the point $E$, while $B$ is the moment of inertia relative to the longitudinal link axis.

The equations of motion of this mechanical system have the form

$$(T + B + (m\rho^2 + C - B)\sin^2\varphi_1)\ddot{\varphi} + (m\rho^2 + C - B)\sin 2\varphi_1\dot{\varphi}_1\dot{\varphi} = M$$

$$(m\rho^2 + A)\ddot{\varphi}_1 - \frac{1}{2}(m\rho^2 + C - B)\sin 2\varphi_1\dot{\varphi}^2 = M^1 + mg\rho\sin\varphi_1$$

where $g$ is acceleration of gravity.

The following constraints are to be imposed on the control torques:

$$|M(t)| \le M_0,\ |M^1(t)| \le M_0^1.$$

At the initial moment of the time, the system is in the given configuration

$$\varphi(0) = 0,\ \dot{\varphi}(0) = 0,\ \varphi_1(0) = \varphi_1,\ \dot{\varphi}_1(0) = 0.$$

It is required to find the control functions, assuring the transfer of the manipulator in minimal possible time $T$ to the assigned final configuration

$$\varphi(T) = \varphi_T,\ \dot{\varphi}(T) = 0,\ \varphi_1(T) = \varphi_{1T},\ \dot{\varphi}_1(T) = 0.$$

Additionally, assume that $\varphi_1(0) = \varphi_1(T)$. This condition is analogous to condition (1.9) and we can find the solution satisfying symmetry conditions (1.12).

Investigations, similar to the above, have been carried out for a manipulator with a rotating link. Simplified equations, obtained from the initial ones after neglecting the centrifugal force moment, were analyzed. Singular movement for this model consists of turning the base with a vertically positioned link. It is analytically proved that the movement with one switching of $M(t)$ and a finite number of switchings of $M^1(t)$ containing a singular mode cannot be optimal for the simplified equations and for the complete ones as well. The control and movement, satisfying Pontryagin's maximum principle, have been numerically designed both for the initial and simplified systems with some values of dimensionless parameters. In this movement the link oscillates relative to the vertical and the movement $M(t)$ switches over once.

The comparison of the results obtained for two kinds of manipulators shows that their optimal movements have the same form.

## V. Conclusion

In this note, the true structure of the time-optimal motions of a manipulator has been presented. True optimal motions of industrial robots are important from two points of view. First, if the optimal control is not too complex, it can be realized. Second, if the optimal control cannot be realized because of its complexity, it can be compared to a simple but perhaps nonoptimal one.

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# Almost Disturbance Decoupling with Internal Stability: Frequency Domain Conditions

A. BÜLENT ÖZGÜLER

***Abstract*—This note considers the almost disturbance decoupling problem with internal stability. The problem is that of determining a dynamic measurement feedback which makes the $H^\infty$-norm of the disturbance input-to-regulated output transfer matrix arbitrarily small while achieving internal stability. It is shown that the solvability condition in frequency domain for this problem is a purely algebraic one and can be formulated in terms of a two-sided matrix matching equation involving polynomial system matrices. This is known to be a *zero cancellation condition*. A synthesis procedure for the compensator in frequency domain is also given.**

## I. Introduction

A good deal of control theory literature has been concerned in recent years with modifying the behavior of a two-channel system in transfer matrix representation

$$\begin{bmatrix} y_m \\ y_r \end{bmatrix} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} u_c \\ u_d \end{bmatrix} \tag{1}$$

where the *control input* is $u_c$, the *disturbance input* is $u_d$, the *measured output* is $y_m$, and the output to be controlled or regulated, the *regulated output*, is $y_r$. The control scheme for the plant is a dynamic output feedback at the control channel $u_c$ to $y_m$. Thus,

$$u_c = -Z_c y_m + u_e$$

where $Z_c$ is the transfer matrix representation of the *compensator* and $u_e$ is an *external input*. The control objective consists of various requirements on the closed-loop transfer matrix between $u_d$ and $y_r$ denoted by $Z_{dr}$. In the *disturbance decoupling problem*, the objective is to achieve $Z_{dr} = 0$ by appropriate choice of the compensator. The *regulator problem* aims at placing the poles of $Z_{dr}$ in a given stability region. In the *standard $H^\infty$-optimization problem*, one of the objectives is to minimize the $H^\infty$-norm of $Z_{dr}$ by a suitable choice of the compensator. (See, e.g., Francis [2].) Finally, in the *almost disturbance decoupling* (AD) *problem* of Willems [9], the objective is to find, for every given positive $\epsilon$, a compensator $Z_c(\epsilon)$, possibly depending on $\epsilon$, such that the $H^\infty$-norm of $Z_{dr}(\epsilon) \le \epsilon$. A second control objective in all these problems is to attain *internal stability* in the closed-loop system.

Some of the four problems listed above with or without internal stability have been well investigated and solutions by many different approaches exist. We refer the reader to [1], [3], [9], [2], respectively, and to the references therein, for a fairly up-to-date bibliography. The exceptions

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The author is with the Department of Electrical Engineering, Bilkent University, Ankara, Turkey.

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are *the regulator problem with internal stability* (RIS), in the generality posed above, and *the almost decoupling problem with internal stability* (ADIS). This paper is concerned with ADIS, where the stability region is the closed left half complex plane. The problem ADIS has been formulated by Willems [9], after a development of *almost controlled and conditionally invariant subspaces*, who also gave frequency domain and geometric solvability conditions for AD. Taking this as our model, we derive the frequency domain conditions for ADIS. However, unlike [9], the synthesis procedure for the compensator is purely in the frequency domain. (The geometric conditions are obtained in [5], in an indirect manner "translating" the frequency domain condition via the use of *polynomial models* of Fuhrmann.)

### A. Notation

We follow the notation and terminology of [6] closely. The reader is referred to [7] for various properties of $H^\infty$-matrices that we use. Some of the special notation used in this note is listed below: $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the real and complex numbers, $\boldsymbol{R}[z]$ and $\boldsymbol{R}(z)$ denote polynomials and rational functions of real coefficients in the indeterminate $z$, as usual. Further,

$$|c|\text{: magnitude of } c \in \boldsymbol{C}$$

$$\boldsymbol{C}_- := \{c \in \boldsymbol{C}\colon \operatorname{Re}(c) < 0\},\ \boldsymbol{C}_{jw} := \{jw\colon w \in \boldsymbol{R}\}$$

$$\boldsymbol{C}_+ := \{c \in \boldsymbol{C}\colon \operatorname{Re}(c) > 0\}$$

$$\deg(a) := \deg(p) - \deg(q),\ a = p/q \in \boldsymbol{R}(z),\ p, q \in \boldsymbol{R}[z]$$

$$\boldsymbol{R}(z)_-\text{: strictly proper rational functions}$$

$$\boldsymbol{R}(z)_0\text{: proper rational functions}$$

$$\Omega, \omega\text{: stability regions (or sets)}$$

$$\boldsymbol{R}(z)_\Omega\text{: rational functions with all poles in } \Omega$$

$$\boldsymbol{R}(z)_{-\Omega} := \boldsymbol{R}(z)_\Omega \cap \boldsymbol{R}(z)_-$$

$$\boldsymbol{R}(z)_{0\Omega} := \boldsymbol{R}(z)_\Omega \cap \boldsymbol{R}(z)_0$$

$$\mathbb{S}^{m\times n}\text{: matrices of size } m \times n \text{ with entries in the set } \mathbb{S}$$

$$\deg(X) := \max_{i,j}\{\deg(x_{ij})\},\ X \in \boldsymbol{R}(z)^{m\times n}.$$

$$\bar{\sigma}\{A\}\text{: largest singular value of } A \in \boldsymbol{C}^{m\times n}$$

$$\|W\|_\infty := \sup_{\operatorname{Re}(z)\geq 0} \bar{\sigma}\{W(z)\},\ W \in \boldsymbol{R}(z)_{0\omega}^{m\times n},\ \omega := \boldsymbol{C}_-.$$

The elements of $\boldsymbol{R}(z)_\Omega^{m\times n}$ are called *$\Omega$-stable rational matrices*. If a nonsingular matrix $M \in \boldsymbol{R}(z)_\Omega^{m\times m}$ is such that $\det M$ is a unit of $\boldsymbol{R}(z)_\Omega$, i.e., its inverse is also $\Omega$-stable rational, then $M$ is said to be an *$\Omega$-bistable rational matrix*. A nonsingular polynomial matrix is said to be *$\Omega$-stable* if its determinant has all its zeros in $\Omega$. Whenever the stability region is clear from the context, we omit its specification in the above terms. Any polynomial matrix $\Pi \in \boldsymbol{R}[z]^{m\times n}$ can be factorized through its Smith form into $\Pi = \Pi_u \Pi_s = \Pi_\sigma \Pi_v$, where $\Pi_u$ and $\Pi_v$ are nonsingular with all their zeros strictly outside $\Omega$, and all invariant zeros of $\Pi_s$ and $\Pi_\sigma$ are in $\Omega$. We shall refer to these factorizations as *unstable-stable* and *stable-unstable factorizations*, respectively. If $\Pi$ is of full row rank, then $\Pi_u$ is unique up to right multiplications by a unimodular (polynomial) matrix. Similarly for $\Pi_v$, if $\Pi$ is of full column rank.

## II. Problem Definitions and Preliminary Results

In this section, we give the problem definitions and state certain preliminary results that will be used in obtaining the main results of Section III.

Our main object of study, *the plant*, is the two-channel system (1) in transfer matrix representation, where $Z_1 \in \boldsymbol{R}(z)^{p\times m}$, $Z_2 \in \boldsymbol{R}(z)^{p\times s}$, $Z_3 \in \boldsymbol{R}(z)^{q\times m}$, $Z_4 \in \boldsymbol{R}(z)^{q\times s}$. The plant is to be controlled by the application of a dynamic output feedback at the control channel $u_c$ to $y_m$. Thus, $u_c = -Z_c y_m + u_e$, where $Z_c \in \boldsymbol{R}(z)_0^{m\times p}$ and $u_e$ is an external input. The resulting closed-loop system is given by

$$\begin{bmatrix} y_m \\ y_r \end{bmatrix} = \begin{bmatrix} Z_1 - Z_1Y_cZ_1 & Z_2 - Z_1Y_cZ_2 \\ Z_3 - Z_3Y_cZ_1 & Z_4 - Z_3Y_cZ_2 \end{bmatrix} \begin{bmatrix} u_e \\ u_d \end{bmatrix} \tag{2}$$

where $Y_c := Z_c(I + Z_1Z_c)^{-1} \in \boldsymbol{R}(z)_0^{m\times p}$. A central control objective in various problems considered below is that *the compensator $Z_c$ internally $\Omega$-stabilizes the plant*. In order to define what we mean by this, we consider a coprime Rosenbrock polynomial fraction representation of the plant

$$\begin{bmatrix} y_m \\ y_r \end{bmatrix} = \left( \begin{bmatrix} P \\ T \end{bmatrix} Q^{-1} [R \quad S] + \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \right) \begin{bmatrix} u_c \\ u_d \end{bmatrix} \tag{3}$$

where $Q \in \boldsymbol{R}[z]^{r\times r}$ is nonsingular and $P, T, R, S, W_i$, $i = 1, 2, 3, 4$ are all polynomial matrices of appropriate sizes. We also choose the representation such that

A) $Q^{-1}$ is proper and $PQ^{-1}$, $Q^{-1}R$ are strictly proper.

It is well known that such a coprime fractional representation satisfying A) exists; see, e.g., [6, Remark 2.7]. In fact,

$$\begin{bmatrix} y_m \\ y_r \end{bmatrix} = \begin{bmatrix} H \\ C \end{bmatrix} (zI - F)^{-1} [G \quad B] \begin{bmatrix} u_c \\ u_d \end{bmatrix} \tag{4}$$

where $[H'\colon C']'(zI - F)^{-1}[G\colon B]$ is a canonical realization of the plant transfer matrix and $Z$ is one such representation satisfying A). By coprimeness of the representation (3), we have the existence of polynomial matrices $A_i$, $B_i$, $i = 0, 1, 2$ satisfying

$$A_0Q + A_1P + A_2T = I,\ QB_0 + RB_1 + SB_2 = I. \tag{5}$$

*Definition 1:* A compensator $Z_c \in \boldsymbol{R}(z)_0^{m\times p}$ is said to internally $\Omega$-stabilize the plant iff: i) $D := \operatorname{gclf}(Q, R)$ and $E := \operatorname{gcrf}(P, Q)$ are both $\Omega$-stable polynomial matrices; and ii) $Y_c$, $Z_1Y_c$, $Y_cZ_1$, $Z_1 - Z_1Y_cZ_1$ are all matrices over $\boldsymbol{R}(z)_\Omega$, where $Y_c = Z_c(I + Z_1Z_c)^{-1}$.

We can now define the regulator and the almost decoupling problems formally. Let the transfer matrix between the disturbance input $u_d$ and the regulated output $y_r$ of the closed-loop system (2) be denoted by $Z_{dr} := Z_4 - Z_3Y_cZ_2$.

*Definition 2 RIS($\Omega, \omega$):* Given the plant (1) and two stability regions $\Omega$, $\omega$ such that $\omega \subseteq \Omega$, determine (the conditions for the existence of) a compensator $Z_c$ which internally $\Omega$-stabilizes the plant and which achieves $Z_{dr} \in \boldsymbol{R}(z)_\omega^{q\times s}$.

*ADIS($\Omega$):* Given the plant (1) and a stability region $\Omega$, determine the conditions under which for every real number $\epsilon > 0$ there exists a compensator $Z_c(\epsilon)$ which internally $\Omega$-stabilizes the plant and for which

$$Z_{dr}(\epsilon) := Z_4 - Z_3Y_c(\epsilon)Z_2;\ Y_c(\epsilon) := Z_c(\epsilon)[I + Z_1Z_c(\epsilon)]^{-1}$$

satisfies $\|Z_{dr}(\epsilon)\|_\infty \leq \epsilon$. Further, give a synthesis procedure for such a compensator $Z_c(\epsilon)$ for a given $\epsilon > 0$, when the problem is solvable.

Among the three results stated below, Lemmas 1 and 2 give a sufficient condition for the solvability of RIS$(\Omega, \omega)$ for all $\omega \subseteq \Omega$ and Lemma 3 yields verifiable conditions for the solvability of a matrix matching equation.

Consider the plant in polynomial fractional representation (3) satisfying A), and let

$$E := \operatorname{gcrf}(P, Q),\ D := \operatorname{gclf}(Q, R).$$

Further, define the following polynomial *system matrices*:

$$\Pi_2 := \begin{bmatrix} Q & S \\ -P & W_2 \end{bmatrix},\ \Pi_3 := \begin{bmatrix} Q & R \\ -T & W_3 \end{bmatrix},\ \Pi_4 := \begin{bmatrix} Q & S \\ -T & W_4 \end{bmatrix}.$$

*Lemma 1:* The problem RIS$(\Omega, \omega)$ is solvable if $D$ and $E$ are $\Omega$-stable polynomial matrices and there exist matrices $M$, $N$ over $\boldsymbol{R}(z)_\Omega$

and matrices $U$, $V$ over $\boldsymbol{R}(z)_\omega$ satisfying

$$M\Pi_2 = E[I \quad U], \; \Pi_3 N = \begin{bmatrix} I \\ V \end{bmatrix} D. \tag{6}$$

*Proof:* It is enough, by the problem definition, to produce a *proper* $Y_c$ satisfying Definition 1ii) and achieving $Z_{dr} \in \boldsymbol{R}(z)_\omega^{q\times s}$. Then, $Z_c := Y_c(I - Z_1Y_c)^{-1}$ is clearly a solution to the problem. Neglecting the constraint of properness for a moment, consider the choice $Y_c := -N_2D^{-1}QE^{-1}M_2$, where $N_2 \in \boldsymbol{R}(z)_\Omega^{m\times r}$ and $M_2 \in \boldsymbol{R}(z)_\Omega^{r\times p}$ are defined by the partitions $M := [M_1 : M_2]$, $N' := [N_1' : N_2']$. By straightforward computations using (6), one obtains $Z_1Y_c = -[P - (PN_1 - W_1N_2)D^{-1}Q]E^{-1}M_2, Y_cZ_1 = -N_2D^{-1}[QE^{-1}(M_1R+M_2W_1)-R]$, $Z_1-Z_1Y_cZ_1 = W_1+(PN_1-W_1N_2)D^{-1}[R-QE^{-1}(M_1R+M_2W_1)]+PE^{-1}(M_1R + M_2W_1)$, which are all matrices over $\boldsymbol{R}(z)_\Omega$. Moreover, $Z_{dr} = W_4 + TU - VS + VQU$, which is over $\boldsymbol{R}(z)_\omega$. Consequently, the above choice of $Y_c$ satisfies all requirements save properness. It is now easy to show that given matrices $M$, $N$, $U$, $V$ over the stated rings satisfy (6), there also exist matrices $\hat{M}$, $\hat{U}$ such that $\hat{M}$, $N$, $\hat{U}$, $V$ satisfying (6) and $\hat{Y}_c := -N_2D^{-1}QE^{-1}\hat{M}_2$ is proper. This is achieved by introducing a sufficiently large number of $\omega$-stable poles into $M$, $U$. We omit further details. ●

*Lemma 2:* Let $E$ and $D$ be $\Omega$-stable polynomial matrices and let $\Pi_4 = \Pi_3X\Pi_2$ have a solution $X$ in $\boldsymbol{R}(z)_\Omega^{(r+m)\times(r+p)}$. Then, the problem RIS $(\Omega, \omega)$ is solvable for all $\omega \subseteq \Omega$.

*Proof:* Let

$$X := \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \tag{7}$$

be a partitioning of $X$ such that $X_1$ is $r \times r$. Let the polynomial matrices $A_i$, $B_i$ be as in (5). Set $M_1 := A_0(QX_1 + RX_3) + A_2(TX_1 - W_3X_3)$, $M_2 := A_0(QX_2 + RX_3) + A_2(TX_2 - W_3X_4) - A_1$, $U := A_0S - A_2W_4 - A_1W_2$, $N_1 := (X_1Q - X_2P)B_0 + (X_1S + X_2W_2)B_2$, $N_2 := (X_3Q - X_4P)B_0 + (X_3S + X_4W_2)B_2 + B_1$, $V := -TB_0 + W_4B_2 + W_3B_1$ and note that $[M_1 : M_2] := M$ and $[N_1' : N_2'] := N'$ are matrices over $\boldsymbol{R}(z)_\Omega$ and $U$, $V$ are polynomial matrices. Further, $M$, $N$, $U$, $V$ can be verified by direct substitution to satisfy (6). Since a polynomial matrix is $\omega$-stable rational for any $\omega$, it follows that RIS $(\Omega, \omega)$ is solvable for all $\omega$ in $\Omega$. ●

Given $\Pi_i$, $i = 1, 2, 3, 4$, let $\hat{U}$, $\hat{V}$ be unimodular polynomial matrices such that

$$\Pi_2\hat{V} = [\hat{C} \quad 0], \; \hat{U}\Pi_3 = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \; \hat{U}\Pi_4\hat{V} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}$$

with $\hat{B}$ of full row rank and $\hat{C}$ of full column rank. Let $\hat{B} = \hat{B}_u\hat{B}_s$, $\hat{C} = \hat{C}_s\hat{C}_u$ be stable-unstable and unstable-stable factorizations of $\hat{B}$ and $\hat{C}$, respectively, where $\hat{B}_u$ and $\hat{C}_u$ are nonsingular, with respect to $\Omega$.

*Lemma 3:* The following are equivalent.

a) There exists $X \in \boldsymbol{R}(z)_\Omega^{(r+m)\times(r+p)}$ such that $\Pi_4 = \Pi_3X\Pi_2$.

b) $\hat{A}_2 = 0$, $\hat{A}_3 = 0$, $\hat{A}_4 = 0$, and $\hat{B}_u^{-1}\hat{A}_1\hat{C}_u^{-1}$ is polynomial.

*Proof:* See [6, Remark 3.11]. ●

## III. A Frequency Domain Solution to Almost Decoupling

In this section, we give a frequency domain solvability condition and a synthesis procedure for the solution. The solvability condition is what might have been expected from the results in [9] and [6]. There should be a stable rational solution to $\Pi_4 = \Pi_3X\Pi_2$, whereas the proof is somewhat more involved than the proofs of the results for the exact version and for almost decoupling without internal stability.

Throughout this section $\Omega := \boldsymbol{C}_- \cup \boldsymbol{C}_{j\omega}$ and $\omega := \boldsymbol{C}_-$.

The following result constitutes a crucial step in the construction of an almost decoupling compensator. The style of its proof is borrowed from an analogous result of Van Der Woude [10, Theorem (3.4)] on "almost solvability" of the equation $Z_4 = Z_3X$ over $\boldsymbol{R}(z)$.

*Lemma 4:* Consider the transfer matrices $Z_1, Z_2, Z_3, Z_4$ of (1). If there exist $Y_c \in \boldsymbol{R}(z)_{0\Omega}^{m\times p}$ and $X \in \boldsymbol{R}(z)_\Omega^{m\times p}$ such that the matrices $Z_1Y_c, Y_cZ_1, Z_1 - Z_1Y_cZ_1, Z_1X, XZ_1, Z_1 - Z_1XZ_1$ are over $\boldsymbol{R}(z)_\Omega$, and

$$W := Z_4 - Z_3XZ_2 = 0, \tag{8}$$

$$W_c := Z_4 - Z_3Y_cZ_2 \in \boldsymbol{R}(z)_\omega^{q\times s} \tag{9}$$

then for every real number $\epsilon > 0$, there exists a matrix $X_c(\epsilon) \in \boldsymbol{R}(z)_{0\Omega}^{m\times p}$ such that the matrices $Z_1X_c(\epsilon)$, $X_c(\epsilon)Z_1$, $Z_1 - Z_1X_c(\epsilon)Z_1$ are over $\boldsymbol{R}(z)_\Omega$ and

$$\|Z_4 - Z_3X_c(\epsilon)Z_2\|_\infty \leq \epsilon. \tag{10}$$

*Proof:* Let $\mu := \deg(X)$ and $\eta := \|W_c\|_\infty$. If $\mu \leq 0$, then one can set $X_c := X$ to prove the claim. So, let $\mu > 0$. Given any $\epsilon > 0$, if $\eta := \|W_c\|_\infty \leq \epsilon$, then set $X_c := Y_c$. Otherwise, $\eta > \epsilon$ and we proceed as follows to choose $X_c$. Since $W_c$ is strictly proper, there exists a real number $\rho > 0$ for which

$$\sup_{|w|>\rho} \bar{\sigma}\{W_c(jw)\} < \epsilon 2^{-\mu}.$$

Let a real number $\lambda$ be such that

$$0 < \lambda < \epsilon 2^{-\mu}/[\rho(\eta^2 - \epsilon^2 2^{-2\mu})^{1/2}]$$

and consider the strictly proper $\omega$-stable rational function $f(z) := (z\lambda + 1)^{-1}$. Note that $\|f\|_\infty = 1$. We now claim that

$$X_c := f^\mu X + (1 - f^\mu)Y_c \tag{11}$$

satisfies all the requirements. First note that $\deg(X_c) \leq \deg(f^\mu X) + \deg[(1 - f^\mu)Y_c] = -\mu + \deg(X) + \deg(Y_c) \leq 0$ by the choice of $\mu$ and by properness of $Y_c$. Hence, $X_c$ is a proper $\Omega$-stable rational stability followed by the stability of every term in its definition. Moreover, $Z_1X_c = f^\mu Z_1X + (1 - f^\mu)Z_1Y_c$, $X_cZ_1 = f^\mu XZ_1 + (1 - f^\mu)Y_cZ_1$, $Z_1 - Z_1X_cZ_1 = f^\mu(Z_1 - Z_1XZ_1) + (1 - f^\mu)(Z_1 - Z_1Y_cZ_1)$, where all are matrices over $\boldsymbol{R}(z)_\Omega$ by the use of the hypothesis. Finally, we show (10) by induction on $\mu$. Let $\mu = 1$. Note that $T_c := Z_4 - Z_3X_cZ_2 = (1 - f)W_c$ by (8) and (11). By exactly the same manipulations as in [10], it can be shown that

$$\|T_c\|_\infty = \|(1 - f)W_c\|_\infty \leq \max\{\epsilon, \lambda\rho\eta/(1 + \rho^2\lambda^2)^{1/2}\} = \epsilon.$$

This establishes (10) for $\mu = 1$. Now let (10) hold for all $1 \leq \mu < \tau$ and consider the case $\mu = \tau$. In this case, $T_c = Z_4 - Z_3[f^\tau X + (1 - f^\tau)Y_c]Z_2 = (1 - f^\tau)W_c = (1 - f)W_c + f(1 - f^{\tau-1})W_c$. Therefore, using the fact that $\|f\|_\infty = 1$ and $\tau > 1$, by the induction hypothesis we have

$$\|T_c\|_\infty \leq \|(1 - f)W_c\|_\infty + \left\|(1 - f^{\tau-1})W_c\right\|_\infty \leq \epsilon/2 + \epsilon/2^{\tau-1} \leq \epsilon.$$

This proves (10) for arbitrary $\mu$. ●

We can now state and prove the main result.

*Theorem 1:* The problem ADIS $(\Omega)$ is solvable if and only if the polynomial matrices $E$ and $D$ are $\Omega$-stable and there exists $X \in \boldsymbol{R}(z)_\Omega^{(r+m)\times(r+p)}$ satisfying

$$\Pi_4 = \Pi_3X\Pi_2. \tag{12}$$

*Proof—"Only If":* Let the problem be solvable so that, by problem definition, for every real positive $\epsilon$, there exists a proper rational matrix $Z_c(\epsilon)$ internally $\Omega$-stabilizing the plant and achieving, with $Y_c(\epsilon) := Z_c(\epsilon)[I + Z_1Z_c(\epsilon)]^{-1}$,

$$\|Z_{dr}(\epsilon)\|_\infty = \|Z_4 - Z_3Y_c(\epsilon)Z_2\|_\infty \leq \epsilon. \tag{13}$$

Hence, by Definition 1, $E$ and $D$ are $\Omega$-stable and one has all the matrices $Y_c(\epsilon)$, $Z_1Y_c(\epsilon)$, $Y_c(\epsilon)Z_1$, $Z_1 - Z_1Y_c(\epsilon)Z_1$ over $\boldsymbol{R}(z)_\Omega$. Consider the $(r + m) \times (r + p)$ matrix

$$Y(\epsilon) := \begin{bmatrix} Q^{-1} - Q^{-1}RY_c(\epsilon)PQ^{-1} & -Q^{-1}RY_c(\epsilon) \\ Y_c(\epsilon)PQ^{-1} & Y_c(\epsilon) \end{bmatrix}$$

which is proper by properness of $Y_c(\epsilon)$ and by A). We now show that $Y(\epsilon)$ is also $\Omega$-stable rational. By $\Omega$-stability of $E = \text{gcrf}(P, Q)$ and

$D = \mathrm{gclf}(Q, R)$, there exist $\Omega$-stable rational $K_i$, $L_i$, $i = 1, 2$ satisfying $K_1Q + K_2P = I$, $QL_1 + RL_2 = I$. Note that

$$Q^{-1}RY_c(\epsilon) = (K_1R - K_2W_1)Y_c(\epsilon) + K_2Z_1Y_c(\epsilon),$$

$$Y_c(\epsilon)PQ^{-1} = Y_c(\epsilon)(PL_1 - W_1L_2) + Y_c(\epsilon)Z_1L_2,$$

$$\begin{aligned} Q^{-1} - Q^{-1}RY_c(\epsilon)PQ^{-1} = {} & L_1 + K_1 - K_1QL_1 - K_2W_1L_2 \\ & + K_2(Z_1 - Z_1Y_c(\epsilon)Z_1)L_2 \\ & + (K_1R - W_1L_2)Y_c(\epsilon)[(PL_1 - W_1L_2) \\ & - K_2Z_1Y_c(\epsilon)(PL_1 - W_1L_2) - Z_1L_2] \end{aligned}$$

which are all $\Omega$-stable rational. Thus, $Y(\epsilon)$ is in $\boldsymbol{R}(z)_{0\Omega}^{(r+m)\times(r+p)}$. Next note that $Y(\epsilon)$ satisfies

$$\Pi_4 - \Pi_3Y(\epsilon)\Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & Z_{dr}(\epsilon) \end{bmatrix}.$$

In view of (13), we have that given any $\epsilon > 0$, there exists $Y(\epsilon) \in \boldsymbol{R}(z)_{0\Omega}^{(r+m)\times(r+p)}$ such that

$$\|\Pi_4 - \Pi_3Y(\epsilon)\Pi_2\|_\infty \leq \epsilon. \tag{14}$$

It immediately follows by [9, Appendix] that

$$\mathrm{Ker}\,\Pi_2 \subseteq \mathrm{Ker}\,\Pi_4,\ \mathrm{Im}\,\Pi_4 \subseteq \mathrm{Im}\,\Pi_3 \tag{15}$$

over $\boldsymbol{R}(z)$. Consequently, there exist unimodular polynomial matrices $\hat{U}$ and $\hat{V}$ such that

$$\hat{U}\Pi_3 = [\hat{B}' : 0]',\ \Pi_2\hat{V} = [\hat{C} : 0]$$

where $\hat{B}$ is of full row rank $b$ and $\hat{C}$ is of full column rank $c$ and $\hat{U}\Pi_4\hat{V}$ is in the form

$$\hat{U}\Pi_4\hat{V} = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix}$$

with $\hat{A} \in \boldsymbol{R}[z]^{b\times c}$. Let $\hat{B} := \hat{B}_u\hat{B}_s$, $\hat{C} := \hat{C}_s\hat{C}_u$ be unstable–stable and stable–unstable factorizations of $\hat{B}$, $\hat{C}$, respectively, with $\hat{B}_u$ and $\hat{C}_u$ nonsingular.

Let $\sigma$ be an $\omega$-stable polynomial of degree

$$\deg(\sigma) > \max\{\deg(\hat{A}), \deg(\hat{B}), \deg(\hat{C}), \deg(\hat{U}), \deg(\hat{V})\}$$

and define the $H^\infty$-matrices $A := \hat{A}\sigma^{-2}$, $B := \hat{B}\sigma^{-1}$, $C := \hat{C}\sigma^{-1}$, $U := \hat{U}\sigma^{-1}$, $V := \hat{V}\sigma^{-1}$. Note that, in view of (14), we have $\|A - BY(\epsilon)C\|_\infty \leq \epsilon\|U\|_\infty\|V\|_\infty$. Since $\|U\|_\infty$, $\|V\|_\infty$ are both independent and $\epsilon$, it follows that if ADIS ($\Omega$) is solvable, then for every $\epsilon > 0$ there exists $Y(\epsilon) \in \boldsymbol{R}(z)_{0\Omega}^{(r+m)\times(r+p)}$ such that

$$\|A - BY(\epsilon)C\|_\infty < \epsilon. \tag{16}$$

Consider the inner–outer factorizations [7] of $B$ and $C$ of the type $B = B_iB_o$, $C = C_oC_i$, where $B_i$ and $C_i$ are square inner matrices and $B_o$ and $C_o'$ are outer matrices. Note that $B_i = \hat{B}_uM$, $C_i = N\hat{C}_u$ for some $\Omega$-bistable matrices $M$ and $N$ (i.e., $B_i$ and $\hat{B}_u$ have the same zeros in $\boldsymbol{C}_+$). We now show that $B_i^{-1}AC_i^{-1}$ is $\Omega$-stable rational from which it immediately follows that $(\hat{B}^-)_u^{\dagger}\hat{A}\hat{C}_u^{-1}$ is $\Omega$-stable rational, or equivalently, polynomial. By the properties of inner matrices, we have $B_i^{-1}(z) = B_i^*(z)$, $C_i^{-1}(z) = C_i^*(z)$ for all $z \in \boldsymbol{C}$ and, by (16),

$$\sup_{w\in \boldsymbol{R}} \|K(jw) - L(jw)\| = \|A - BY(\epsilon)C\|_\infty \leq \epsilon \tag{17}$$

where $K := B_i^*AC_i^*$, $L := B_oY(\epsilon)C_o$. Thus, for all $w \in \boldsymbol{R}$, we have

$$\|K(jw)\| \leq \epsilon + \|L(jw)\| \leq \epsilon + \|L\|_\infty \tag{18}$$

since by (17) it holds that

$$|\|K(jw)\| - \|L(jw)\|| \leq \|K(jw) - L(jw)\| \leq \|K - L\|_\infty \leq \epsilon.$$

Now, if $K$ has a pole $z_0 \in \boldsymbol{C}_+$, then consider a region R in $\boldsymbol{C}_+$ having $z_0$ and a segment of $jw$-axis on its boundary and in which $K$ is analytic. Such a region exists since $K$ has only a finite number of poles. Since $z_0$ is a pole of $K$, there exists $z_1 \in$ R sufficiently close to $z_0$ for which $\|K(z_1)\| > \epsilon + \|L\|_\infty$. By the maximum modulus theorem, it follows that for some $jw$ on the boundary of R, one has $\|K(jw)\| > \epsilon + \|L\|_\infty$, which clearly contradicts (18). Therefore, $K = B_i^{-1}AC_i^{-1}$ is free of $\boldsymbol{C}_+$ poles as we set out to prove. By this, inclusions (14), and by Lemma 3, it follows that (12) admits an $\Omega$-stable rational solution.

*"If":* Let $E$ and $D$ be $\Omega$-stable polynomial matrices and let there exist an $\Omega$-stable rational $X$ such that (12) holds. We first show that there exists $Y \in \boldsymbol{R}(z)_\Omega^{m\times p}$ such that $Z_1Y$, $YZ_1$, $Z_1 - Z_1YZ_1$ are matrices over $\boldsymbol{R}(z)_\Omega$ and $Z_4 = Z_3YZ_2$. By $\Omega$-stability of $E$ and $D$ and by A), there exist matrices $Y_i$, $i = 1, 2, 3, 4$ over $\boldsymbol{R}(z)_\Omega$ such that $QY_1 + RY_3 = I$, $Y_1Q - Y_2P = I$, $QY_2 + RY_4 = 0$, $Y_3Q - Y_4P = 0$, by [6, Lemma 4.1]. Let $X$ be partitioned as in (7), and define, as in [6, p. 762] (correcting a sign mistake in the definition of $\hat{X}_3$) the matrices $\hat{X}_i$, $i = 1, 2, 3, 4$, by

$$\hat{X}_1 := 2Y_1 + X_1 - Y_1QX_1 - X_1QY_1 - Y_1RX_3 + X_2PY_1,$$

$$\hat{X}_2 := Y_2 + X_2 - Y_1RX_4 - X_1QY_2 - Y_1QX_2 + X_2PY_2,$$

$$\hat{X}_3 := Y_3 + X_3 - Y_3RX_3 - X_3QY_1 + X_4PY_1 - Y_3QX_1,$$

$$\hat{X}_4 := X_4 - X_3QY_2 - Y_3RX_4 + X_4PY_2 - Y_3QX_2$$

which are all matrices over $\boldsymbol{R}(z)_\Omega$. It follows that

$$\hat{X} := \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_4 \end{bmatrix}$$

satisfies $\Pi_4 = \Pi_3\hat{X}\Pi_2$ and further $Q\hat{X}_2 + R\hat{X}_4 = 0$, $\hat{X}_3Q - \hat{X}_4P = 0$. Now let $Y := \hat{X}_4$ and note by $\Pi_4 = \Pi_3\hat{X}\Pi_2$ that $Z_4 = Z_3YZ_2$, and also by the last two equalities above, the matrices $Z_1Y$, $YZ_1$, $Z_1 - Z_1YZ_1$ are all over $\boldsymbol{R}(z)_\Omega$ as can be verified by a routine computation. In addition to the existence of $Y$ as above, our hypothesis also yields, by Lemma 2 in Section II, the existence of $Y_c \in \boldsymbol{R}(z)_{0\Omega}^{m\times p}$ such that $Z_1Y_c$, $Y_cZ_1$, $Z_1 - Z_1Y_cZ_1$ are matrices over $\boldsymbol{R}(z)_\Omega$ and such that $Z_4 - Z_3Y_cZ_2$ is in $\boldsymbol{R}_\omega^{q\times s}$. Therefore, by Lemma 4 of this section, for every $\epsilon > 0$, there exists $X_c(\epsilon) \in \boldsymbol{R}(z)_{0\Omega}^{m\times p}$ such that $Z_c(\epsilon) := X_c(\epsilon)[I - Z_1X_c(\epsilon)]^{-1}$, internally $\Omega$-stabilizes the plant while achieving $\|Z_4 - Z_3X_c(\epsilon)Z_2\|_\infty \leq \epsilon$. Therefore, ADIS ($\Omega$) is solvable. ●

*Comments:* 1) After the submission of this note, the author became aware of an alternative solution by Wieland and Willems [8] who give geometric solvability conditions for the same problem. In the meantime, a paper by Linnemann, Postlethwaite, and Anderson [4] has appeared in which the authors give a solution to the problem for the more fundamental stability set $\Omega = \boldsymbol{C}_-$. The frequency domain solvability condition of [4] is stated on the stable plant obtained after the application of an initial stabilizing feedback. 2) The technique of this note also easily yields conditions for the pole-placement version of ADIS ($\Omega$). 3) It is possible to solve ADIS ($\Omega$) with the more general internal stability constraint "the pair $(Z_1, Z_c(\epsilon))$ is internally stable" [6]. 4) It is not true in general that "$E$ and $D$ are $\Omega$-stable and the equation $Z_4 = Z_3XZ_2$ has an $\Omega$-stable solution implies the equation $\Pi_4 = \Pi_3\hat{X}\Pi_2$ has an $\Omega$-stable rational solution." The reason is that for unstable open-loop plants there may be unstable cancellations between $(T, Q)$ or between $(Q, S)$.


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# Pole Placement Direct Adaptive Control for Time-Varying Ill-Modeled Plants


FOUAD GIRI, MOHAMED M'SAAD, JEAN-MICHEL DION, AND LUC DUGARD



*Abstract*—**This note addresses the problem of preserving stability of pole placement direct adaptive control in spite of output bounded disturbances, time-varying plant model parameters, and unmodeled dynamics, assumed to be small in the mean.**

**The controller parameter estimates are shown to track, in the mean, their true (time-varying) parameter values. Such a convergence property is achieved using an ad hoc, internally generated, excitation sequence which ensures persistent excitation.**

**Furthermore, in the ideal case the convergence of the parameter estimates is exponential, avoiding, in particular, possible chaotic phenomena.**


## I. Introduction

The fundamental practical feature which motivates the adaptive control theory is how to achieve acceptable performance vis-à-vis time-varying dynamics and plant model uncertainties. The latter are mainly due to external disturbances and unmodeled dynamics. However, most of the early investigations have been devoted to linear time-invariant systems, possibly subject to well-modeled disturbances, which will be referred to as the ideal case throughout this note.

Robustness studies in adaptive control have been undertaken in the last few years, e.g., [13], [14], [10], [1], [8], [6]. In the previous references, robustness issues in indirect adaptive control are addressed. On the other hand, the problem of robustness in direct adaptive control has been dealt with, based on minimum phase assumption, e.g., [2], [9], [11].

In this note, we propose a new solution to the closed-loop pole placement direct adaptive control. The controller parameters estimation is performed following the design philosophy proposed in [3] for the ideal case. The involved robust stability is achieved by using a least-squares-type algorithm with data normalization [13], parameter projection, and adaptation gain monitoring. Furthermore, the adaptive control law is modified adding an internally generated impulse excitation sequence following a nonlinear feedback excitation approach [12].


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The authors are with the Laboratoire d'Automatique de Grenoble, ENSIEG, Saint-Martin-d'Hères, France.



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## II. Preliminaries

*Definition 2.1:* Let $\alpha$ be a real number. A real sequence $\{s(t)\}$ is said to be *$\alpha$-asymptotically small in the mean* ($\alpha$ – ASM), if

$$\limsup_{k\to\infty}\limsup_{l\to\infty}\left[\frac{1}{k}\sum_{t=l+1}^{l+k}s(t)\right]\le\alpha$$

the set of all such sequences is denoted by $S_a(\alpha)$. $\nabla$

*Proposition 2.1:*

a) Any $\alpha$-ASM nonnegative real sequence is uniformly bounded.

b) If $\alpha_1 \le \alpha_2$, then $S_a(\alpha_1)$ is included in $S_a(\alpha_2)$.

c) If $\{s(t)\}\in S_a(\alpha)$ and $\{s'(t)\}\in S_a(\alpha')$, then for any $\lambda, \lambda' \ge 0$, $\{\lambda s(t)+\lambda' s'(t)\}\in S_a(\lambda\alpha+\lambda'\alpha')$. $\nabla$

The proof of this proposition is evident from Definition 2.1.

*Proposition 2.2:* Let $\{s(t)\}$ be a real sequence. For any $\epsilon > 0$ and $p \in N - \{0\}$, there exist an integer sequence $\{t_k\}$ and a finite integer $T$ such that, if $\{s(t)\}\in S_a(\alpha)$, then for any $k \in N$

$$\text{a) } |t_k - t_{k-1}| \le T$$

$$\text{b) } s(t_k - j) \le p\alpha + \epsilon, \qquad j = 1,\cdots,p. \qquad \nabla$$

The proof can be found in [5].

*Proposition 2.3:* Let $\{s(t)\}$ and $\{r(t)\}$ be nonnegative real sequences such that for any $t \in N$

$$r(t+1) \le s(t)r(t) + K_r \tag{2.1a}$$

$$\{s(t)\}\in S_a(\alpha) \tag{2.1b}$$

where $\alpha$ and $K_r$ are nonnegative real constants. If $0 \le \alpha < 1$, then

a) $\{r(t)\}$ is uniformly bounded;

b) in addition, if $K_r = 0$, then $\{r(t)\}$ converges to zero. $\nabla$

Part a) of the above proposition has been proved in [14]. Part b) can be established following closely the proof of part a).

## III. The Plant Representation

The plant to be controlled is assumed to be represented by the time-varying discrete-time model

$$A(\theta^*(t), q^{-1})\xi(t) = u(t) \tag{3.1a}$$

$$y(t) = B(\theta^*(t), q^{-1})\xi(t) + \eta(t) \tag{3.1b}$$

where $\{u(t)\}$, $\{y(t)\}$, and $\{\xi(t)\}$ are the input, output, and "partial state," respectively. $\{\eta(t)\}$ is the plant unmodeled response. $\{\theta^*(t)\}$ is a $2n$-vector sequence ($n \in N - \{0\}$) and

$$A(\theta^*(t), q^{-1}) = 1 + \theta_1^*(t)q^{-1} + \cdots + \theta_n^*(t)q^{-n} \tag{3.1c}$$

$$B(\theta^*(t), q^{-1}) = \theta_{n+1}^*(t)q^{-1} + \cdots + \theta_{2n}^*(t)q^{-n} \tag{3.1d}$$

where $q^{-1}$ is the backward shift operator and $\theta_i^*(t)$ ($i = 1,\cdots,2n$) is the $i$th component of $\theta^*(t)$.

It is assumed that an integer $n$ is known, such that the resulting sequences $\{\theta^*(t)\}$ and $\{\eta(t)\}$, satisfy the following assumptions.

*A1*: $\{\|\theta^*(t)\|\}$ is uniformly bounded by a known real $R^*$.

*A2*: There exists a positive real $\epsilon_c$ such that for any $t \in N$: $|\det M_s[A(\theta^*(t), q^{-1}), B(\theta^*(t), q^{-1})]| \ge \epsilon_c$, where $M_s[\cdot,\cdot]$ denotes the Sylvester matrix.

*A3*: There exists a real $\nu$ such that: $\{|\eta(t)|/m(t)\}\in S_a(\nu)$, where $\{m(t)\}$ is defined by

$$m(t) = \sigma m(t-1) + \max\{m_0, |u(t-1)| + |y(t-1)|\} \tag{3.2}$$

and $m(0) > 0$, $0 < \sigma < 1$ and $m_o \ge 0$ are arbitrarily chosen.

Now, let $\nu'$ be a real such that

$$\{\|\theta^*(t) - \theta^*(t-1)\|\}\in S_a(\nu'). \tag{3.3a}$$