manuscripta math. 72, 325 - 334 (1991)

manuscripta
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THE PAIR CORRELATION OF ZEROS OF DIRICHLET L-FUNCTIONS AND PRIMES IN ARITHMETIC PROGRESSIONS

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We define a function which correlates the zeros of two Dirichlet L-functions to the modulus q and we prove an asymptotic estimate for averages of the pair correlation functions over all pairs of characters to (mod q) . An analogue of Montgomery's pair correlation conjecture is formulated as to how this estimate can be extended to a greater domain for the parameters that are involved. Based on this conjecture we obtain results about the distribution of primes in an arithmetic progression (to a prime modulus q) and gaps between such primes.

1. Introduction and statement of results

In 1973 Montgomery's approach [7] provided a new direction for research on the Riemann zeta–function, $\zeta(s)$, and the distribution of primes. Assuming the Riemann Hypothesis Montgomery defined the pair correlation function of the critical zeros of $\zeta(s)$

$$F(\alpha,T) \;=\; \Big(\frac{T\,\log T}{2\pi}\Big)^{-1} \sum_{\substack{0<\gamma,\gamma'\le T\\ \zeta(\frac12+i\gamma)=0\\ \zeta(\frac12+i\gamma')=0}} T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma') \tag{1}$$

(where $w(u) = \frac{4}{4+u^2}$ is a weighting function which serves to diminish the contribution from those pairs of zeros with large differences) and he proved that (see [7] and [2])

$$F(\alpha,T) \;=\; (1+o(1))T^{-2\alpha}\log T + \alpha + o(1) \tag{2}$$

as $T \to \infty$, uniformly for $0 \le \alpha \le 1$. Montgomery also conjectured that

$$F(\alpha,T) \;=\; 1+o(1) \tag{3}$$

for $\alpha \ge 1$, uniformly in bounded intervals. This statement has become known as the pair correlation conjecture. He then used (3) to show that almost all zeros of $\zeta(s)$ are simple.

Assuming RH and the pair correlation conjecture in various forms Heath-Brown [5] proved that

$$\sum_{\substack{p_n\le x\\ d_n\ge\Delta}} d_n \;\ll\; \frac{x}{\Delta}\log x,$$

(where $d_n = p_{n+1} - p_n$ and p_n is the n-th prime) and that for functions f such that $f(x) \to \infty$ as $x \to \infty$, almost all intervals $[x, x + f(x)\log x]$ contain a prime. Heath-Brown also showed that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0.$$

Goldston and Heath-Brown [3] proved that RH and the pair correlation conjecture together imply $d_n = o((p_n \log p_n)^{\frac{1}{2}})$.

Moreover, Goldston and Montgomery established an equivalence between an asymptotic result for the distribution of primes and the pair correlation conjecture (see [4], Theorem 2).

In this paper we apply the ideas of the pair correlation conjecture to the distribution of primes in an arithmetic progression. Let,

$$G_{\chi_1,\chi_2}(x,T) = 2\pi \sum_{0<\gamma_1,\gamma_2\le T} x^{i(\gamma_1-\gamma_2)} w(\gamma_1 - \gamma_2) \tag{4}$$

where, assuming GRH, $\frac{1}{2} + i\gamma_j$ runs through the zeros of the Dirichlet L-functions $L(s,\chi_j)$, $(j = 1,2)$, and χ_j are characters to the modulus q. One may say that $G_{\chi_1,\chi_2}(x,T)$ correlates the critical zeros of $L(s,\chi_1)$ with those of $L(s,\chi_2)$.

In the following we suppose $(a,q) = 1$. Capital letters A, B, C will denote arbitrary fixed positive numbers. We prove

Theorem 1. *Assume GRH. As $x \to \infty$ we have, uniformly for*

$$1 \le q \le x^{\frac{1}{2}} \log^{-3} x$$

when T is in the range

$$\frac{x}{q}\log x \le T \le e^{\sqrt[4]{x}},$$

$$\sum_{\chi_1,\chi_2 (\mathrm{mod}\, q)} \bar{\chi}_1(a)\chi_2(a) G_{\chi_1,\chi_2}(x,T) \sim \phi(q) T \log x.$$

For T smaller relative to x we assume the following

Conjecture. *Under GRH, as $x \to \infty$, it holds uniformly for*

$$q \le \min(x^{\frac{1}{2}} \log^{-3} x, x^{1-\eta} \log x)$$

and

$$x^{\eta} \le T \le \frac{x}{q}\log x$$

where q is prime or 1 *and* $0 < \eta \le 1$ *is fixed, that*

$$\sum_{\chi_1,\chi_2 (\mathrm{mod}\ q)} \bar{\chi}_1(a)\chi_2(a) G_{\chi_1,\chi_2}(x,T) \sim \phi(q) T \log qT.$$

The restriction to prime moduli q in the Conjecture is made to avoid the presence of imprimitive characters.

Let

$$\psi(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a(\mathrm{mod}\ q)}} \Lambda(n)$$

where Λ is the von Mangoldt function. Upon the Generalized Riemann Hypothesis (GRH) the prime number theorem for arithmetic progressions is, for $q \le x$, ([1],p.125)

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(x^{\frac{1}{2}}\log^2 x) \tag{5}$$

Assuming the Conjecture we obtain the following asymptotic result for an individual arithmetic progression with prime modulus.

Theorem 2. *Assume GRH. Let α_1 and α_2 be fixed, $0 < \alpha_1 \le \alpha \le \alpha_2 \le 1$ and $\delta = x^{-\alpha}$. Also let $0 < \eta < \alpha_1$ be fixed. Assume that the Conjecture holds uniformly for*

$$q \le \min(x^{\frac{1}{2}}\delta^{\frac{1}{2}}(\log x)^A, \delta^{-1}x^{-\eta})$$

where q is prime or 1 and

$$\frac{x^{\alpha_1}}{\phi(q)} \log^{-3} x \le T \le \phi(q)x^{\alpha_2}\log^3 x.$$

Then

$$\int_x^{2x} \{\psi(u+u\delta;\ q,a) - \psi(u;\ q,a) - \frac{u\delta}{\phi(q)}\}^2 du \sim \frac{3}{2}\ \frac{\delta x^2}{\phi(q)}\ \log\frac{q}{\delta}\ ,$$

as $x \to \infty$, uniformly for $x^{-\alpha_2} \le \delta \le x^{-\alpha_1}$ and such moduli q.

The following Corollaries, all resting on GRH, may be deduced from Theorem 2. Let $p_n(a,q)$ denote the n-th prime congruent to $a(\mathrm{mod}\, q)$ and $d_n(a,q) = p_{n+1}(a,q) - p_n(a,q)$. The modulus q is always supposed to be prime or 1. For $q = 1$ all of the results presented here yield the known estimates ([2],[3],[4],[5],[7]) mentioned above. By the weaker form of the Conjecture it is meant that there is $\ll$ in place of the asymptotic estimate in the Conjecture. This implies the same change in the estimate for the integral in Theorem 2.

Corollary 1. *Let $4 \le \Delta \le x^{1-\epsilon}$ and $\varepsilon > 0$ be fixed as $x \to \infty$. Also let $0 < \eta < 1 - \frac{\log\Delta}{\log x}$ be fixed. Assume the weaker form of the Conjecture for*

$$q \le \min\Big(\Delta^{\frac{1}{2}}(\log x)^A, \frac{x^{1-\eta}}{\Delta}\ \log^{-3} x\Big)$$

and

$$\frac{x}{\phi(q)\Delta}\ \log^{-3} x \le T \le \frac{\phi(q)x}{\Delta}\ \log^3 x.$$

Then

$$\sum_{\substack{x\le p_n(a,q)\le \frac{3}{2}x \\ d_n(a,q)\ge\Delta}} d_n(a,q) \ll \phi(q)\frac{x}{\Delta}\ \log x$$

uniformly for such q .

Corollary 1 is non-trivial if $q\log x \le \Delta$. It follows readily from Corollary 1 that

Corollary 2. *Let* $q \le (\log x)^A$ *and* $f(x) \to \infty$ *arbitrarily slowly as* $x \to \infty$. *Assume the weaker form of the Conjecture for*

$$\frac{x(\log x)^{-(4+2A)}}{f(x)} \le T \le x\log^2 x\ .$$

Then

$$\sum_{\substack{x\le p_n(a,q)\le \frac{3}{2}x \\ d_n(a,q)\ge\phi(q)f(x)\log x}} d_n(a,q)\ \ll\ \frac{x}{f(x)}\ .$$

Hence for $q \le (\log x)^A$ *almost all intervals* $[x, x+\phi(q)f(x)\log x]$ *contain a prime congruent to* $a(\mathrm{mod}\ q)$.

Corollary 3. *Assume that the asymptotic estimate of the Conjecture holds uniformly for* $q \le x^{\frac{1}{3}-\eta}$ *and*

$$\sqrt{\frac{x}{q^3}}\ \log^{-\frac{7}{2}} x \le T \le \sqrt{qx}\log^2 x.$$

Then, for $p_n(a,q) \asymp x$,

$$d_n(a,q)\ =\ o(\phi(q)p_n(a,q)\log p_n(a,q))^{\frac{1}{2}}$$

uniformly for such q.

Corollary 4. *Assume that the asymptotic estimate of the Conjecture holds uniformly for* $q \le (\log x)^A$ *and*

$$x(\log x)^{-(4+2A)} \le T \le x\log^3 x,$$

as $x \to \infty$. *Let* $p_n(a,q) \asymp x$. *Then*

$$\liminf_{n\to\infty}\ \frac{d_n(a,q)}{\phi(q)\log p_n(a,q)}\ =\ 0\ .$$

2. Explicit formulae and the proof of Theorem 1

Paralleling Landau's derivation of an explicit formula ([6], p. 353) that provides a link between the zeros of $\zeta(s)$ and primes one gets in the case of Dirichlet L-functions

$$\sum_{n\le x}{}' \frac{\Lambda(n)\chi(n)}{n^r}\ =\ \sum_{\ell=0}^{\infty}\frac{x^{-2\ell-r-\mathbf{a}}}{2\ell+r+\mathbf{a}} - \sum_{\rho}\frac{x^{\rho-r}}{\rho-r} - \frac{L'}{L}(s,\chi)\ , \tag{6}$$

for $x > 1$ (the prime on the summation means only half of the term with $n = x$ is included in the sum) and $r \neq \rho$, $r \neq -(2q + \mathbf{a})$ where for primitive $\chi(\text{mod } q)$

$$\mathbf{a} = \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1 . \end{cases}$$

The sum over the non-trivial zeros ρ of $L(s, \chi)$ in (6) is interpreted in the symmetric sense as $\lim_{T\to\infty} \sum_{|\gamma|<T} \frac{x^{\rho-r}}{\rho-r}$.

The explicit formula (6) combined with the functional equation of $\frac{L'}{L}(s, \chi)$, as in the proof of the Lemma of Montgomery [7], leads to another explicit formula

$$\begin{aligned}(2\sigma - 1)\sum_{\rho} \frac{x^{i\gamma}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \quad = \quad & -x^{-\frac{1}{2}}\Big[\sum_{n\leq x} \Lambda(n)\chi(n)(\frac{x}{n})^{1-\sigma+it} \\ & + \sum_{n>x} \Lambda(n)\chi(n)(\frac{x}{n})^{\sigma+it}\Big] \\ & + x^{\frac{1}{2}-\sigma+it}(\log q\tau + O_\sigma(1)) \\ & + O(x^{-\frac{1}{2}-\mathbf{a}}\tau^{-1}) . \end{aligned} \tag{7}$$

This last explicit formula is valid, under GRH for $L(s, \chi)$, for all $x \geq 1$.

From (7) we can write for $(a, q) = 1$

$$\begin{aligned} & \Big| \sum_{\chi(\text{mod } q)} \bar{\chi}(a) \sum_{\gamma} \frac{2x^{i\gamma}}{1 + (t - \gamma)^2}\Big|^2 = \\ & \Big| \sum_{\chi(\text{mod } q)} \bar{\chi}(a) \Big\{-x^{-\frac{1}{2}}\Big(\sum_{n\leq x} \Lambda(n)\chi(n)(\frac{x}{n})^{-\frac{1}{2}+it} + \sum_{n>x} \Lambda(n)\chi(n)(\frac{x}{n})^{\frac{3}{2}+it}\Big) \\ + \; & x^{-1+it}(\log q'\tau + O(1)) + O(x^{-\frac{1}{2}-\mathbf{a}}\tau^{-1}) \\ + \; & x^{\frac{1}{2}}\Big(\sum_{\substack{n\leq x \\ \chi(n)\neq\chi'(n)}} \Lambda(n)\chi'(n)(\frac{x}{n})^{-\frac{1}{2}+it} + \sum_{\substack{n>x \\ \chi(n)\neq\chi'(n)}} \Lambda(n)\chi'(n)(\frac{x}{n})^{\frac{3}{2}+it}\Big)\Big\}\Big|^2 , \end{aligned} \tag{8}$$

where $\chi(\text{mod } q)$ is a character induced by the primitive character $\chi'(\text{mod } q')$. The last term in the parantheses is a correction term for nonprimitive χ. The terms with $\chi(n) \neq \chi'(n)$ can exist only when $(n, q) > 1$ and when summed over all $\chi(\text{mod } q)$ the contribution of the correction term vanishes.

We integrate both sides of (8) from $t = 0$ to $t = T$. From the left side we obtain (cf. Eq.s (23)-(26) [7] and (4) above)

$$\begin{aligned} & \int_0^T \sum_{\chi_1,\chi_2} \bar{\chi}_1(a)\chi_2(a) \sum_{\gamma_1,\gamma_2} \frac{4x^{i(\gamma_1-\gamma_2)}}{[1 + (t - \gamma_1)^2][1 + (t - \gamma_2)^2]}dt = \\ & \sum_{\chi_1,\chi_2} \bar{\chi}_1(a)\chi_2(a)G_{\chi_1,\chi_2}(x, T) + O(\phi(q)^2 \log T \; \log^2 qT). \end{aligned} \tag{9}$$

Writing

$$a_n = \begin{cases} \Lambda(n)(\frac{x}{n})^{-\frac{1}{2}}, & \text{if} \quad n \le x, \\ \Lambda(n)(\frac{x}{n})^{\frac{3}{2}}, & \text{if} \quad n > x, \end{cases}$$

the first term in parentheses on the right-hand side of (8) contibutes

$$\frac{\phi(q)^2}{x} \Big| \sum_{n\equiv a(\text{mod } q)} a_n n^{-it} \Big|^2 .$$

By the Montgomery-Vaughan mean value theorem for Dirichlet series ([8]) in the form

$$\int_0^T \Big| \sum_{n\equiv a(\text{mod } q)} a_n n^{-it} \Big|^2 dt = T \sum_{n\equiv a(\text{mod } q)} |a_n|^2 + O\Big(q \sum_{\substack{n<q \\ n\equiv a(\text{mod } q)}} \frac{|a_n|^2}{n} + \frac{1}{q} \sum_{\substack{n>q \\ n\equiv a(\text{mod } q)}} n|a_n|^2\Big)$$

and using (5) we find that for $q \le x^{\frac{1}{2}} \log^{-3} x$ as T, and $x \to \infty$,

$$\frac{\phi(q)^2}{x} \int_0^T \Big| \sum_{n\equiv a(\text{mod } q)} a_n n^{-it} \Big|^2 dt = \phi(q) T \log x (1 + o(1)) + O(x \log x) . \qquad (10)$$

The remaining contributions to the integral of the right side of (8) are

$$\int_0^T \frac{\phi(q)^2}{x} \log^2 q\tau \; dt \ll \phi(q)^2 \frac{T}{x} \log^2 qT$$

$$\int_0^T \frac{\phi(q)^2}{x\tau^2} dt \ll \frac{\phi(q)^2}{x}$$

and the cross-terms. If $(\log T)^4 \le x$ then the main term is (10) and Theorem 1 holds.

3. Proof of Theorem 2

The method of Goldston and Montgomery [4] about primes in short intervals based on the pair correlation conjecture for $\zeta(s)$ is adaptable to primes in arithmetic progressions in short intervals. We have the following lemmas the first two of which have been slightly modified from their original statements in [4].

Lemma 1.([4] Lemma 2) *Let ϕ satisfy $1 \le \phi \le (\frac{1}{\kappa})^B$ where $B \ge 0$ is fixed as $\kappa \to 0^+$. Let $f(t)$ be a continuous, non-negative function defined for $t > 0$ such that*

$$f(t) \ll \phi^2 \log^2 \phi\tau .$$

Assume further that

$$J(T) = \int_0^T f(t) dt = (1 + \varepsilon(T)) \phi T \log \phi T ,$$

$|\varepsilon(T)|$ *being small (that is, given* $\varepsilon_0 > 0$, $|\varepsilon(T)| < \varepsilon_0$ *for sufficiently large* T*) uniformly for*

$$\frac{1}{\phi\kappa \log^2 \kappa} \le T \le \frac{\phi}{\kappa} \log^2 \frac{\phi}{\kappa} .$$

Then,

$$\int_0^\infty (\frac{\sin \kappa u}{u})^2 f(u) du \sim (\frac{\pi}{2} + \varepsilon'(\kappa))\phi\kappa \log \frac{\phi}{\kappa} ,$$

where $|\varepsilon'(\kappa)|$ *is small as* $\kappa \to 0^+$ *(that is, given* $\varepsilon_0 > 0$, $|\varepsilon'(\kappa)| < \varepsilon_0$ *for sufficiently small* κ*). This result is uniform in* ϕ.

Lemma 2.([4] Lemma 10) *Let* $\omega(s) = \dfrac{(1+\delta)^s - 1}{s}$ *where* $\delta \in (0,1]$. *Then*

$$\int_{-\infty}^{\infty} |\omega(it)|^2 \Big| \sum_{\chi (\text{mod } q)} \bar\chi(a) \sum_\gamma \frac{x^{i\gamma}}{1+(t-\gamma)^2}\Big|^2 dt =$$

$$\int_{-\infty}^{\infty} \Big| \sum_{\chi (\text{mod } q)} \bar\chi(a) \sum_{|\gamma| \le Z} \frac{\omega(\frac{1}{2} + i\gamma) x^{i\gamma}}{1+(t-\gamma)^2}\Big|^2 dt$$

$$+ O(\phi(q)^2 \delta^2 \log^3 \frac{2q}{\delta}) + O(\phi(q)^2 Z^{-1} \log^3 Z)$$

provided that $Z \ge \delta^{-1}$.

Lemma 3.([4] Lemma 1) *If* $\int_{-\infty}^{\infty} e^{-2|y|} f(Y+y) dy = 1 + \epsilon(Y)$ *and if* $f(y) \ge 0$ *for all* y, *then for any Riemann integrable function* $R(y)$

$$\int_a^b R(y) f(Y+y) dy = \Big(\int_a^b R(y) dy\Big)(1 + \epsilon'(Y)) .$$

If R *is fixed then , as* $Y \to \infty$, $|\epsilon'(Y)|$ *is small if* $|\epsilon(y)|$ *is small uniformly for* $Y + a - 1 \le y \le Y + b + 1$.

In Lemma 1 we take

$$f(t) = \Big| \sum_{\chi (\text{mod } q)} \bar\chi(a) \sum_\gamma \frac{2x^{i\gamma}}{1+(t-\gamma)^2}\Big|^2$$

(trivially $f(t) \ll \phi(q)^2 \log^2 q\tau$) and $\phi = \phi(q)$. Let $\kappa = \frac{1}{2}\log(1+\delta)$ and assume that the asymptotic value $\phi(q)T \ \log qT$ of the Conjecture is valid, uniformly for

$$\frac{1}{\phi(q)\kappa \ \log^2 \kappa} \le T \le \frac{\phi(q)}{\kappa} \log^2 \frac{q}{\kappa}$$

as $\kappa \to 0^+$. This range of T is consistent with the range of T for which we may assume the asymptotic estimate of the Conjecture and Theorem 1, if for $0 < \eta < \alpha_1$

$$q \le \min(x^{\frac{1}{2}} \kappa^{\frac{1}{2}} (\log x)^A, \kappa^{-1} x^{-\eta}). \tag{11}$$

For such T and q we have, as $x \to \infty$,

$$\int_0^T f(t)dt \sim \phi(q)T\log qT$$

and therefore by Lemma 1 as $\kappa \to 0^+$

$$\int_0^\infty (\frac{\sin \kappa t}{t})^2 f(t)dt \sim \frac{\pi}{2}\,\phi(q)\kappa\,\log\frac{q}{\kappa}.$$

But $(\frac{\sin \kappa t}{t}) = \frac{\omega(it)}{4}$ and applying Lemma 2 with $Z = x(\log x)^C$, $C > 3$ and fixed,

$$\int_{-\infty}^{\infty} \Big| \sum_{\chi(\text{mod } q)} \bar{\chi}(a) \sum_{|\gamma|\le Z} \frac{\omega(\rho)x^{i\gamma}}{1+(t-\gamma)^2}\Big|^2 dt \sim \frac{\pi}{2}\,\phi(q)\delta\log\frac{q}{\delta}\,.$$

By Plancherel's identity the last integral is transformed into

$$\pi^2 \int_{-\infty}^{\infty} \Big| \sum_{\chi(\text{mod } q)} \bar{\chi}(a) \sum_{|\gamma|\le Z} \omega(\rho)x^{i\gamma}e(-\gamma u)^2\Big|^2 e^{-4\pi|u|}\,du\,.$$

Let $Y = \log x$ and $y = -2\pi u$, so that $x \to \infty$

$$\int_{-\infty}^{\infty} \Big| \sum_{\chi(\text{mod } q)} \bar{\chi}(a) \sum_{|\gamma|\le Z} \omega(\rho)e^{i\gamma(Y+y)}\Big|^2 e^{-2|y|}\,dy \sim \phi(q)\delta\log\frac{q}{\delta}\,.$$

We now use Lemma 3 with

$$R(y) = \begin{cases} e^{2y}, & \text{if } \quad 0 \le y \le \log 2, \\ 0, & \text{otherwise} \end{cases}$$

and letting $u = e^{Y+y}$ we have

$$\int_x^{2x} \Big| \sum_{\chi(\text{mod } q)} \bar{\chi}(a) \sum_{|\gamma|\le Z} \omega(\rho)u^{\rho}\Big|^2 du \sim \frac{3}{2}\,\phi(q)x^2\delta\log\frac{q}{\delta}\,. \tag{12}$$

Let now

$$S_\chi = \begin{cases} 1, & \text{if } \quad \chi = \chi_0, \\ 0, & \text{otherwise}\,. \end{cases}$$

We recall the identity

$$\int \{\psi((1+\delta)u;\ q,a) - \psi(u;\ q,a) - \frac{u\delta}{\phi(q)}\}^2 du =$$
$$\frac{1}{\phi(q)^2} \sum_{\chi_1,\chi_2(\text{mod } q)} \bar{\chi}_1(a)\chi_2(a) \int \{\psi((1+\delta)u;\ \chi_1) - \psi(u;\ \chi_1) - \delta u\, S_{\chi_1}\}$$
$$\cdot\{\psi(1+\delta)u;\ \bar{\chi}_2) - \psi(u;\ \bar{\chi}_2) - \delta u\, S_{\chi_2}\}du \tag{13}$$

and the explicit formula ([D], p. 117)

$$\begin{aligned}\psi((1+\delta)u;\ \chi)-\psi(u;\ \chi)-\delta u\, S_\chi &= -\sum_{|\gamma|\le Z}\omega(\rho)u^\rho+O(\delta)+O(\frac{u}{Z}\log^2 uqZ) \qquad (14)\\ &\quad + O(\log u\cdot\min(1,\frac{u}{Z}(\frac{1}{\|u\|}+\frac{1}{\|(1+\delta)u\|})))\\ &\quad + \text{correction for nonprimitive } \chi\,.\end{aligned}$$

We insert (14) in (13) and when the sums over χ_1, χ_2 are carried out the correction terms for nonprimitive characters disappear as in (8). Given $A > 0$ we choose $C = C(A)$ (recall that $Z = x(\log x)^C$ so that the integral in (12) multiplied by $\frac{1}{\phi(q)^2}$ is the main term and all other terms are $o(\frac{x^2\delta}{\phi(q)}\log x)$ as $x\to\infty$, uniformly for q subject to (11). The proof of Theorem 2 is now complete.

We observe that Theorem 2 remains valid if ψ's are replaced by θ's, i.e. only the contribution from the primes is counted. We will abbreviate $p_n(a,q)$ and $d_n(a,q)$ as p_n and d_n. To deduce Corollary 1 let $x \le p_n \le \frac{3}{2}x, \Delta = 4\delta x$ and $\Delta \le d_n(a,q)$. Then $\theta(u+u\delta;\ q,a) = \theta(u;\ q,a)$ for $p_n \le u \le p_n + \frac{d_n}{2}$. By Theorem 2, which holds for q in the specified range,

$$\frac{x^2\delta}{\phi(q)}\log\frac{q}{\delta} \gg \sum_{\substack{x\le p_n\le\frac{3}{2}x\\ d_n\ge\Delta}} \int_{p_n}^{p_n+\frac{d_n}{2}} (\frac{u\delta}{\phi(q)})^2 du \gg \Big(\sum_{\substack{x\le p_n\le\frac{3}{2}x\\ d_n\ge\Delta}} d_n\Big)\frac{x^2\delta^2}{\phi(q)^2}$$

This proves Corollary 1.

To show Corollary 3 suppose that the assertion is false, given any fixed $\epsilon > 0$ there exist p_n and p_{n+1} such that $x \le p_n < p_{n+1} \le (1+\epsilon^{\frac{3}{2}})x$ and $d_n \ge \epsilon\sqrt{\phi(q)x\log x}$ however large x may be. Let $\delta = \frac{\epsilon}{3}\sqrt{\frac{\phi(q)\log x}{x}}$ and $H = 3x\delta$. The range in which the Conjecture is assumed corresponds to this value of δ. Then from Theorem 2

$$\int_x^{(1+\epsilon^{\frac{3}{2}})x} \{\theta((u+u\delta);\ q,a)-\theta(u;\ q,a)-\frac{u\delta}{\phi(q)}\}^2 du \ll \epsilon^{\frac{3}{2}}\frac{\delta x^2}{\phi(q)}\log\frac{q}{\delta}.$$

We also have

$$\int_{p_n}^{p_{n+1}-\frac{H}{2}} \{\theta((u+u\delta);\ q,a)-\theta(u;\ q,a)-\frac{u\delta}{\phi(q)}\}^2 du = \int_{p_n}^{p_{n+1}-\frac{H}{2}} (\frac{u\delta}{\phi(q)})^2 du \gg \frac{\delta^2x^2}{\phi(q)^2}d_n\,.$$

Hence $d_n \ll \sqrt{\epsilon\phi(q)x\log x}$. According to Theorem 2 with the above choice of δ this holds uniformly for $q \le x^{\frac{1}{3}-\eta}$ for any $\eta > 0$ fixed. To conclude the proof of Corollary 3 we let $\epsilon\to 0^+$ sufficiently slowly.

To prove Corollary 4 it is suitable to have a version of Theorem 2 with fixed difference H rather than the varying difference $u\delta$ in the integrand. Applying the method of Goldston and Montgomery [4] one obtains, for $H \to \infty$ with x and $q \le (\log x)^A$,

$$\int_x^{2x} \{\theta(u+H;\ q,a) - \theta(u;\ q,a) - \frac{H}{\phi(q)}\}^2 du \ = \ (1+o(1))\frac{Hx}{\phi(q)} \log \frac{qx}{H}. \qquad (15)$$

Remark: In deriving (15) one uses the unconditional estimate

$$\int_x^{2x} \{\theta(u+u\delta;\ q,a) - \theta(u;\ q,a) - \frac{u\delta}{\phi(q)}\}^2 du \ \ll \ \frac{x^3\delta^2}{\phi(q)^2} + \frac{\delta x^2}{\phi(q)} \log x\ ,$$

which is valid if $0 \le \delta \le 1$ and $q < x^{1-\epsilon}$, for the integration over $0 \le \delta \le \frac{1}{x}$.

Expanding the integrand in (15) and using the Siegel-Walfisz theorem yields

$$2 \sum_{\substack{x<p<r\le 2x+H \\ 0<r-p\le H \\ p\equiv r\equiv a (\mathrm{mod}\ q)}} (H+r-p)\log p \log r \ = \ \frac{H^2 x}{\phi(q)^2}\,(1+o(1)) + \frac{Hx}{\phi(q)}(\log \frac{q}{H} + o(\log \frac{qx}{H}))\ .$$

With the choice $H = \epsilon\phi(q)\log x$ the right-hand side is a positive quantity for any fixed $\epsilon > 0$ if x is sufficiently large. This completes the proof of Corollary 4.

Acknowledgement

I thank Professor J.B. Friedlander, my thesis advisor, for his encouragement and comments while I worked on this topic at the University of Toronto. I am grateful to Professor D.A. Goldston for the argument used in the proof of Corollary 3.

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(Received May 3, 1991)