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Dynamic Boundary Control of a Euler–Bernoulli Beam

Ömer Morgül

Abstract—**We consider a flexible beam clamped to a rigid base at one end and free at the other end. To stabilize the beam vibrations, we propose a dynamic boundary force control and a dynamic boundary torque control applied at the free end of the beam. We prove that with the proposed controls, the beam vibrations decay exponentially. The proof uses a Lyapunov functional based on the energy functional of the system.**

I. Introduction

Many mechanical systems, such as spacecraft with flexible appendages or robots with flexible arms can be modeled as coupled rigid and elastic parts. In such systems, if a good performance of the overall system is desired, the dynamical effect of the flexible parts on the overall system has to be taken into account. Thus, modeling, control, and stabilization of such flexible structures are important for the control of such mechanical systems.

In recent years, the boundary control of flexible systems has become an important research area. This idea is first applied to the systems described by the wave equation (i.e., strings) [1], and recently extended to the Euler–Bernoulli beam equation [2], [3] and to the Timoshenko beam equation [5]. In particular, it has been proven that in a cantilever beam, a single *nondynamic* actuator applied at the free end of the beam is sufficient to uniformly stabilize the beam vibrations, [2], [3]. A good source of references to papers in which boundary stabilization problems are treated using the Lyapunov approach can be found in [6].

In this note, we consider a beam clamped to a rigid base at one end and free at the other end. To stabilize the beam vibrations, we propose a dynamic boundary force control and a dynamic boundary torque control law (i.e., *dynamic* actuators) at the free end of the beam. Under some assumptions, mainly the positive realness of the actuator transfer functions, we prove that the beam vibrations exponentially decay to 0. This generalizes an earlier result due to Chen [2].

II. Equations of Motion

We consider a Euler–Bernoulli beam, clamped to a rigid base at one end and free at the other, as shown in Fig. 1. In Fig. 1, the horizontal and vertical axes are assumed to be fixed in an inertial frame N. The beam is clamped to a rigid base fixed in N at the point O, which is the origin of the inertial frame N.

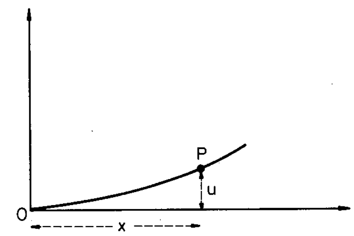

Fig. 1. A flexible beam.

We assume that the beam is initially straight along the horizontal axis and this configuration is referred to as the reference configuration for the beam. Let P be a material point on the beam whose distance from O in the reference configuration is x and let u denote the vertical displacement of P from its position in the reference configuration. We also assume that the beam is inextensible, i.e., no displacement along the horizontal axis, and that the beam is homogeneous with uniform cross-sections.

Assuming small displacements, neglecting the effect of gravity and rotatory inertia of the beam cross-sections and using the Euler–Bernoulli beam model, we obtain the following equations, see [10]: for all $t \geq 0$

$$EIu_{xxxx} + \rho u_{tt} = 0, \qquad 0 < x < L \tag{1}$$

$$u(0, t) = 0, \qquad u_x(0, t) = 0 \tag{2}$$

$$EIu_{xxx}(L, t) = f_1(t) \tag{3}$$

$$-EIu_{xx}(L, t) = f_2(t) \tag{4}$$

where a subscript denotes the partial differential with respect to the corresponding variable, EI is the flexural rigidity of the beam, ρ is the mass per unit length of the beam, L is the length of the beam; (1) is the balance of forces at x along the vertical axis, (2) gives the boundary conditions at the clamped end, (3) and (4) are the boundary conditions at the free end, $f_1(t)$ and $f_2(t)$ are the boundary control force and the boundary control torque, both applied to the free end, respectively.

Our control objective is this: find a control law for the control force $f_1(t)$ and the control torque $f_2(t)$ such that the solutions of the system given by (1)–(4) satisfy the following asymptotic relations: for all $0 < x < L$

$$\lim_{t \to \infty} u(x, t) = 0 \tag{5}$$

$$\lim_{t \to \infty} u_t(x, t) = 0. \tag{6}$$

To stabilize the system given by (1)–(4), we propose the following feedback control laws: for $i = 1, 2$

$$\dot{w}_i = A_i w_i + b_i u_i(t) \tag{7}$$

$$f_i(t) = c_i^T w_i + d_i u_i(t) \tag{8}$$

where, for $i = 1, 2$, $w_i \in \boldsymbol{R}^{n_i}$ is the actuator state, $A_i \in \boldsymbol{R}^{n_i \times n_i}$ is a constant matrix, b_i, $c_i \in \boldsymbol{R}^{n_i}$ are constant column vectors, the superscript T stands for the transpose, $d_i \in \boldsymbol{R}$ is a constant real number, and $u_i(t)$ is defined as

$$u_1(t) = u_t(L, t), \qquad u_2(t) = u_{xt}(L, t), \qquad t \in \boldsymbol{R}. \tag{9}$$

We note that for $i = 1$ ($i = 2$, respectively) (7) and (8) give the

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equations for the actuator whose input is $u_t(L,t)$ ($u_{xt}(L,t)$, respectively) and the output is the boundary control force $f_1(t)$ (the boundary control torque $f_2(t)$, respectively).

We assume the following throughout this work.

Assumptions: For $i = 1, 2$

1) all eigenvalues of the matrix A_i are in the open left-half plane,
2) the triplet (A_i, b_i, c_i) is both observable and controllable,
3) $d_1 > 0$, $d_2 \geq 0$; furthermore for some $\gamma_1 > 0$, $\gamma_2 \geq 0$ such that $d_1 > \gamma_1$, $d_2 \geq \gamma_2$, we have the following:

$$\mathscr{R}e\{g_i(j\omega)\} > \gamma_i, \qquad i = 1,2, \qquad \omega \in \mathbf{R} \tag{10}$$

where, for $i = 1, 2$, $g_i(s) = d_i + c_i^T(sI - A_i)^{-1}b_i$. □

The assumption 3) implies that, for $i = 1, 2$, the actuator transfer function $g_i(s)$ is a strictly positive real function.

Let the assumptions 1)–3) stated above hold. Then it follows from the Kalman–Yacubovitch lemma that, for $i = 1, 2$, given any symmetric positive definite matrix $Q_i \in \mathbf{R}^{n_i \times n_i}$, there exist a symmetric positive definite matrix $P_i \in \mathbf{R}^{n_i \times n_i}$ and a vector $q_i \in \mathbf{R}^{n_i}$ satisfying

$$A_i^T P_i + P_i A_i = -q_i q_i^T - \epsilon_i Q_i \tag{11}$$

$$P_i b_i - \frac{1}{2} c_i = \sqrt{(d_i - \gamma_i)}\, q_i \tag{12}$$

provided that $\epsilon_i > 0$ is sufficiently small, see [13, p. 201].

To analyze the system given by (1)–(4), (7)–(9), we first define the function space $\mathscr{H}$ as follows:

$$\mathscr{H} := \{(u\ u_t\ w_1\ w_2)^T \mid u \in \mathbf{H}_0^2,\ u_t \in \mathbf{L}^2,\ w_1 \in \mathbf{R}^{n_1},\ w_2 \in \mathbf{R}^{n_2}\} \tag{13}$$

where the spaces $\mathbf{L}^2$ and $\mathbf{H}_0^k$ are defined as follows:

$$\mathbf{L}^2 = \left\{ f : [0, L] \to \mathbf{R} \mid \int_0^L f^2\, dx < \infty \right\} \tag{14}$$

$$\mathbf{H}_0^k = \{ f \in \mathbf{L}^2 \mid f, f', f'', \cdots, f^{(k)} \in \mathbf{L}^2, \qquad f(0) = f'(0) = 0 \}. \tag{15}$$

Equations (1)–(4), (7)–(9) can be written in the following abstract from:

$$\dot{z} = Az, \qquad z(0) \in \mathscr{H} \tag{16}$$

where $z = (u\ u_t\ w_1\ w_2)^T \in \mathscr{H}$, the operator A: $\mathscr{H} \to \mathscr{H}$ is a linear unbounded operator defined as

$$Az = \begin{bmatrix} u_t \\ -\dfrac{EI}{\rho} \dfrac{\partial^4}{\partial x^4} \\ A_1 w_1 + b_1 u_t(L) \\ A_2 w_2 + b_2 u_{xt}(L) \end{bmatrix}. \tag{17}$$

The domain $D(A)$ of the operator A is defined as:

$$\begin{aligned} D(A) := \{ (u\ u_t\ w_1\ w_2)^T \mid u \in \mathbf{H}_0^4, \quad & u_t \in \mathbf{H}_0^2, \\ w_1 \in \mathbf{R}^{n_1}, \quad & w_2 \in \mathbf{R}^{n_2} \\ -EIu_{xxx}(L) + c_1^T w_1 + & d_1 u_t(L) = 0, \\ EIu_{xx}(L) + c_2^T w_2 + & d_2 u_{xt}(L) = 0 \}. \end{aligned} \tag{18}$$

Let the assumptions 1)–3) hold, let, for $i = 1, 2$, $Q_i \in \mathbf{R}^{n_i \times n_i}$ be an arbitrary symmetric positive definite matrix and let $P_i \in \mathbf{R}^{n_i \times n_i}$, $q_i \in \mathbf{R}^{n_i}$ be the solutions of (11) and (12) where P_i is also a symmetric and positive definite matrix. In $\mathscr{H}$, we define the following "energy" inner-product:

$$\langle z, \hat{z} \rangle_E = \frac{1}{2} \int_0^L \rho u_t \hat{u}_t\, dx + \frac{1}{2} \int_0^L EIu_{xx}\hat{u}_{xx}\, dx + \hat{w}_1^T P_1 w_1 + \hat{w}_2^T P_2 w_2 \tag{19}$$

where $z = (u\ u_t\ w_1\ w_2)^T$, $\hat{z} = (\hat{u}\ \hat{u}_t\ \hat{w}_1\ \hat{w}_2)^T$. We note that $\mathscr{H}$, together with the energy inner-product given by (19) becomes a Hilbert space [2], [7]. The "energy" norm induced by (19) is

$$E(t) = \| z(t) \|_E^2 = \frac{1}{2} \int_0^L \rho u_t^2\, dx + \frac{1}{2} \int_0^L EIu_{xx}^2\, dx + w_1^T P_1 w_1 + w_2^T P_2 w_2 \tag{20}$$

where the first term is the beam kinetic energy, the second term is the beam potential energy [10]. The last two terms are a measure of the actuator "energy."

Note that $E(t) = 0$ on a time interval $[a, b]$ if and only if $u(x, t) = 0$ and $u_t(x, t) = 0$ almost everywhere on $[0, L]$, for all $t \in [a, b]$. This follows from (20) and (2).

Proposition: Consider the system given by (1)–(4), (7)–(9). Let the assumptions 1)–3) hold. Then the energy $E(t)$ given by (20) is a nonincreasing function of time along the *classical* solutions of (1)–(4), (7)–(9). (For the terminology of partial differential equations and semigroup theory, the reader is referred to, e.g., [11].)

Proof: By differentiating (20) with respect to time, using (1)–(4), (7)–(9) and integrating by parts, we obtain

$$\begin{aligned} \dot{E}(t) = & -EIu_{xxx}(L,t)u_t(L,t) + EIu_{xx}(L,t)u_{xt}(L,t) \\ & + w_1^T(A_1^T P_1 + P_1 A_1)w_1 + w_2^T(A_2^T P_2 + P_2 A_2)w_2 \\ & + 2w_1^T P_1 b_1 u_t(L,t) + 2w_2^T P_2 b_2 u_{xt}(L,t) \\ = & -d_1 u_t^2(L,t) - d_2 u_{xt}^2(L,t) - \epsilon_1 w_1^T Q_1 w_1 \\ & - \epsilon_2 w_2^T Q_2 w_2 \\ & - c_1^T w_1 u_t(L,t) - c_2^T w_2 u_{xt}(L,t) \\ & - w_1^T q_1 q_1^T w_1 - w_2^T q_2 q_2^T w_2 \\ & + 2w_1^T P_1 b_1 u_t(L,t) + 2w_2^T P_2 b_2 u_{xt}(L,t) \\ = & -\gamma_1 u_t^2(L,t) - \gamma_2 u_{xt}^2(L,t) \\ & - \epsilon_1 w_1^T Q_1 w_1 - \epsilon_2 w_2^T Q_2 w_2 \\ & - \left(\sqrt{d_1 - \gamma_1}\, u_t(L,t) - w_1^T q_1\right)^2 \\ & - \left(\sqrt{d_2 - \gamma_2}\, u_{xt}(L,t) - w_2^T q_2\right)^2 \end{aligned} \tag{21}$$

where to obtain the first equation we differentiated (20), used (1), (7), integrated by parts twice and used (2); to obtain the second equation we used (8) and (11); and to obtain the last equation, we used (12). Since $\dot{E}(t) \leq 0$, it follows that the energy $E(t)$ is a nonincreasing function of time along the *classical* solutions of (1)–(4), (7)–(9). □

Theorem 1: Consider the system given by (16), where the operator A is given by (17) and the spaces $\mathscr{H}$ and $D(A)$ are given by (13) and (18), respectively. Then the operator A generates a C_0 semigroup $T(t)$ in $\mathscr{H}$. Moreover, if $z(0) \in D(A)$, then $z(t) = T(t)z(0)$, $t \geq 0$, is the unique *classical* solution of (16) and $z(t) \in D(A)$, $t \geq 0$. (For the terminology on semigroup theory, the reader is referred to [11].)

Proof: From (21) it follows that A is dissipative on $\mathscr{H}$. It can also be shown that the operator $\lambda I - A$: $\mathscr{H} \to \mathscr{H}$ is onto for all $\lambda > 0$ (see [7] for similar proofs). Therefore, it follows that $D(A)$ is dense in $\mathscr{H}$, see [11, p. 16]. Hence from the Lumer–Phillips theorem [11, p. 14], we conclude that A generates a C_0 semigroup $T(t)$ on $\mathscr{H}$. That $z(t) = T(t)z(0)$, $t \ge 0$, is the unique *classical* solution of (1)–(4), (7)–(9) when $z(0) \in D(A)$, and that $z(t) \in D(A)$, $t \ge 0$, follows from the semigroup property, see [11]. □

Next, we prove that the energy $E(t)$ decays exponentially to zero along the *classical* solutions of (16).

Theorem 2: Consider the system given by (16). Let the assumptions 1)–3) hold. Then the energy $E(t)$ given by (20) decays exponentially to zero along the classical solutions of (16).

Proof: We consider the following function

$$V(t) = 2(1-\varepsilon)tE(t) + 2\int_0^L \rho x u_t u_x \, dx \tag{22}$$

where $\varepsilon \in (0, 1)$ is an arbitrary constant.

In the sequel, we need the following inequalities:

$$u^2(s,t) \le L\int_0^L u_x^2\,dx \le L^2\int_0^L u_{xx}^2\,dx,$$

$$u_x^2(s,t) \le L\int_0^L u_{xx}^2\,dx \qquad s \in [0, L] \tag{23}$$

$$ab \le \delta^2 a^2 + b^2/\delta^2, \qquad a, b, \delta \in \boldsymbol{R}, \delta \ne 0 \tag{24}$$

where (23) follows from boundary conditions and Jensen's inequality [12, p. 110]. Using (20), (23), and (24), it follows that there exists a constant $C > 0$ such that the following inequality holds for all $t \ge 0$

$$[2(1-\varepsilon)t - C]E(t) \le V(t) \le [2(1-\varepsilon)t + C]E(t). \tag{25}$$

Differentiating (22) with respect to time, we obtain

$$\begin{aligned}\dot V(t) &= 2(1-\varepsilon)E(t) + 2(1-\varepsilon)t\dot E(t)\\ &\quad + 2\int_0^L \rho x u_{tt}u_x\,dx + 2\int_0^L \rho x u_t u_{xt}\,dx\\ &= 2(1-\varepsilon)E(t) + 2(1-\varepsilon)t\dot E(t)\\ &\quad - 2EI\int_0^L x u_{xxxx}u_x\,dx + 2\int_0^L \rho x u_t u_{xt}\,dx.\end{aligned} \tag{26}$$

Using integration by parts, (24), and the boundary conditions (2)–(4), we obtain

$$\begin{aligned}A_1 &= -2EI\int_0^L x u_{xxxx}u_x\,dx\\ &= -2EILu_x(L,t)u_{xxx}(L,t) + 2EIu_x(L,t)u_{xx}(L,t)\\ &\quad + EILu_{xx}^2(L,t) - 3EI\int_0^L u_{xx}^2\,dx\\ &\le (2L\delta_1^2 + 2\delta_2^2)u_x^2(L,t) + (4Ld_1^2/\delta_1^2)u_t^2(L,t)\\ &\quad + (4/\delta_2^2 + 2L/EI)d_2^2u_{xt}^2(L,t)\\ &\quad + (4L/\delta_1^2)(c_1^Tw_1)^2\\ &\quad + (4/\delta_2^2 + 2L/EI)(c_2^Tw_2)^2 - 3EI\int_0^L u_{xx}^2\,dx\end{aligned} \tag{27}$$

$$A_2 := 2\int_0^L \rho x u_t u_{xt}\,dx = \rho L u_t^2(L,t) - \int_0^L \rho u_t^2\,dx \tag{28}$$

where $\delta_1 \ne 0$, $\delta_2 \ne 0$ are arbitrary constants.

Using (21), (27), and (28) in (26), we obtain

$$\begin{aligned}\dot V(t) &= 2(1-\varepsilon)E(t) + 2(1-\varepsilon)t\dot E(t) + A_1 + A_2\\ &\le -\varepsilon\int_0^L \rho u_t^2\,dx - \big[2(1-\varepsilon)t\epsilon_1 w_1^TQ_1w_1\\ &\quad -2(1-\varepsilon)w_1^TP_1w_1 - (4L/\delta_1^2)(c_1^Tw_1)^2\big]\\ &\quad -\big[2(1-\varepsilon)t\epsilon_2w_2^TQ_2w_2 - 2(1-\varepsilon)w_2^TP_2w_2\\ &\quad -(4/\delta_2^2 + 2L/EI)(c_2^Tw_2)^2\big]\\ &\quad -\big[2(1-\varepsilon)t\gamma_1 - 4Ld_1^2/\delta_1^2 - \rho L\big]u_t^2(L,t)\\ &\quad -\big[2(1-\varepsilon)t\gamma_2 - (4/\delta_2^2 + 2L/EI)d_2^2\big]u_{xt}^2(L,t)\\ &\quad - 2(1-\varepsilon)t\big(\sqrt{d_1-\gamma_1}\,u_t(L,t) - w_1^Tq_1\big)^2\\ &\quad - 2(1-\varepsilon)t\big(\sqrt{d_2-\gamma_2}\,u_{xt}(L,t) - w_2^Tq_2\big)^2\\ &\quad -\left[(\varepsilon+2)EI\int_0^L u_{xx}^2\,dx - (2L\delta_1^2 + 2\delta_2^2)u_x^2(L,t)\right].\end{aligned} \tag{29}$$

By choosing δ_1 and δ_2 sufficiently small, the last term in (29) can be made negative (see (23)). Hence, from (29) it follows that there exists a $T \ge 0$, such that the following holds:

$$\dot V(t) \le 0, \qquad t \ge T. \tag{30}$$

This, together with (25) implies that

$$E(t) \le \frac{V(T)}{(2(1-\varepsilon)t - C)}, \qquad t \ge T. \tag{31}$$

By (21), $E(t) \le E(0)$; hence (25) implies that $V(T) \le K_0E(0)$ for some constant $K_0 > 0$. Therefore, from (21) and (31), we conclude that for some constant $K_1 > 0$, the following holds:

$$\int_0^\infty \|T(t)z(0)\|^4\,dt \le K_1\|z(0)\|^4 \qquad \forall z(0) \in D(A) \tag{32}$$

where $E(t) = \|z(t)\|^2$ (see (20)) and $z(t) = T(t)z(0)$ (see the Theorem 1). Then, by density of $D(A)$ in $\mathscr{H}$, we conclude that the following also holds:

$$\int_0^\infty \|T(t)z(0)\|^4\,dt < \infty \qquad \forall z(0) \in \mathscr{H}. \tag{33}$$

Exponential decay follows from a result due to Pazy, [11, p. 116], that is there exist constants $0 < M < \infty$ and $\delta > 0$ such that the following holds:

$$E(t) \le Me^{-\delta t}, \qquad t \ge 0. \tag{34}$$

In fact, by density, (32) will hold on $\mathscr{H}$, hence exponential decay obtains along *all finite energy solutions*, not just classical solutions. □

Remark 1: Assume that the feedback laws which give $f_1(t)$ and $f_2(t)$ are given by two transfer functions $g_1(s)$ and $g_2(s)$, instead of (7) and (8). In this case, a minimal realization of $g_1(s)$ and $g_2(s)$ in the form of (7) and (8) can always be found. Because of minimality, for $i = 1, 2$, the eigenvalues of A_i are the same as the poles of $g_i(s)$. Hence, if the feedback laws are specified by two proper transfer functions $g_1(s)$ and $g_2(s)$, the assumptions 1)–3) can be replaced by the following two assumptions: for $i = 1, 2$

Assumptions:

i) The poles of $g_i(s)$ are in the open left-half plane.

ii) There exist constants $\gamma_1 > 0$, $\gamma_2 \ge 0$ such that the following holds:

$$\mathscr{Re}\{g_i(j\omega)\} > \gamma_i, \qquad \omega \in \boldsymbol{R}. \qquad \square \quad (35)$$

Remark 2: Note that the result of Theorem 2 still holds even if $g_2(s) = 0$ (see (10)), that is if $f_2(t) = 0$, $t \geq 0$ (see (4)). In this case (7) will not exist for $i = 2$ and (8) will yield $c_2 = 0$, $d_2 = 0$. Hence, we have $\gamma_2 = 0$, and consequently from (21) and (29) we conclude the exponential decay of the energy. □

Remark 3: In the case of *nondynamic* control laws, for $i = 1, 2$, (7) will not exist and (8) reduces to

$$f_1(t) = d_1 u_t(L, t), \qquad f_2(t) = d_2 u_{xt}(L, t), \qquad d_1 > 0, \quad d_2 \geq 0. \quad (36)$$

This is the case considered by Chen [2]. Hence, the results presented here may be considered as a generalization of some results of [2], and consequently, we obtain a wider class of exponentially stabilizing controllers for the system given by (1)–(4).

One way of implementing the control laws given by (36) is to use actuators whose inputs are $u_t(L, t)$ and $u_{xt}(L, t)$, and whose outputs are $f_1(t)$ and $f_2(t)$, where the actuator transfer functions are given by d_1 and d_2. From a practical point of view, however, most actuators show some dynamic behavior, at least over a frequency range [14]. In this case, Theorem 2 provides a sufficient condition to ensure exponential stability, whereas the results of [2] do not apply.

Also note that the proposed dynamic control (7)–(8), as well as the standard nondynamic one (36), change the frequency-domain characteristic of the system; that is the eigenvalues of the operator given by (17) are completely different than the eigenvalues of the uncontrolled system. This change in the spectrum, although limited, can be used for some control applications, such as eigenvalue assignment, disturbance rejection (see [8]). Note that the dynamic control offers more degrees of freedom to change the spectrum of the operator given by (17), than the standard nondynamic one. This point is still under investigation and will be the subject of a forthcoming paper. We also note that the use of dynamic control for a rotating flexible structure yields some interesting results, such as it may be possible to achieve fast rotation rates without destroying stability, [9].

One way of implementing the dynamic boundary control presented here is to use gas jets at the tip of the beam and to control the gas pressure by a dynamic actuator. (In [4], it has been noted that the pressurized gas tanks with servo-controlled actuators can be used to significantly reduce the dynamic response of tall buildings.) Also, a classical way of employing such control laws is to use parallel/series combinations of standard mass-spring-dampers, see [10], [4]. □

III. Conclusion

In this note, we considered the stabilization of a clamped-free Euler–Bernoulli beam using dynamic boundary control. Under some assumptions, one of which is the positive realness of the actuator transfer functions corresponding to the dynamic boundary controls, we proved that the energy of the beam-actuator configuration decays exponentially to zero. We also give an equivalent characterization of our assumptions which guarantees the exponential stability of the beam-actuator configuration in the Remark 1. This latter set of assumptions are in the frequency domain and are easy to check. Our results are a generalization of an earlier result due to Chen [2], and provide a wider class of stabilizing actuators for the clamped-free Euler–Bernoulli beam.

An interesting research topic would be the characterization of other, if possible all, finite-dimensional exponentially stabilizing controllers for the clamped-free Euler–Bernoulli beam.

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Time-Varying Riccati Differential Equations with Known Analytic Solutions

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Abstract—**This note presents several examples of large-scale time-varying stiff as well as nonstiff Riccati differential equations (RDE's) with known analytic solutions. These examples are useful for testing the accuracy and efficiency of algorithms for solving such equations. Analytic expressions of the eigenvalues of the solutions of the RDE's are also found. Eigenvectors of some of the solutions are also given.**

I. Introduction

Many algorithms have been proposed to solve matrix Riccati differential equations. Most of these algorithms were tested only

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