

# Static spherically symmetric solutions to Einstein–Maxwell dilaton field equations in $D$ dimensions

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**Abstract.** We classify the spherically symmetric solutions of the Einstein–Maxwell dilaton field equations in  $D$  dimensions and find some exact solutions of the string theory at all orders of the string tension parameter. We also show the uniqueness of the black-hole solutions of this theory in static axially symmetric spacetimes.

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## 1. Introduction

Recently we have observed an increase of interest in constructing exact solutions of string theory [1–24]. There are several ways of performing such a construction [1]. One of these is to start with an exact solution of the leading-order field equations (low-energy limit of string theory) and showing that this solution also solves the string equations at all orders of the inverse power of the string tension parameter. Plane-wave spacetimes with appropriate gauge and scalar fields have been shown to be exact solutions of the string theory [2–4]. In these solutions the higher-order terms in the field equations disappear due to the special form of the plane-wave metrics. Levi-Civita–Bertotti–Robinson spacetimes in four dimensions are also an exact solution of the string theory [8,21]. In this solution the higher-order terms do not disappear but give some algebraic constraints among the constants of solutions [21].

To construct exact solutions in this approach one must first study the low-energy limit of the string theory with some symmetry assumptions like spherical symmetry. Among these solutions one searches for those satisfying the field equations at all orders. Hence we need a classification of the solutions under a symmetry of the low-energy limit of the string theory. In this work, instead of working directly with the low-energy limit of the string theory we consider the Einstein–Maxwell dilaton field theory with arbitrary dilaton coupling constant  $\alpha$ . For some fixed values of  $\alpha$  this theory reduces to the low-energy limit of the string theory and Kaluza–Klein-type theories.

With the assumption of a symmetry, static spherical symmetry for instance, some of the solutions may describe black-hole spacetimes. Non-rotating black-hole solutions of this theory, in spherically symmetric spacetimes, found so far [17–20] carry mass  $M$ , electric charge (or magnetic charge)  $Q$  and a scalar charge  $\phi$ . Out of these parameters only two of

them are independent. In four dimensions we have the Garfinkle–Horowitz–Strominger (GHS) solution [18]. Gibbons and Maeda (GM) [17] have given black-hole solutions in

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arbitrary  $D$  dimensions. In four dimensions the GM solution describes the region  $r > r_1$  of the GHS metric. In this work we find and classify the solutions of the dilatonic Einstein–Maxwell (and hence of the low-energy limit of the string theory) for static and spherically symmetric spacetimes.

Recently [22] the most general asymptotically flat spherically symmetric static solution of the low-energy limit of string theory in four dimensions has been found. The uniqueness of the GHS–GM black-hole solutions in four-dimensional static spherically symmetric spacetimes is manifest in this work. Following this work we find the most general  $D$ -dimensional spherically symmetric static solution of the Einstein–Maxwell dilaton field theory.

The uniqueness of these black-hole solutions with the assumed symmetry ( $D$ -dimensional spherical symmetry) is understood, because the solutions of the field equations are completely classified. On the other hand without solving the field equations it is desirable to know the existence of other black-hole solutions under relaxed symmetries. Here we show that the GHS–GM metric in four dimensions is the unique asymptotically flat static axially symmetric black-hole solution of the Einstein–Maxwell dilaton field theory.

In this work we assume  $D$ -dimensional static spherically symmetric spacetimes. We first write, in section 2, the geometrical quantities like Riemann and Ricci tensors in terms of the volume form of the sphere  $S^{D-2}$  or in terms of its dual form. With such a representation it is relatively easier to work with the higher-order curvature terms. In the third section we classify the static spherically symmetric solutions of the Einstein–Maxwell dilaton field equations. In the fourth section we prove that the GHS+GM metrics are the unique static axially symmetric black holes of the Einstein–Maxwell dilaton field theory. In the last section we give some theorems for the  $D$ -dimensional static spherically symmetric spacetimes which are crucial to finding the exact solutions of the string theory. In this section we give some possible exact solutions of string theory in  $D$  dimensions.

## 2. Static spherically symmetric Riemannian geometry

The line element of a static and spherically symmetric spacetime is given by

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 d\Omega_{D-2}^2 \quad (1)$$

be rewritten as where  $A, B$  and  $C = -$ depend only  $t_i = \delta_i^t, k_i = \delta_i^r, h_{ij} = \text{on } A^2 dt^2 + B^2 dr^2 + C^2 d\Omega_{D-2}^2$  is the line element on, where  $S^{D-2}$ .

The metric can be written on a

$D - 2$  sphere for  $i, j \geq 2, h_{0i} = h_{i0} = 0$ .

Components of the Christoffel symbol are given by

$$\Gamma_{jmi}^0 = -A A_{,0} (t_{jk} + t_{kj}) + A A_{,0} k_i t_m + B B_{,0} k_i k_j k_m - C C_{,0} h_{ijm}$$

where  $0_{(s)}$  is the Christoffel symbol on a  $D - 2$  sphere.

The Riemann tensor is given by

$$R^i_{jml} = \Gamma^i_{jl,m} - \Gamma^i_{jm,l} + \Gamma^i_{nm}\Gamma^n_{jl} - \Gamma^i_{nl}\Gamma^n_{jm} \quad (3)$$

We find that

$$R_{ijml} = \left( AA'' - \frac{AA'B'}{B} \right) t_{[i}k_j]t_{[m}k_{l]} + \left( CC'' - \frac{CC'B'}{B} \right) k_{[i}h_{j][m}k_{l]} + \frac{AA'CC'}{B^2} t_{[m}h_{l][j}t_{i]} - \frac{C^2C'^2}{B^2} h_{i[m}h_{l]j} + C^2 R_{(s)ijml} \quad (4)$$

where  $R_{(s)ijml} = h_{i[m}h_{l]j}$  are the components of the Riemann tensor on  $S_{D-2}$ . The Riemann tensor in (4) may be rewritten as

$$S_{ij} = \eta_0 M_{ij} + \eta_1 k_i k_j + \frac{1}{2} \eta_3 g_{ij}$$

$$R_{ijml} = g_{il}S_{jm} - g_{im}S_{jl} + g_{im}S_{jl} - g_{il}S_{mj} + \eta_2 H_{ijk\dots n} H_{ml} \quad \text{where } \begin{matrix} M = H & H^{m\dots n} & 1 \\ & & H^2 \end{matrix} \quad (5)$$

$$(6)$$

$$\text{which turns out to be } \frac{2(D-2)}{j} \quad \begin{matrix} ij \\ - \\ g_{ij} \end{matrix} \quad \text{im}\dots n$$

$$(7)$$

$$M_{ij} = \frac{(D-3)!}{C^{2(D-3)}} \left( h_{ij} - \frac{1}{2C^2} g_{ij} \right) \quad (8)$$

Here the tensor  $H_{ij\dots k}$  is the volume form of  $S_{D-2}$ , i.e.

$$H_{ij\dots k} = -\sqrt{h} \epsilon_{ij\dots k} \quad (9)$$

where  $h$  denotes  $\text{deth}_{ij}$ .

Here the scalars  $\eta_i$  are given by

$$\eta_0 = \frac{C^{2(D-2)}}{(D-3)!} \left( \frac{A''}{AB^2} - \frac{A'B'}{AB^3} - \frac{C''}{CB^2} + \frac{B'C'}{B^3C} \right) \quad (10)$$

$$\eta_1 = \frac{A'C'}{AC} - \frac{C''}{C} + \frac{B'C'}{BC} \quad (11)$$

$$\eta_2 = \frac{C^{2(D-3)}}{(D-4)!} \left( 1 - \frac{C'^2}{B^2} + \frac{A'C'C}{AB^2} - \frac{A''C^2}{AB^2} + \frac{A'B'C^2}{AB^3} + \frac{CC''}{B^2} - \frac{B'C'C}{B^3} \right) \quad (12)$$

$$\eta_3 = -\frac{A'C'}{AB^2C} \quad (13)$$

The Ricci tensor, Ricci scalar and Einstein tensor can be computed from Riemann as follows:

$$(14)$$

$$R_{ij} = \left[ \frac{(D-3)!}{2C^{2(D-2)}} (\eta_0(D-4) + \eta_2) + \frac{1}{B^2} \eta_1 + \eta_3(D-1) \right] g_{ij} + \eta_1(D-2)k_i k_j + [\eta_0(D-2) + \eta_2]M_{ij} \tag{15}$$

$$R = \frac{(D-3)!}{C^{2(D-2)}} [\eta_0(D-4)(D-1) + \eta_2(D-2)] + \frac{2\eta_1(D-1)}{B^2} + \eta_3 D(D-1)$$

$$G_{ij} = - \left[ \frac{(D-3)!}{2C^{2(D-2)}} (\eta_0(D-4)(D-2) + \eta_2(D-3)) + \frac{1}{B^2} \eta_1(D-2) + \frac{1}{2} \eta_3(D-1)(D-2) \right] g_{ij} + \eta_1(D-2)k_i k_j + [\eta_0(D-2) + \eta_2]M_{ij} . \tag{16}$$

The covariant derivatives

of  $H_{ij\dots k}$  and  $k_i$  are given as

$$\begin{aligned} \nabla_l H_{ij\dots m} &= -\rho[(D-2)H_{ij\dots m}k_l + k_i H_{lj\dots m} + k_j H_{il\dots m} + \dots + k_m H_{ij\dots l}] \\ \nabla_i k_j &= \rho_1 g_{ij} + \rho_2 M_{ij} + \rho_3 k_i k_j \end{aligned} \tag{17}$$

(18)

where

$$\begin{aligned} \rho &= \frac{C'}{C} & \rho_1 &= \frac{A'C + AC'}{2AB^2C} \\ \rho_2 &= \frac{C^{2(D-2)}}{(D-3)!} \frac{A'C - AC'}{AB^2C} & \rho_3 &= -\frac{(AB)'}{AB} \end{aligned} \tag{19}$$

In equation (5) we have given the Riemann tensor of a static, spherically symmetric spacetime in terms of the volume form of the  $S_{D-2}$  ( $D-2$ )-brane field. We may also write the Riemann tensor in the form

$$R_{ijml} = g_{jl}S_{im} - g_{jm}S_{il} + g_{im}S_{jl} - g_{il}S_{mj} + e_2 F_{ij} F_{ml} \tag{20}$$

where

$$F_{ij} = \frac{AB}{C^2} (t_i k_j - t_j k_i) . \tag{21}$$

Other

case, i.e.

$$S_{ij} = e_0 \widetilde{M}_{ij} + e_1 k_i k_j + \frac{1}{2} e_3 g_{ij} \tag{22}$$

$$\widetilde{M}_{ij} = F_{mj} F_i^m - \frac{1}{4} F^2 g_{ij} \tag{22}$$

$$\widetilde{M}_{ij} = \frac{1}{C^2} \left( h_{ij} - \frac{1}{2C^2} g_{ij} \right) . \tag{23}$$

(24) The

scalars are given as

$$\begin{aligned} e_0 &= C^2 - \frac{C'^2 C^2}{B^2} + \frac{C^3 C' A'}{AB^2} \\ e_1 &= \eta_1 \\ e_2 &= -C^2 \left( 1 - \frac{C'^2}{B^2} + \frac{A'C'C}{AB^2} - \frac{A''C^2}{AB^2} + \frac{A'B'C^2}{AB^3} + \frac{CC''}{B^2} - \frac{B'C'C}{B^3} \right) \\ e_3 &= \eta_3 . \end{aligned} \tag{25}$$

Notice that when  $D = 4$ ,  $\widetilde{M}_{ij} = M_{ij}$  and  $e_2 = -\eta_2$ .

The covariant derivatives of  $F_{ij}$ ,  $k_i$  and  $t_i$  are given as

$$\nabla_l F_{ij} = -\frac{3C'}{C} F_{ij} k_l - \frac{AC'}{BC^3} (t_j g_{il} - t_i g_{jl}) \quad (26)$$

$$\nabla_i k_j = \rho_1 g_{ij} + \tilde{\rho}_2 \widetilde{M}_{ij} + \rho_3 k_i k_j \quad (27)$$

$$\nabla_i t_j = \frac{A'}{A} (t_i k_j + t_j k_i) \quad (28)$$

where

$$\tilde{\rho}_2 = \frac{(D-3)!}{C^{2(D-4)}} \rho_2.$$

### 3. Solutions of the Einstein–Maxwell dilaton field equations

The field equations of the Einstein–Maxwell dilaton theory can be obtained from the following Lagrangian:

$$L = \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{4}{(D-2)\kappa^2} (\nabla\phi)^2 - \frac{1}{4} e^{-\alpha_e \phi} F^2 \right] \quad (29)$$

The field equations are

$$G_{ij} = \frac{8}{(D-2)} \left[ \partial_i \phi \partial_j \phi - \frac{1}{2} (\nabla\phi)^2 g_{ij} \right] - \kappa^2 e^{-\alpha_e \phi} \left[ F_i^m F_{jm} - \frac{1}{4} F^2 g_{ij} \right], \quad (30)$$

$$\nabla_i (e^{-\alpha_e \phi} F^{ij}) = 0,$$

$$\partial_i (\sqrt{-g} g^{ij} \partial_j \phi) + \frac{(D-2)\kappa^2 \alpha_e \sqrt{-g}}{32} e^{-\alpha_e \phi} F^2 = 0, \quad (31)$$

$$(32)$$

where  $F_{ij}$  is the Maxwell and  $\phi$  is the dilaton field. Here  $i, j = 1, 2, \dots, D \geq 4$ .

In static spherically symmetric spacetimes, gravitational field equations first lead to

$$\frac{\eta_0 (D-4)(D-2)!}{2C^{2(D-2)}} + \frac{\eta_2 (D-3)(D-3)!}{2C^{2(D-2)}} + \frac{\eta_1 (D-2)}{B^2} + \frac{\eta_3 (D-1)(D-2)}{2} - \frac{4\phi'^2}{(D-2)B^2} = 0 \quad (33)$$

$$\eta_1 (D-2) - \frac{8\phi'^2}{(D-2)} = 0$$

$$\eta_0 (D-2)! + \eta_2 (D-3)! - \frac{\kappa^2 Q^2 e^{\alpha_e \phi}}{A_{D-2}^2} = 0. \quad (34)$$

(35) The dilaton equation is

$$\frac{8}{D-2} \left[ \frac{A}{B} C^{D-2} \phi' \right]' - \frac{\alpha_e A B \kappa^2 Q^2 e^{\alpha_e \phi}}{2C^{D-2} A_{D-2}^2} = 0. \quad (36)$$

From equations (33) and (34) we obtain

$$\left[ \frac{(AC^d)' C}{B} \right]' = d^2 ABC^{d-1} \quad (37)$$

where  $d = D - 3$ . Using the freedom in choosing the  $r$  coordinate we can let

$$ABC^{d-1} = r^{d-1}. \quad (38)$$

Using equations (37) and (38) we obtain

$$A_2 C_{2d} = r_{2d} - 2b_1 r_d + b_2 \quad (39)$$

where  $b_1$  and  $b_2$  are integration constants.

A combination of the dilaton (32) and gravitational field equations (30) give

$$dT^2 - \frac{d-1}{r} T + T' + \frac{8}{(d+1)^2} \phi'^2 = 0, \quad (40)$$

where  $T$  is defined as

$$T = \frac{(r^d + c_1)r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} - \frac{16\phi'}{(d+1)^2 \alpha_e}. \quad (41)$$

Here  $c_1$  is an integration constant. Now defining a new function  $\psi(\rho)$  as follows:

$$\alpha_e \phi' = \frac{k_1 dr^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} \psi(\rho) \quad (42)$$

equation (40) becomes

$$r_{2d} - \frac{2b_1 r^d + b_2}{dr^{d-1}} \frac{d\psi}{d\rho} \frac{d\rho}{dr} = (\psi + \mu)^2 + v^2 \quad (43)$$

The constants are given by

$$a = \frac{(d+1)^2 \alpha_e^2}{32d} \quad \mu = -(c_1 + b_1) \quad v^2 = a\mu^2 - 1(a+1). \quad (44)$$

Now, if we solve the auxiliary equation

$$\frac{r^{2d} - 2b_1 r^d + b_2}{dr^{d-1}} \frac{d\rho}{dr} = \rho \quad (45)$$

and insert  $\rho$  above in (43), we obtain

$$-\frac{d\psi}{d\rho} = (\psi + \mu)^2 + v^2. \quad (46)$$

Note that  $\rho$  depends on the sign of  $\Delta = b_1^2 - b_2$ . Hence  $\phi$  can be found from  $\psi$  by

$$\phi = \frac{k_1}{\alpha_e} \int \frac{\psi(\rho)}{\rho} d\rho + \phi_0 \quad (47)$$

where  $k_1 = \frac{2a}{a+1}$  and  $\phi_0$  is an arbitrary constant. The metric function  $C$  is connected to  $\phi$  as

$$\frac{C'}{C} = \frac{(r^d + c_1)r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} - \frac{16\phi'}{(d+1)^2 \alpha_e}. \quad (48)$$

This gives us

$$\ln\left(\frac{C}{c_0}\right)^d = \left(\int \frac{u + b_1 + c_1}{u^2 - \Delta} du\right) - \frac{\alpha_e}{2a} \phi \quad (49)$$

$u = r^d - b$   $c$  is an arbitrary integration constant. where  $-1$  and  $0$

Metric functions A and B can be found from C through (38) and (39). We have three different cases according to the sign of 1: case 1:

$$\begin{aligned}
 & \frac{1}{r^d} \quad \frac{2}{r^d} \quad \frac{2}{r^d} \\
 & \frac{3}{r^d} \quad \frac{b}{r^d} \quad \frac{\pi}{r^d} \\
 & r_1^d = b_1 + \sqrt{\Delta}, r_2^d = b_1 - \sqrt{\Delta}, r_3^d = b_1. \\
 \Delta > 0 \quad \ln \rho(r) &= \frac{1}{r^d - r_1^d} \ln \left( \frac{r^d - r_1^d}{r^d - r_2^d} \right) \\
 \text{case 2: } \Delta = 0 \quad \ln \rho(r) &= -\frac{1}{r^d - r_1^d} \\
 \text{case 3: } \Delta < 0 \quad \ln \rho(r) &= \frac{1}{\sqrt{-\Delta}} \left[ \arctan \left( \frac{r^d - r_1^d}{\sqrt{-\Delta}} \right) - \frac{\pi}{2} \right]
 \end{aligned} \tag{50}$$

where

The integration constants  $c_0$  and  $\phi_0$  are determined by taking the asymptotic behaviour of the functions C(r) and  $\phi(r)$  to be

$$\begin{aligned}
 \lim_{r \rightarrow \infty} \phi &= \phi_0 = 0 \\
 \lim_{r \rightarrow \infty} \frac{C(r)}{r} &= c_0 = 1
 \end{aligned} \tag{51}$$

$$\tag{52}$$

To determine the remaining integration constants, we restrict the asymptotic behaviour of the metric, scalar field and the tensor field  $F_{ij}$  as follows:

$$\begin{aligned}
 \lim_{r \rightarrow \infty} r^d (A^2 - 1) &= -\frac{2\kappa^2 M}{A_{d+1}(d+1)} \\
 \lim_{r \rightarrow \infty} r^{d+1} \phi' &= -\frac{\kappa \sqrt{d+1}}{2A_{d+1}} \Sigma \\
 \lim_{r \rightarrow \infty} r^{d+1} F_{tr} &= \frac{Q}{A_{d+1}}
 \end{aligned} \tag{53}$$

where M,  $\phi$  and Q are the mass, dilaton and electric charges, respectively [17]. From the definition of these constants it is clear that we are interested in the asymptotically flat solutions of the Einstein–Maxwell dilaton field equations: *Case 1*

$$1 > 0 \tag{54}$$

which means there are two roots to (39). According to the sign of  $v^2$  we have three distinct solutions:

*Type 1*

$$v^2 < 0 \quad \lambda^2 = -v^2 \quad \psi = \frac{\lambda - \mu + c_2(\lambda + \mu)\rho^{2\lambda}}{1 - c_2\rho^{2\lambda}}$$

The metric functions become

$$A^2 = \frac{(r^d - r_1^d)(r^d - r_2^d)}{C^{2d}} \quad B^2 = \frac{r^{2d-2}C^2}{(r^d - r_1^d)(r^d - r_2^d)} \tag{55}$$

$$C^d = (r^d - r_2^d) \left( \frac{1 - c_2 \rho^{2\lambda}}{1 - c_2} \right)^{k_2} \rho^{k_3}. \quad (56) \text{ The dilaton field is given as}$$

$$e^{\alpha_e \phi} = \left[ \frac{(1 - c_2) \rho^{\lambda - \mu}}{1 - c_2 \rho^{2\lambda}} \right]^{k_1}. \quad (57)$$

The constants are given by

$$k_2 = \frac{1}{(a+1)}, \quad k_3 = \frac{(r_1^d - r_2^d)}{2} - \frac{(v + a\mu)}{(a+1)}. \quad (58)$$

Undetermined integration constants are  $r_1, r_2, c_1$  and  $c_2$ . From the boundary condition (53) we find that

$$2M = \left[ \frac{1 + c_2}{1 - c_2} \lambda + a\mu \right] e_1 \quad \Sigma = \left[ \frac{1 + c_2}{1 - c_2} \lambda - \mu \right] e_2 \quad Q = \lambda \frac{\sqrt{c_2}}{1 - c_2} e_3. \quad (59)$$

The constants  $e_i$  are given by

$$e_1 = \frac{2A_{d+1}(d+1)}{(a+1)\kappa^2} \quad e_2 = \frac{A_{d+1}\alpha_e(d+1)^{3/2}}{8(a+1)\kappa} \quad e_3 = \frac{2A_{d+1}}{\kappa} \sqrt{\frac{(d+1)d}{(a+1)}}. \quad (60)$$

Note that, we have four integration constants ( $c_1, c_2, r_1^d$  and  $r_2^d$ ) but there exist only three equations to determine them. Also note that  $c_1$  does not appear in the solution directly, so we have a freedom in  $c_1$  ( $c_2 \neq 1$ ). In order to complete the solution, we need to determine the integration constants in terms of the physical parameters  $M, \mu$  and  $Q$ . Let us define some auxiliary variables to solve the set of algebraic equations (59):

$$T_1 = \frac{1}{a+1} \left( \frac{a\Sigma}{e_2} + \frac{2M}{e_1} \right) \quad T_2 = \frac{1}{a+1} \left( \frac{2M}{e_1} - \frac{\Sigma}{e_2} \right) \\ T_3 = \sqrt{1 - \frac{4Q^2}{e_3^2 T_1^2}}. \quad (61)$$

Then the integration constants are

$$\lambda = T_3 T_1 \quad \mu = T_2 \quad c_2 = \frac{1 - T_3}{1 + T_3}. \quad (62)$$

Also

$$\Delta = \frac{a\mu^2 + \lambda^2}{a+1}. \quad (63)$$

The reality of  $T_3$  imposes

$$M + g\mu \geq s|Q| \quad (64)$$

where  $g$  and  $s$  are given by

$$g = \frac{\alpha_e(d+1)^{3/2}}{4d\kappa}, \quad s = \frac{1}{\kappa} \sqrt{\frac{(d+1)(a+1)}{d}}. \quad (65)$$

Such an inequality has been found by Gibbons and Wells for  $D = 4$ .

When  $1 > 0$  we have two roots to (39). In general these roots are the singular points of the spacetime. If the integration constants satisfy some additional constraints one of these roots becomes regular. An invariant of the spacetime is the scalar curvature which is given by



$$R = A_1 \frac{\rho^{z_1} \left[ -\mu + \lambda + \frac{2c_2 \lambda \rho^{2\lambda}}{1 - c_2 \rho^{2\lambda}} \right]^2}{[(r^d - r_1^d)(r^d - r_2^d)]^{1+\frac{1}{a}} (1 - c_2 \rho^{2\lambda})^{z_2}} + A_2 \frac{\rho^{z_1+2\lambda} \left( \frac{1-c_2}{1-c_2 \rho^{2\lambda}} \right)^{z_2+2}}{[(r^d - r_1^d)(r^d - r_2^d)]^{1+\frac{1}{a}}} \quad (66)$$

where

$$z_1 = \frac{2}{d(a+1)}(a\mu + \lambda) \quad z_2 = \frac{2}{d(a+1)} \quad (67)$$

$$A_1 = \frac{8}{d+1} \frac{k_1^2 d^2}{\alpha_e^2} \quad A_2 = \frac{\kappa^2 (D-4)}{2(D-2)}.$$

As  $r \rightarrow r_1$  we have a singularity unless we choose  $\mu = \lambda$ . This choice, by the use of (44), gives

$\mu = \lambda = \frac{r_1^d - r_2^d}{2}$ . Inserting this in (66), we find that as  $r \rightarrow r_1$ ,

$$R = \tilde{A}_1 (r^d - r_1^d) + \tilde{A}_2 \quad (68)$$

where  $\tilde{A}_1$  and  $\tilde{A}_2$  are constants, so the horizon is regular. The choice  $\mu = \lambda$  gives  $T_2 = T_1 T_3$

from (62), which means

$$Q^2 = \frac{8d\Sigma}{\alpha_e} \left[ \frac{2\kappa}{(d+1)^{3/2}} M + \frac{8(a-1)}{\alpha_e (d+1)^2} \Sigma \right] \quad (69)$$

The connection between physical parameters and integration constants is

$$2M = \frac{r_1^d - r_2^d}{2} \left( \frac{1+c_2}{1-c_2} + a \right) e_1$$

$$\Sigma = \frac{r_1^d - r_2^d}{2} \frac{2c_2}{1-c_2} e_2$$

$$Q = \frac{r_1^d - r_2^d}{2} \frac{\sqrt{c_2}}{1-c_2} e_3. \quad (70)$$

The metric is

$$ds^2 = - \frac{(r^d - r_1^d)(r^d - r_2^d)}{C^{2d}} dt^2 + \frac{r^{2d-2} C^2}{(r^d - r_1^d)(r^d - r_2^d)} dr^2 + C^2 d\Omega_{D-2}^2 \quad (71)$$

where

$$C^d = (r^d - r_2^d) \left( 1 + \frac{\Sigma/e_2}{r^d - r_2^d} \right)^{k_2} \quad (72)$$

$$e^{\alpha_e \phi} = \left( 1 + \frac{\Sigma/e_2}{r^d - r_2^d} \right)^{-k_1} \quad (73)$$

$$F_{tr} = \frac{Q}{A_{d+1} r^{d+1}} \quad (74)$$

$$\frac{r_1^d - r_2^d}{2} = T_2.$$

and

If we choose the integration constant  $r_2$  as zero, these black-hole metrics are identical to the metrics given by Gibbons and Maeda [17].

Although the extreme limit ( $1 = 0$ ), in general, is going to be studied in case 2 we obtain the extreme case of the above solution by setting  $c_2 = (r_2/r_1)^d$  and  $r_2 = r_1$ . Then we obtain

$$2M = r_1^d e_1 \quad \Sigma = r_1^d e_2 \quad 2Q = r_1^d e_3. \quad (75)$$

$$v^2 > 0 \quad \Psi = v \tan(c_2 + v \ln \rho) - \mu \quad (76)$$

$$(77) \quad \int \frac{\Psi(\rho)}{\rho} d\rho = -\ln[\cos(c_2 + v \ln \rho)] - \mu \ln \rho + c_3$$

After similar steps to the previous type, we arrive at the solution

$$A^2 = \frac{(r^d - r_1^d)(r^d - r_2^d)}{C^{2d}} \quad B^2 = \frac{r^{2d-2} C^2}{(r^d - r_1^d)(r^d - r_2^d)} \quad (78)$$

(79) The scalar field is given as

$$C^d = (r^d - r_2^d) \rho^{\frac{r_1^d - r_2^d}{2} - \frac{\mu k_1}{2}} \left[ \frac{\cos(c_2 + v \ln \rho)}{\cos c_2} \right]^{k_2}$$

$$e^{\alpha_e \phi} = \left[ \frac{\cos c_2}{\rho^\mu \cos(c_2 + v \ln \rho)} \right]^{k_1} \quad (80)$$

Physical parameters are found using (53):

$$2M = (a\mu + v \tan c_2) e_1 \quad \Sigma = (-\mu + v \tan c_2) e_2 \quad Q = \frac{v}{\cos c_2} e_3 \quad (81)$$

$$\mu = T_2 \quad \sin c_2 = \frac{e_3 T_1}{Q} \quad v \tan c_2 = T_1 \quad (82)$$

The condition  $|\text{sinc}_2| < 1$  imposes

$$M + g_6 < s|Q| \quad (83)$$

where  $g$  and  $s$  are defined in (65). We also have to check the sign of 1:

$$(a + 1)\Delta = \frac{1}{a + 1} \left( \frac{a\Sigma^2}{e_2^2} + \frac{4M^2}{e_1^2} \right) - \frac{Q^2}{e_3^2} \quad (84)$$

Here the sign of 1 puts a constraint on the physical variables.

Type 3

$$(85) \quad \Psi = -\mu - \frac{1}{\ln \rho + c_2} \quad v^2 = 0$$

$$\int \frac{\Psi(\rho)}{\rho} d\rho = -\ln(\ln \rho + c_2) - \mu \ln \rho + c_3 \quad (86)$$

$$e^{\alpha_e \phi} = \left[ \frac{c_2}{\rho^\mu (c_2 + \ln \rho)} \right] \quad (87)$$

$$C^d = \left( \frac{c_2 + \ln \rho}{c_2} \right)^{k_2} [(r^d - r_1^d)(r^d - r_2^d)]^{1/2} \rho^{-\frac{\mu k_1}{2}} \quad (88)$$

Physical parameters are found using (53)

$$2M = \left( a\mu - \frac{1}{c_2} \right) e_1 \quad \Sigma = \left( -\mu - \frac{1}{c_2} \right) e_2 \quad Q = \frac{e_3}{2c_2} \quad (89)$$

The solution is

$$\mu = -T_2 \quad -\frac{1}{c_2} = T_1 \quad (90)$$

which gives

$$\frac{2Q}{e_3} = T_1 \quad (91)$$

This is the equality case of the inequality (64).

Case 2

$$1 = 0. \quad (92)$$

Then there is one root to (39). Denote it by  $r_1^d$ ,

$$AC^d = (r^d - r_1^d) \quad (93)$$

$$A^2 = \frac{(r^d - r_1^d)^2}{C^{2d}} \quad B^2 = \frac{r^{2d-2}C^2}{(r^d - r_1^d)^2} \quad (94)$$

$$C^d = (r^d - r_1^d)\rho^{-\frac{\mu k_1}{2}} \left[ \frac{\cos(c_2 + \nu \ln \rho)}{\cos(c_2)} \right]^{k_2} \quad (95)$$

The scalar field is given as

$$e^{\alpha_e \phi} = \left[ \frac{\cos c_2}{\rho^{\mu} \cos(c_2 + \nu \ln \rho)} \right]^{k_1} \quad (96)$$

Case 3

$$1 < 0. \quad (97)$$

This case is similar to the previous one:

$$e^{\alpha_e \phi} = \left[ \frac{\cos c_2}{\rho^{\mu} \cos(c_2 + \nu \ln \rho)} \right]^{k_1} \quad (98)$$

$$A^2 = \frac{(r^{2d} - 2b_1 r^d + b_2)}{C^{2d}} \quad B^2 = \frac{r^{2d-2}C^2}{(r^{2d} - 2b_1 r^d + b_2)} \quad (99)$$

$$C^d = (r^{2d} - 2b_1 r^d + b_2)^{1/2} \rho^{-\frac{\mu k_1}{2}} \left[ \frac{\cos(c_2 + \nu \ln \rho)}{\cos(c_2)} \right]^{k_2} \quad (100)$$

In cases 2 and 3, the relation of physical parameters to integration constants are exactly the same as case 1, type 2. In addition we have (84) with the sign of 1 chosen according to the case.

It is perhaps noticed that we solve field equations exactly without assuming any special ansatz†. In that respect we have all the possible integration constants in the solutions, hence we have the following assertion: in D dimensions the metric given in (71) and (72) is the only asymptotically flat metric corresponding to a spherically symmetric static black hole carrying mass M, dilaton charge 6 and electric charge Q and covered by a regular horizon. † After this work was completed we became aware of a recent paper [23], which similarly solves the field equations (30)–(32) without any special ansatz, but in a completely different way to those used here.

In D dimensions the Gibbons–Maeda metric is diffeomorphic to the region  $r > r_1$  where  $r_1$  is the location of the outer horizon. Relaxing the condition of asymptotical flatness there is a possibility of having a black-hole (with a regular horizon and singularity located at the origin) solution of the Einstein–Maxwell dilaton field equations [24].

Expressing the boundary conditions of the black-hole solution (72) (the behaviour of the metric as  $r$  approaches the outer horizon  $r_1$  and to infinity) as the black-hole boundary conditions in static axially symmetric four dimensions one can prove that the GHS or Gibbons–Maeda solution also describes the unique asymptotically flat static black-hole solution in Einstein–Maxwell dilaton gravity. In the next section we give a proof of this statement.

#### 4. Uniqueness of black-hole solutions

In this section we are interested in the black-hole solutions of the Einstein–Maxwell dilaton theory in the static axially symmetric spacetimes in four dimensions. Using a different approach the uniqueness of a static charged dilaton black hole (GHS+GM black hole) has been shown recently in [27]. In this proof the dilaton coupling constant was taken as fixed ( $a = 2$ ) which corresponds to the low-energy limit of the string theory. Here we shall show that the static black holes of the Einstein–Maxwell dilaton theory are unique for arbitrary values of the dilaton coupling constant  $a$ . We are not going to solve the corresponding field equations but first formulate these field equations as a sigma model in two dimensions and use this formulation in the proof of the uniqueness of the solutions under the same boundary conditions. Uniqueness of the stationary black-hole solutions of the Einstein theory is now a very well established concept [27–34]. This proof is based on the sigma model formulation of the stationary axially symmetric Einstein–Maxwell field equations [30–37]. Here we shall follow the approach given by [30, 34].

The line element of a static axially symmetric four-dimensional spacetime is given by

$$ds^2 = e^{2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] - e^{-2\psi}dt^2. \quad (101)$$

The field equations of the Einstein–Maxwell dilaton field theory with the above metric and  $A_\mu = (A, 0, 0, 0)$  are

$$\nabla^2 \psi + \frac{1}{2} \kappa^2 e^{2\psi - a\phi} (\nabla A)^2 = 0 \quad (102)$$

$$\nabla^2 \phi - a \frac{1}{8} \kappa^2 e^{2\psi - a\phi} (\nabla A)^2 = 0$$

$$\nabla^2 A + \nabla(2\psi - a\phi) \nabla A = 0 \quad (103)$$

$$\frac{1}{\rho} \gamma_{,z} = 2\psi_{,\rho} \psi_{,z} + 4\phi_{,\rho} \phi_{,z} - \kappa^2 e^{2\psi - a\phi} A_{,\rho} A_{,z} = 0$$

$$\frac{2}{\rho} \gamma_{,\rho} = 2\psi_{,\rho}^2 - 2\psi_{,z}^2 + 4\phi_{,\rho}^2 - 4\phi_{,z}^2 - \kappa^2 e^{2\psi - a\phi} (A_{,\rho}^2 - A_{,z}^2). \quad (104)$$

(105)

(106)

Let  $E = \psi - \frac{a}{2} \phi$  and  $B = \frac{\kappa'}{\sqrt{2}} A$  we then find

$$\nabla^2 E + e^{2E} (\nabla B)^2 = 0 \quad (107)$$

$$\nabla B + 2\nabla E \nabla B = 0 \quad (108)$$

where

2

$$\kappa' = \kappa \sqrt{1 + \frac{a^2}{8}}. \quad (109)$$

We wish to write (107) and (108) as a single complex equation by introducing a complex potential. In order to achieve this we introduce pseudopotential  $\omega$  by use of (108),

$$\omega_\rho = \rho e^{-2EBz}, \quad \omega_z = -\rho e^{-2EB\rho} \quad (110)$$

then the resulting equations can be written as the following single complex equation (the Ernst equation) for  $\varepsilon = \rho e^E + i\omega$ :

$$\text{Re}(\varepsilon) \nabla^2 \varepsilon = \nabla \varepsilon \nabla \varepsilon. \quad (111)$$

Hence the above complex equation represents (107) and (108) if we let  $\varepsilon = \rho e^E + i\omega$ . The above Ernst equation (111) defines a sigma model on  $SU(2)/U(1)$  with the equation of motion

$$\vec{\nabla}(P^{-1} \vec{\nabla} P) = 0 \quad (112)$$

where

$$P = \frac{1}{\rho e^E} \begin{pmatrix} 1 & -B \\ -B & \rho^2 e^{2E} + B^2 \end{pmatrix}. \quad (113)$$

In the following we assume enough differentiability for the components of the matrix  $P$  in  $V \cup \partial V$ . Here  $V$  is a region in  $M$  with boundary  $\partial V$ . In our case  $V$  is the region  $r > 0$  (see section 2, type I solutions) and hence  $\partial V$  has two disconnected components.

We also assume that  $P$  is positive definite. Let  $P_1$  and  $P_2$  be two different solutions of (112). The difference of their equations satisfy

$$\vec{\nabla}(P_1^{-1}(\vec{\nabla} Q)P_2) = 0 \quad (114)$$

where  $Q = P_1 P_2^{-1}$ . Multiplying both sides by  $Q^\dagger$  (Hermitian conjugation) and taking the trace we obtain

$$\vec{\nabla}[\text{tr}(Q^\dagger P_1^{-1}(\vec{\nabla} Q)P_2)] = \text{tr}[(\vec{\nabla} Q^\dagger)P_1^{-1}(\vec{\nabla} Q)P_2] \quad (115)$$

The left-hand side of the above equation can be simplified further and we obtain

$$\nabla^2 q = \text{tr}[(\vec{\nabla} Q^\dagger)P_1^{-1}(\vec{\nabla} Q)P_2] \quad (116)$$

where  $q = \text{tr}(Q)$ . Using the Hermiticity and positive definiteness properties of the matrices  $P_1$  and  $P_2$  we may let

$$P_i = A_i A_i^\dagger, \quad (i = 1, 2) \quad (117)$$

where  $A_1$  and  $A_2$  are non-singular  $n \times n$  matrices given by

$$A_i = \frac{1}{\sqrt{\rho e^{\frac{E_i}{2}}}} \begin{pmatrix} 1 & 0 \\ -B_i & \rho e^{E_i} \end{pmatrix}. \quad (118)$$

With the aid of (117), equation (116) reduces to

$$\nabla^2 q = \text{tr}(J E^\dagger J) E \quad (119)$$

$$\vec{J} = A_1^{-1}(\vec{\nabla} Q)A_2. \tag{120}$$

Equation (119) is a crucial step towards the proof of the uniqueness theorems. It is clear that the right-hand side is positive definite at all point of  $V$ . Before going on let us give the scalar function  $q$ :

$$q = 2 + \frac{1}{\rho^2 e^{E_1 + E_2}} [\rho^2 (e^{E_1} - e^{E_2})^2 + (B_1 - B_2)^2]. \tag{121}$$

It is clear that  $q = 2$  and its first derivatives vanish on the boundary  $\partial V$  of  $V$ .

Let  $M$  be a two-dimensional manifold with local coordinates  $(\rho, z)$ . Let  $V$  be a region in  $M$  with boundary  $\partial V$ . Let  $P$  be a Hermitian positive-definite  $2 \times 2$  matrix with unit determinant and let  $P_1$  and  $P_2$  be two such matrices satisfying (112) in  $V$  with the same boundary conditions on  $\partial V$  then we have  $P_1 = P_2$  at all points in region  $V$ . The proof is as follows. Integrating (119) in  $V$  we obtain

$$\int_{\partial V} \vec{\nabla} q \, d\vec{\sigma} = \int_V \text{tr}(\vec{J}^\dagger \vec{J}) \, dV \tag{122}$$

and using the boundary condition  $q = 2$  on  $\partial V$  we get

$$\int_V \text{tr}(\vec{J}^\dagger \vec{J}) \, dV = 0. \tag{123}$$

Then the integrand in (123) vanishes at all points in  $V$ . This implies the vanishing of  $JE$  which implies that  $Q = Q_0 = a$  constant matrix in  $V$ . Since  $Q$  is the identity matrix  $I$  on  $\partial V$  then  $Q = I$  in  $V$ . Hence  $P_1 = P_2$  at all points in  $V$ . Another way to obtain this result is to use (119) directly. The vanishing of the integrand in (123) implies that  $q$  is a harmonic function in  $V$ . Since  $q = 2$  on the boundary  $\partial V$  of  $V$  then it must be equal to the same constant in  $V$  as well. This implies that  $P_1 = P_2$  in  $V$ .

In four dimensions for the Einstein–Maxwell dilaton field theory a static black hole should carry mass  $M$ , electric charge  $Q$  and dilaton charge  $\phi$ . Such a black-hole solution exists and was found by Gibbons and Maeda for an arbitrary dilaton coupling parameter  $a$ . Here the above proof implies that all those solutions with the same black-hole boundary conditions (asymptotically flat and regular horizons) as the GM solution are the same everywhere in spacetime.

### 5. Exact solutions of string theory

In this section we introduce some theorems for static spherically symmetric  $D$ -dimensional spacetimes which will be utilized to obtain exact solutions of string theory. This section is a natural extension of the work in [21] to arbitrary dimensions. We shall not give the proofs of the theorems. The necessary tools for the proofs are given in the second section.

The covariant derivatives of  $H_{ij\dots k}$  and  $k_i$  given in (17) and (18) are expressed only in terms of themselves and the metric tensor. Hence we have the following theorem:

*Theorem 1.* Covariant derivatives of the Riemann tensor  $R_{ijkl}$ , the tensor  $H_{ij\dots k}$  and the vector  $k_i$  at any order are expressible only in terms of  $H_{ij\dots k}, g_{ij}, k_i$ .

Since contraction of  $k^i$  with  $H_{ij\dots k}$  vanishes, the only symmetric tensors constructable out of  $H_{ij\dots k}, g_{ij}$  and  $k_i$  are  $M_{ij}$ , the metric tensor  $g_{ij}$  and  $k_i k_j$ . Then the following theorem holds:

*Theorem 2.* Any second rank symmetric tensor constructed out of the Riemann tensor, antisymmetric tensor  $H_{ij\dots k}$ , dilaton field  $\varphi = \varphi(r)$  and their covariant derivatives is a linear combination of  $M_{ij}, g_{ij}$  and  $k_i k_j$ .

Let this symmetric tensor be  $E_{ij}^0$ . Then we have

$$E_{ij}^0 = \sigma_1 M_{ij} + \sigma_2 g_{ij} + \sigma_3 k_i k_j \quad (124)$$

where  $\sigma_1, \sigma_2$  and  $\sigma_3$  are scalars which are functions of the metric functions, invariants constructed out of the curvature tensor  $R_{ijkl}, H_{ij\dots k}$  and the dilaton field  $\varphi(r)$  and their covariant derivatives.

*Theorem 3.* Any vector constructed out of the Riemannian tensor  $R_{ijkl}, H_{ij\dots k}$  the dilaton field  $\varphi = \varphi(r)$  and their covariant derivatives is proportional to  $k_i$ .

Let this vector be  $E'_i$ . Hence

$$E'_i = \sigma k_i \quad (125)$$

where  $\sigma$  is a scalar like  $\sigma_1, \sigma_2, \sigma_3$ .

The covariant derivatives of  $F_{ij}$ ,  $k_i$  and  $t_i$  are expressed in terms of themselves, a metric tensor. So we have a similar theorem:

*Theorem 4.* Covariant derivatives of the Riemann tensor  $R_{ijkl}$ , the antisymmetric tensor  $F_{ij}$ , the vectors  $k_i$  and  $t_i$  at any order are expressible only in terms of  $F_{ij}, k_i, t_i$  and  $g_{ij}$ .

As an example to an exact solution of the low-energy limit of the string theory in D dimensions with constant dilaton field, we propose the LCBR (Levi-Civita Bertotti–Robinson) metric which is given by

$$g_{ij} = \frac{q^2}{r^2} (-t_i t_j + c_0^2 k_i k_j - r^2 h_{ij}) \quad (126)$$

where  $h_{ij}$  is the metric on  $S_{D-2}$ ,  $t_i = \delta_i^t$ ,  $k_i = \delta_i^r$ ,  $q$  and  $c_0$  are constants. Then the antisymmetric

tensor  $F_{ij}$  defined in (21) becomes

$$F_{ij} = \frac{c_0}{r^2} (t_i k_j - t_j k_i) \quad (127)$$

$$\widetilde{M}_{ij} = \frac{c_0}{r^2} (t_i k_j - t_j k_i) - \frac{1}{4} g_{ij} \quad (128)$$

$$e_1 = e_3 = 0 \Rightarrow S_{ij} = e_0 \widetilde{M}_{ij}, e_0 = q^2$$

with

$$R_{ijkl} = q^2 [g_{jl} \widetilde{M}_{ik} - g_{jk} \widetilde{M}_{il} + g_{ik} \widetilde{M}_{jl} - g_{il} \widetilde{M}_{kj}] + \frac{q^2 (c_0^2 - 1)}{c_0^2} F_{ij} F_{kl} \quad (129)$$

$$R_{ij} = q^2 \left[ (D-3) + \frac{1}{c_0^2} \right] \widetilde{M}_{ij} + \frac{1}{2q^2} \left[ (D-3) - \frac{1}{c_0^2} \right] g_{ij} \quad (130)$$

$$G_{ij} = q^2 \left[ (D-3) + \frac{1}{c_0^2} \right] \widetilde{M}_{ij} + \frac{1}{2q^2} \left[ \frac{1}{c_0^2} - (D-3)^2 \right] g_{ij}. \quad (131)$$

It is easily seen that

$$\nabla_{\mathbf{l}} F_{ij} = 0, \quad \nabla_{\mathbf{m}} R_{ijkl} = 0. \quad (132)$$

Here the origin of the antisymmetric tensor  $F_{ij}$  is geometrical. It is the volume form of the two-dimensional part  $ds^2 = -A^2 dt^2 + B^2 dr^2$  of the  $D$ -dimensional spacetime line element (1). We can define the electromagnetic tensor field,  $F_{ij}^{el}$ , simply as  $F_{ij}^{el} = QF_{ij}$ . Here  $Q$  is a constant (in asymptotically flat cases  $Q$  plays the role of electric charge). Equation (131) becomes

$$G_{ij} = \frac{q^2}{Q^2} \left[ (D-3) + \frac{1}{c_0^2} \right] M_{ij}^{el} + \frac{1}{2q^2} \left[ \frac{1}{c_0^2} - (D-3)^2 \right] g_{ij} \quad (133)$$

where  $M_{ij}^{el}$  ( $= Q^2 \widetilde{M}_{ij}$ ) is the energy-momentum tensor of the electromagnetic field tensor  $F_{ij}^{el}$ .

In  $D$  dimensions the LCBR metric is the solution of the Einstein-Maxwell equations with a cosmological constant. This constant vanishes when  $(D-3) = \frac{1}{c_0^2}$ . In this case spacetime is conformally flat only when  $D = 4$ . For this metric, the field equations of a most general Lagrangian (for instance, the Lagrangian of the low-energy limit of the string theory at all orders in string tension parameter) yields

$$E_{ij} = \sigma_2 g_{ij} + \sigma_1 M_{ij} = G_{ij}, \quad (134)$$

$$E_i = \sigma k_i, \quad (135)$$

$$E = \sigma^0, \quad (136)$$

where  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma$  and  $\sigma^0$  are constants. This means that, Einstein's equations reduce to algebraic equations. The Lagrangian of the most general theory is a scalar containing the curvature tensor, metric tensor, matter fields and their derivatives, contractions and multiple products of all orders. According to the theorem 2 given above all second-rank symmetric tensors constructed out of these are expressible in terms of  $g_{ij}$ ,  $M_{ij}$  and  $k_i k_j$ . So whatever the theory is we have three equations (under our assumptions all the terms containing  $k_i$  drop out) for five constants. Two constants ( $c_0, q$ ) come from the definition of the  $D$ -dimensional LCBR metric (126). One constant  $Q$  comes from the definition of the electromagnetic field tensor  $F_{ij}^{el}$ . The other two constants are the constant dilaton field  $\phi_0$  and the cosmological constant  $\lambda$ . In some cases  $\lambda$  is set to zero. In these cases as well, the number of equations are less than the number of unknown constants. In the general case, under our assumptions the field equations reduce to the following algebraic equations for  $q, c_0, Q, \phi_0$  and  $\lambda$ :

$$\frac{q^2}{Q^2} \left( D - 3 + \frac{1}{c_0^2} \right) = \sigma_2(q, c_0, Q, \phi_0, \lambda) \quad (137)$$

$$\frac{1}{2q^2} \left( -(D-3)^2 + \frac{1}{c_0^2} \right) = \sigma_4(q, c_0, Q, \phi_0, \lambda) \quad (138)$$

$$0 = \sigma(q, c_0, Q, \phi_0, \lambda) \quad (139)$$

$$0 = \sigma'(q, c_0, Q, \phi_0, \lambda) \quad (140)$$

where  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma$  and  $\sigma^0$  are defined in (134)–(136). Choosing  $\phi$  ( $= \phi_0$ ) and the metric function  $C$  ( $= c_0$ ) as constants the function  $\sigma$  given above vanishes identically. Hence we have three



algebraic equations for  $q, c_0, Q, \phi_0$  and  $\lambda$ . These algebraic equations may or may not have solutions. For instance, when we consider the Lovelock theory [26] without a cosmological constant these equations have solutions when  $D = 4$  and  $D > 6$ . With a cosmological constant these algebraic equations have solutions for  $D \geq 4$ . In the case of string theory the LCBR metric is a solution of the low-energy limit of the string theory. LCBR spacetime in four dimensions is also known to be an exact solution of the string theory [8,21]. Here we have the generalization of this result for an arbitrary value of  $D > 2$ . If there exists a solution of the above-mentioned algebraic equations then

*Theorem 5.* The LCBR metric is an exact solution of the string theory at all orders.

As in the case of the LCBR spacetime [8] the symmetric spaces,  $\nabla_i R_{jklm} = 0$  and recurrent spaces,  $\nabla_i R_{jklm} = l_i R_{jklm}$  where  $l_i$  is a (gradient) vector, play an effective role in constructing exact solutions of the string theory. These spaces are, in general, product spaces [38]. In the case of static spherically symmetric spacetimes the only possibility to have such spaces is to choose  $C = c_0$ . Then  $D$ -dimensional spacetime becomes  $M_2 \times S^{D-2}$ , where  $M_2$  is a two-dimensional geometry with metric  $ds^2 = -A^2 dt^2 + B^2 dr^2$ .

If we take  $C = c_0$  in the metric (1), then

$$\begin{aligned} \eta_0 &= \frac{c_0^{2(D-2)}}{(D-3)!} \frac{1}{AB} \left( \frac{A'}{B} \right)' & \eta_1 &= 0 \\ \eta_2 &= \xi - (D-3)\eta_0 & \eta_3 &= 0 \end{aligned} \tag{141}$$

$$\xi = \frac{c_0^{2(D-3)}}{(D-4)!}:$$

where

$$\begin{aligned} R_{ij} &= \frac{(D-3)!}{2c_0^{2(D-2)}} [ \eta_0 + \xi ] g_{ij} + [ \eta_0 + \xi ] M_{ij} \\ R &= \frac{(D-3)!}{c_0^{2(D-2)}} [ -2\eta_0 + (D-2)\xi ] \\ G_{ij} &= -\frac{(D-3)!}{2c_0^{2(D-2)}} [ -\eta_0 + (D-3)\xi ] g_{ij} + [ \eta_0 + \xi ] M_{ij} \\ \nabla_l H_{ij\dots m} &= 0 \\ \nabla_i k_j &= \frac{A'}{2AB^2} g_{ij} + \frac{c_0^{2(D-2)}}{(D-3)!} \frac{A'}{AB^2} M_{ij} - \frac{(AB)'}{AB} k_i k_j. \end{aligned} \tag{142}$$

If we let  $\eta_0$  be a constant, then we have to solve the equation  $\frac{1}{AB} \left( \frac{A'}{B} \right)' = \alpha$ . The solution is

$$B^2 = \frac{A'^2}{\alpha A^2 + \beta} \tag{144}$$

where  $\alpha$  and  $\beta$  are constants. Hence the metric (1) can be written as

$$ds^2 = -A^2 dt^2 + B^2 dr^2$$

$$ds = -A dt + \alpha A \dots^2 + \beta + c_0 d \bullet_{D-2}. \tag{145}$$

The case  $\beta = 0$  is identical with the LCBR metric. As in the case of the LCBR metric the metric constants  $\alpha$ ,  $c_0$  and the magnetic charge satisfy three coupled (due to the theorems given before) algebraic equations. If these algebraic equations have solutions then the metric given above with nonvanishing  $\alpha$  and  $\beta$  is a candidate as an exact solution of the string theory.

Above we have given the Riemann tensor in terms of the volume form of  $S_{D-2}(D-2)$ brane field. We may also write the Riemann tensor in the form

$$R_{ijml} = g_{jl}S_{im} - g_{im}S_{jl} + g_{im}S_{jl} - g_{il}S_{mj} + e_2 F_{ij}F_{ml}. \tag{146}$$

In this case

$$\begin{aligned} e_0 &= c_0^2 e_1 = 0 \quad e_2 = -c_0^2 + c_0^2 \zeta \\ e_3 &= 0 \end{aligned} \tag{147}$$

$$\zeta = \frac{c_0^2}{AB} \left( \frac{A'}{B} \right)'$$

where  $R_{ij} = \frac{D-3-\zeta}{2c_0^2} g_{ij} + (D-3+\zeta)c^2 \widetilde{M}_{ij}$

$$\begin{aligned} \nabla_l F_{ij} &= 0 \\ \nabla_i k_j &= \frac{A'}{2AB^2} g_{ij} + c_0^4 \frac{A'}{AB^2} \widetilde{M}_{ij} - \frac{(AB)'}{AB_0} k_i k_j \\ R &= \frac{D^2 - 5D + 6 - 2\zeta}{c_0^2} \\ G_{ij} &= \frac{\zeta - (D-3)^2}{2c_0^2} g_{ij} + (D-3+\zeta)c^2 \widetilde{M}_{ij} \end{aligned} \tag{148}$$

$$\tag{149}$$

The solution given in (144) is not valid when A is set to a constant. In such a case B can be chosen as unity. Therefore letting  $A = B = 1$  and  $C = c_0$ . We find that  $e_0 = -e_2 = c_0^2$  and

$$\nabla_l F_{ij} = 0, \quad \nabla_m R_{ijkl} = 0 \tag{150}$$

$$\nabla_i k_j = 0, \quad \nabla_l i_t j = 0 \tag{151}$$

$$R_{ij} = (D-3)c_0^2 \left[ \widetilde{M}_{ij} + \frac{1}{2c_0^2} g_{ij} \right] \tag{152}$$

$$R = \frac{(D-3)(D-2)}{c_0^2} \tag{153}$$

$$G_{ij} = (D-3)c_0^2 \left[ \tilde{M}_{ij} + \frac{1}{2c_0^2} (3-D)g_{ij} \right] \quad (154)$$

Since the metric contains only one arbitrary constant  $c_0$  it may not be possible to eliminate the coefficient of  $g_{ij}$  (the cosmological constant) in the field equations. Hence the metric given by

$$ds^2 = -dt^2 + dr^2 + c_0^2 d\Omega_{D-2}^2 \quad (155)$$

is a candidate of an exact solution of the string equations with a cosmological constant.

## 6. Conclusion

We have found all possible asymptotically flat solutions of the Einstein–Maxwell dilaton field theory in D-dimensional static spherically symmetric spacetimes with arbitrary dilaton coupling constant. We proved that the asymptotically flat black-hole solutions carrying mass M, electric charge Q and dilaton charge  $\phi$  in four dimensions are unique not only in static spherically symmetric spacetimes but also in static axially symmetric spacetimes. This proof may be extended to arbitrary dimensions and also to spacetimes which are not asymptotically flat [24]. We searched for all possible static spherically spacetimes which may solve the string equations exactly (to all orders in string tension parameter).

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