the American claim C. Chalasani and Jha [3] show that $\Pi^{\downarrow}(C)$ can be obtained as the optimal value of the following optimization problem:

$$\max_{\xi \in \Xi} \{-V_0(\xi) | \exists \tau \in \mathcal{T} \text{ s.t. } V_\tau(\xi) + C_\tau \geq 0\}$$

where Ξ represents the set of self-financing portfolio strategies ξ. Assuming ξ^* is an optimal portfolio strategy and τ^* an optimal exercise rule, the buyer borrows the amount $V_0(\xi^*)$ at time 0 to pay the seller for the contingent claim, and acquires the claim. At the time τ^* of exercise of the claim, the buyer repays his/her debt incurred at time 0. We refer to the optimal portfolio strategy of the buyer as a "sub-hedging strategy". A sub-hedging strategy ξ^* has the property that $V_t(\xi^*) \leq C_t$ on $\{C_t > 0\}$ for all $t \in \mathbb{T}$, and $V_T(\xi^*) \geq 0$; see [3,9,14].

Inspired by [15], for a portfolio strategy ξ we define the failure ratio process ψ^ξ of ξ by

$$\psi_t^\xi := \mathbb{1}_{\{V_t(\xi) \leq C_t\}} + \frac{V_t(\xi)}{C_t} \mathbb{1}_{\{V_t(\xi) > C_t\}} = \frac{V_t(\xi)}{C_t} \vee 1.$$

For an amount $v > \Pi^{\downarrow}(C)$, the lower hedging problem under minimal surplus risk consists in searching among all self-financing portfolio strategies with initial endowment equal to at least v one that minimizes the *maximum of expected surplus* over all stopping times. In other words, the problem of lower hedging in this case is problem [ACSHP]

$$\begin{aligned} \inf \quad & \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\ell(C_\tau(\psi_\tau^\xi - 1))] \\ \text{s.t.} \quad & V_0(\xi) \geq v \end{aligned}$$

where ℓ has exactly the same properties as in Section 2. We are interested in self-financing portfolio strategies ξ with value process V such that $V_t = 0$ on $\{C_t = 0\}$. We define by R the set of $[1, \infty]$-valued adapted processes, i.e.

$$\mathsf{R} = \{\psi = \{\psi_t\}_{t \in \mathbb{T}} : \psi_t \in [1, \infty], \text{ and } \mathcal{F}_t\text{-measurable}, \forall t \in \mathbb{T}\}.$$

For the American claim C we define the subset R_0 of R

$$R_0 = \{\psi \in R : \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau \psi_\tau] \geq v\}.$$

The goal is to solve the following problem referred to as [ACFP] as a proxy to [ACSHP] and recover a solution of [ACSHP] from a solution of [ACFP].

$$\inf \quad \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\ell(C_\tau(\psi_\tau - 1))]$$
$$\text{s.t.} \quad \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau \psi_\tau] \quad \geq \quad v$$
$$\psi \in [1, \infty] \qquad \mathbb{P}\text{-a.s.}$$

The following result makes the relationship between the two problems precise. The proof is in the Appendix.

Theorem 1. *For an optimal solution $\hat{\psi}$ of [ACFP], a sub-hedging strategy $\hat{\xi}$ for the adjusted American claim $\hat{C} = C\hat{\psi}$ is a solution of problem [ACSHP], and both problems have identical optimal value.*

As a consequence of the theorem, for an optimal ψ^*, the lower-hedge policy for the scaled-up claim $C\psi^*$ is the optimal lower minimal surplus hedge corresponding to initial capital v. Next we investigate finite-dimensional optimization formulations of the minimal surplus lower hedging problem in finite-state markets for a piecewise linear surplus function.

4. Minimal surplus lower hedging problem for American claims in finite state markets

In finite state markets, we can transform [ACFP] into a finite-dimensional optimization problem that can be processed numerically by existing optimization algorithms and software. First, we describe the finite-state financial market.

In our financial market security prices and other payments are discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose atoms are sequences of real-valued vectors (asset values) over discrete time periods $t \in \mathbb{T} = \{0, 1, \ldots, T\}$ [10,11]. We further assume that the market evolves as a discrete, non-recombinant scenario tree (hence, suitable for incomplete markets [5,11]) in which the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at level t in the tree. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. While not needed in the finite probability setting, the σ-algebras $\mathcal{F}_t$ generated by the partitions $\mathcal{N}_t$ are such that, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all $0 \leq t \leq T - 1$ and $\mathcal{F}_T = \mathcal{F}$. A stochastic process is said to be $(\mathcal{F}_t)_{t=0}^T$-adapted if for each $t = 0, \ldots, T$, the outcome of the process only depends on the element of $\mathcal{F}_t$ that has been realized at stage t. Similarly, a decision process is said to be $(\mathcal{F}_t)_{t=0}^T$-adapted if for each $t \in \mathbb{T}$, the decision depends on the element of $\mathcal{F}_t$ that has been realized at stage t. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, \ldots, T$ has a unique parent denoted by $\pi(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$ has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. We denote the set of all nodes in the tree by $\mathcal{N}$. The set $\mathcal{A}(n, m)$ denotes the collection of ascendant nodes or the unique path leading to node n (including itself) from node m while $\mathcal{D}(n)$ denotes the set of all descendant nodes of the node n including itself. The probability distribution $\mathbb{P}$ is obtained by attaching positive weights p_n to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non-leaf (intermediate level) node in the tree we have, recursively,

$$p_n = \sum_{m \in \mathcal{C}(n)} p_m, \quad \forall n \in \mathcal{N}_t, \ t = T - 1, \ldots, 0.$$

Hence, each non-leaf node has a probability mass equal to the combined mass of its child nodes.

A random variable X is a real valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathcal{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbb{R}\}$ is either the empty set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathcal{N}_t$ if it can be assigned a value on each node of $\mathcal{N}_t$ that is consistent with its definition on Ω. This kind of random variable is said to be measurable with respect to the information contained in the nodes of $\mathcal{N}_t$. For our purposes in the present paper, it suffices to say that X is a function of the nodes n of the scenario tree. A stochastic process $\{X_t\}$ is a time-indexed collection of random variables such that each X_t is measurable with respect $\mathcal{N}_t$. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^{\mathbb{P}}[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathcal{N}_t$ is given by the expression

$$\mathbb{E}^{\mathbb{P}}[X_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{C}(n)} \frac{p_m}{p_n} X_m.$$

The market consists of a bond (the risk-free asset) and a single risky security with prices at node n given by the two-dimensional vector $Z_n = (Z_n^0, Z_n^1)^T$. The number of shares of securities held by the investor at node $n \in \mathcal{N}_t$ is denoted by $\theta_n \in \mathbb{R}^2$. Therefore, to each state $n \in \mathcal{N}_t$ is associated the two-dimensional real vector θ_n. The value of the portfolio at state
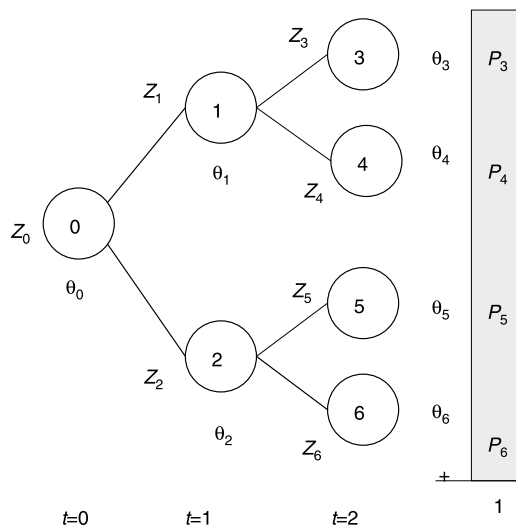
Fig. 1. A sample scenario tree.

n is $Z_n\theta_n$. We assume without loss of generality that prices at all nodes have been scaled so that $Z_n^0 = 1$ for all $n \in \mathcal{N}$. The assumption of a single risky security can be easily relaxed, and the development of the paper can be repeated for multiple securities, *mutatis mutandis*.

An example of scenario with three trading dates is given in Fig. 1.

Definition 1. If there exists a probability measure $\mathbb{Q} = \{q_n\}_{n \in \mathcal{N}_T}$ such that

$$S_t = \mathbb{E}^{\mathbb{Q}}[S_{t+1}|\mathcal{N}_t] \quad (t \leq T - 1)$$

then the process $\{S_t\}$ is called a martingale under $\mathbb{Q}$, and $\mathbb{Q}$ is called a martingale probability measure for the process.

We denote as usual by $\mathcal{Q}$ the set of all equivalent probability measures that make S a martingale over $[0, 1, \ldots, T]$.

4.1. Lower hedging price for the buyer of an American claim

Before describing the transformation of problem [ACFP] into a problem that can be numerically solved, we need an auxiliary result that is important in its own right, and will let us formulate the pricing problem of the buyer as a linear program with all the nice duality theory attached to it. To this end, let us recall that an arbitrage seeking buyer's problem for an American contingent claim F with pay-offs F_n for all $n \in \mathcal{N}$[1] (i.e., the computation of a sub-hedging strategy for F), can be formulated as the following problem that we will refer as AP1 [14].

$$
\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & Z_0 \cdot \theta_0 = F_0 e_0 - V \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \quad \forall n \in \mathcal{N}_t, \ 1 \leq t \leq T \\
& Z_n \cdot \theta_n \geq 0, \ \forall n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n)} e_m \leq 1, \ \forall n \in \mathcal{N}_T \\
& e_n \in \{0, 1\}, \ \forall n \in \mathcal{N}.
\end{aligned}
$$

The optimal value of V gives the quantity $\Pi^{\downarrow}(F)$, and is the largest amount that a potential buyer is willing to disburse for acquiring a given American contingent claim F. The computation of this quantity via the above integer programming problem is carried out by construction of a least costly (adapted) portfolio process replicating the proceeds from the contingent claim by self-financing transactions using the market-traded securities in such a way to avoid any terminal losses. The integer variables and related constraints represent the one-time exercise of the American contingent claim; see [14] for further details. Define the sets

$$E = \left\{ e | e \text{ is } (\mathcal{F}_t)_{t=0}^{T}\text{-adapted}, \sum_{t=0}^{T} e_t \leq 1 \text{ and } e_t \in \{0, 1\} P\text{-a.s.} \right\},$$

---

[1] To avoid possible confusion between C_n and C_t we use the notation F_n for pay-off at node n from this point on. Whenever the pay-off is indexed by t we use C_t.

$$\tilde{E} = \left\{ e | e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted}, \sum_{t=0}^T e_t \le 1 \text{ and } e_t \ge 0 P\text{-a.s.} \right\}.$$

The relation $e_t = 1 \Leftrightarrow \tau = t$ defines a one-to-one correspondence between stopping times and decision variables $e \in E$.

A linear programming relaxation of AP1 is the following problem AP2:

$$
\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & Z_0 \cdot \theta_0 &=& \; F_0 e_0 - V \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) &=& \; F_n e_n, \quad \forall n \in \mathcal{N}_t, \; 1 \le t \le T \\
& Z_n \cdot \theta_n &\ge& \; 0, \quad \forall n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n,0)} e_m &\le& \; 1, \quad \forall n \in \mathcal{N}_T \\
& e_n &\ge& \; 0, \quad \forall n \in \mathcal{N}.
\end{aligned}
$$

Now, we have the following result that is proved in [2,14].

Theorem 2. *There exists an optimal solution to AP2 with $e_n \in \{0, 1\}$, $\forall n \in \mathcal{N}$.*

A direct consequence of the above result is given below. We denote by $\tilde{\mathcal{Q}}$ the set of all martingale measures (not necessarily equivalent to $\mathbb{P}$), which is nothing other than the closure of $\mathcal{Q}$ in discrete-time finite state markets [10].

Assuming no arbitrage in the stock price processes, the buyer's price for American contingent claim C can be expressed as in the following theorem; see [2,14] for a proof. We use OPT(P) to denote the optimal value of an optimization problem P.

Theorem 3.

$$\max_{\tau \in \mathcal{T}} \min_{\mathbb{Q} \in \tilde{\mathcal{Q}}} \mathbb{E}^{\mathbb{Q}}[C_\tau] = \min_{\mathbb{Q} \in \tilde{\mathcal{Q}}} \max_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau]. \tag{1}$$

Corollary 1. $\inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau] = \text{OPT}(AP1) = \text{OPT}(AP2)$.

4.2. Formulation of the minimum surplus hedge for American claims

We now return to the problem [ACFP] with $\ell(x) = x$ for $x > 0$ and $\ell(x) = 0$ for $x \le 0$. Let us first examine the objective function

$$\inf_\psi \sup_{\tau \in \mathcal{T}} \mathbb{E}^P[\ell(F(\psi_\tau - 1))].$$

For fixed ψ, in a finite state probability setting this inner sup is just the problem (since F is non-negative and $\psi \ge 1$)

$$\max_{e \in E} \sum_{n \in \mathcal{N}} p_n e_n(F_n(\psi_n - 1)).$$

However, as $\tilde{E}$ is the convex hull of e's [3,14] we have

$$\max_{e \in E} \sum_{n \in \mathcal{N}} p_n e_n(F_n(\psi_n - 1)) = \max_{e \in \tilde{E}} \sum_{n \in \mathcal{N}} p_n e_n(F_n(\psi_n - 1)).$$

Now, using linear programming duality, we can transform the objective function

$$\max_{e \in \tilde{E}} \sum_{n \in \mathcal{N}} p_n e_n(F_n(\psi_n - 1)) = \min_{\gamma_n} \left\{ \sum_{n \in \mathcal{N}_T} \zeta_n \; \Big| \; \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} \zeta_m \ge p_n F_n(\psi_n - 1) \; \forall n \in \mathcal{N} \right\}.$$

This completes the transformation of the objective function. Next, we transform the constraint

$$\inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[F_\tau \psi_\tau] \ge v.$$

By Theorem 2 the left hand side is equal to the optimal value of problem AP2 for the adjusted claim $F\psi$, i.e., the sub-hedging value for $F\psi$!

The above developments lead to the following result.

Theorem 4. *The minimum surplus lower hedging problem for an American claim F in discrete-time finite-state markets is posed as the problem [BASHP]*

$$\min \quad \sum_{n \in \mathcal{N}_T} \zeta_n$$

$$
\begin{aligned}
s.t. \quad & \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} \zeta_m && \geq && p_n F_n (\psi_n - 1), && \forall n \in \mathcal{N} \\
& -Z_0 \cdot \theta_0 + F_0 e_0 \psi_0 && \geq && v \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) && = && F_n e_n \psi_n, && \forall n \in \mathcal{N}_t, \, 1 \leq t \leq T \\
& Z_n \cdot \theta_n && \geq && 0, && \forall n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n,0)} e_m && \leq && 1, && \forall n \in \mathcal{N}_T \\
& e_n && \geq && 0, && \forall n \in \mathcal{N} \\
& \psi_n && \geq && 1, && \forall n \in \mathcal{N} \\
& \zeta_n && \geq && 0, && \forall n \in \mathcal{N}_T.
\end{aligned}
$$

Furthermore, any optimal portfolio strategy θ^* to [BASHP] solves problem [ACSHP].

Notice that the problem [BASHP] involves only continuous variables! It is a bilinear and hence non-convex problem. On the other hand, we could have equally stated the above result using integer valued variables e_n. While this may appear at first glance as an unnecessary complication, the presence of binary variables is the key to a linear, equivalent formulation.

Theorem 5. *The minimum surplus lower hedging problem for an American claim F in discrete-time finite-state markets is posed as the problem [BBASHP]*

$$\min \quad \sum_{n \in \mathcal{N}_T} \zeta_n$$

$$
\begin{aligned}
s.t. \quad & \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} \zeta_m && \geq && p_n F_n (\psi_n - 1) && \forall n \in \mathcal{N} \\
& -Z_0 \cdot \theta_0 + F_0 e_0 \psi_0 && \geq && v \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) && = && F_n e_n \psi_n, && \forall n \in \mathcal{N}_t, \, 1 \leq t \leq T \\
& Z_n \cdot \theta_n && \geq && 0, && \forall n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n,0)} e_m && \leq && 1, && \forall n \in \mathcal{N}_T \\
& e_n && \in && \{0, 1\}, && \forall n \in \mathcal{N} \\
& \psi_n && \geq && 1, && \forall n \in \mathcal{N} \\
& \zeta_n && \geq && 0, && \forall n \in \mathcal{N}_T.
\end{aligned}
$$

Proof. The proof is identical to that of the previous theorem since we could equally represent the lower hedging no-arbitrage value $\inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[F_\tau \psi_\tau]$ using problem AP1 by Corollary 1. $\square$

The problem [BBASHP] is a non-convex, non-linear mixed-integer programming problem, and some numerical algorithms and codes are available for its numerical solution. On the other hand, an exact linearization that uses the binary nature of the exercise variables e gives a mixed-integer linear formulation, readily and more reliably solvable by modern codes. Consider the following linear mixed-integer program [LBASHP]

$$\min \quad \sum_{n \in \mathcal{N}_T} \zeta_n$$

$$
\begin{aligned}
s.t. \quad & \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} \zeta_m && \geq && p_n F_n (\psi_n - 1), && \forall n \in \mathcal{N} \\
& \gamma_n && \geq && F_n (\psi_n - 1), && \forall n \in \mathcal{N} \\
& -Z_0 \cdot \theta_0 + F_0 w_0 && \geq && v \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) && = && F_n w_n, && \forall n \in \mathcal{N}_t, 1 \leq t \leq T \\
& Z_n \cdot \theta_n && \geq && 0, && \forall n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n,0)} e_m && \leq && 1, && \forall n \in \mathcal{N}_T \\
& e_n && \in && \{0, 1\}, && \forall n \in \mathcal{N} \\
& w_n && \leq && M e_n, && \forall n \in \mathcal{N} \\
& w_n && \leq && \psi_n, && \forall n \in \mathcal{N} \\
& \psi_n && \geq && 1, && \forall n \in \mathcal{N} \\
& \zeta_n && \geq && 0, && \forall n \in \mathcal{N}_T.
\end{aligned}
$$

where M can be chosen to be a suitable positive constant times v in our computational experience.
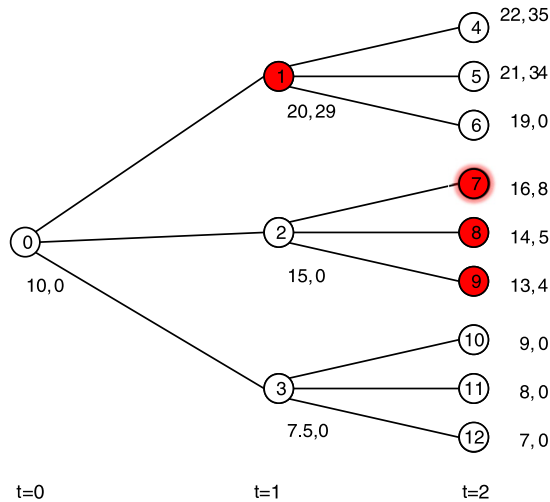
Fig. 2. Scenario tree for the numerical example. The numbers next to the node represent the risky security price and the pay-off value of the contingent claim at that node, respectively. The nodes where it is optimal to exercise the claim are colored. The node 7 is where a scaling of the pay-off occurs with $v = 8/3$ and $v = 7/3$, hence it is represented with a glow effect.

Theorem 6. OPT(BASHP) = OPT(LBASHP).

Proof. Let $\zeta^*, \psi^*, \gamma^*, \Theta^*, e^*$ be feasible for [BASHP]. Assume $e^* \in E$ after recalling that we can equally solve [BASHP] as a mixed-integer nonlinear optimization problem. Define $w_n^* = \psi_n^* e_n^*$ for all $n \in \mathcal{N}$. For suitable M this is feasible for [LBASHP].

For the converse, assume you have $\zeta^L, \psi^L, \gamma^L, \Theta^L, e^L, w^L$ that are feasible for [LBASHP]. If for all $n \in \mathcal{N}$ we have $w_n^L = e_n^L \psi_n^L$, then $\zeta^L, \psi^L, \Theta^L, e^L, w^L$ is clearly feasible for [BASHP]. On the other hand, if there exists $\eta \in \mathcal{N}$ such that $w_\eta^L \neq e_\eta^L \psi_\eta^L$ where $e_\eta^L = 1$, we have that $w_\eta^L \neq \psi_\eta^L$, i.e., $w_\eta^L < \psi_\eta^L$. Consider the corresponding constraint where w_η occurs in the right-hand side:

$$Z_\eta \cdot (\theta_\eta^L - \theta_{\pi(\eta)}^L) = F_\eta w_\eta^L < F_\eta \psi_\eta,$$

i.e. we have

$$Z_\eta \cdot (\theta_\eta^L - \theta_{\pi(\eta)}^L) + \alpha = F_\eta \psi_\eta^L$$

for some $\alpha > 0$. But, since $Z_n^0 = 1$ for all $n \in \mathcal{N}$ we can absorb the slack α by adding α to θ_η^L, i.e. $\theta_\eta^L = \theta_\eta^L + \alpha$. This operation does not affect the portfolio values in the nodes leading to node η, i.e. the nodes in $\mathcal{A}(\eta) \setminus \eta$. The impact on the portfolio positions in the descendant nodes $\mathcal{D}(\eta)$ is an increase in the net portfolio value, which propagates into the leaf nodes' portfolio position as a net increase and thus preserves feasibility. Hence, we obtained a new set of portfolio positions, and hence a feasible solution to [BASHP] with objective function value identical to the objective function value of [LBASHP] at $\zeta^L, \psi^L, \Theta^L, e^L, w^L$. $\square$

4.3. A numerical example

Consider the scenario tree in Fig. 2 corresponding to a financial market with three trading dates $t = 0, 1, 2$ and two securities. The risky security prices at each node are indicated on the figure as the first number next to the node itself. The risk-free security has price equal to one everywhere as we assumed previously. The financial market is arbitrage-free. Assume that all nodes are equally likely to occur (i.e., $p_n = 1/9$ for all $n \in \{4, \ldots, 12\}$). Consider now an American contingent claim with pay-off vector $F = (0, 29, 0, 0, 35, 34, 0, 8, 5, 4, 0, 0, 0)$ at nodes $n = 0, 1, \ldots, 12$ also indicated in Fig. 2 as the second number next to each node. The sub-hedging price $\Pi^\downarrow(F)$ is computed to be equal to 2.

Now, setting the value $v = 8/3$, and solving the linear mixed-integer program [LBASHP] using version 12 of CPLEX through GAMS [1], we obtain an optimal objective function value of 0.33. The optimal exercise policy is to exercise the option at node 1 and nodes 7, 8, and 9 (those nodes are color-filled in Fig. 2), should the stock price process visit any of these nodes. All the ψ_n values which correspond to the scaling of the pay-off are equal to one, except $\psi_7 = 1.375$. Node 7 is represented with a glow effect in Fig. 2.

When we choose $v = 7/3$, a lower value than our previous choice, the optimal value goes down to 0.11. The ψ values equal to one in the previous run remain so while we have $\psi_7 = 1.125$, which is lower compared to the previous run. The optimal exercise policy does not change.

As expected, when we set $v = 2$, the sub-hedging value, all ψ's collapse down to one, and the optimal value becomes equal to zero. We chose $M = 6$ in this example.

5. Conclusions

In this paper we have defined a minimum expected surplus criterion for hedging American contingent claims, and formulated the problem of computing an optimal exercise policy and hedging policy under this criterion as a bilinear integer programming with an exact linearized counterpart in finite state markets. The resulting mixed-integer programming problems have the potential to become very large. The solution of these large instances is an interesting subject for future study.

Appendix A. Proof of Theorem 1

Before giving the proof of the theorem we have the following observation that will be useful in the proof.

Remark 1. Every process $\bar{\psi} \in \mathsf{R}_0$ with $\bar{\psi}_t \leq \hat{\psi}_t$ $\forall t \in \mathbb{T}$ also solves [ACFP].

Proof of Theorem 1. Let ξ be an self-financing portfolio strategy with $V_0(\xi) \geq v$ and $\tau \in \mathcal{T}$ be a stopping time. Using Doob's Stopping Theorem (cf. Theorem 6.17 [9]) on the $\mathbb{Q}$-martingale $V(\xi)$ we have that

$$\mathbb{E}^{\mathbb{Q}}[C_\tau \psi_\tau^\xi] = \mathbb{E}^{\mathbb{Q}}[V_\tau(\xi) \vee C_\tau] \geq \mathbb{E}^{\mathbb{Q}}[V_0(\xi)] \geq v.$$

Hence, $\psi^\xi \in \mathsf{R}_0$, and as a feasible point of [ACFP] it gives

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\ell(C_\tau(\psi_\tau^\xi - 1))] \geq \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\ell(C_\tau(\hat{\psi}_\tau - 1))]. \tag{2}$$

Now, let us consider a sub-hedging strategy $\hat{\xi}$ for the adjusted American claim $\hat{C} \equiv C\hat{\psi}$. Using the corresponding failure ratio $\psi^{\hat{\xi}}$ we have

$$C_t \psi_t^{\hat{\xi}} = C_t \vee V_t(\hat{\xi}) \leq C_t \vee (C_t\hat{\psi}) = C_t\hat{\psi}.$$

Therefore, $\psi_t^{\hat{\xi}}$ is dominated by $\hat{\psi}_t$ on the set $\{C_t > 0\}$. Moreover, any failure ratio is equal to 1 on $\{C_t = 0\}$, and we obtain

$$\psi_t^{\hat{\xi}} \leq \hat{\psi}_t, \quad \mathbb{P}\text{-a.s.}$$

Thus, we have $\psi_t^{\hat{\xi}} \in \mathsf{R}_0$. Now, by Remark 1 above since every process $\bar{\psi} \in \mathsf{R}_0$ such that $\bar{\psi}_t \leq \hat{\psi}_t$ for all $t \in \mathbb{T}$ is also a solution to [ACFP] we obtain that $\hat{\xi}$ solves [ACSHP]. Combined with (2), we have that

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\ell(C_\tau(\psi_\tau^{\hat{\xi}} - 1))] = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\ell(C_\tau(\hat{\psi}_\tau - 1))]. \quad \square$$

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