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Variable coefficient third order Korteweg–de Vries type of equations

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It is shown that the integrable subclasses of the equations $q_{,t}=f(x,t)q_{,3}+H(x,t,q,q_{,1})$ are the same as the integrable subclasses of the equations $q_{,t}=q_{,3}+F(q,q_{,1})$. © *1995 American Institute of Physics.*

Classification of nonlinear partial differential equations possessing infinitely many symmetries in $1+1$ dimensions was started almost two decades ago. So far the complete classification has been done for some evolution types of autonomous equations.[1–5] There are some partial attempts of the classification of the nonautonomous types of equations.[2,6–9] In $1+1$ or $2+0$ dimensions almost all definitions of integrability coincide. But what is important is the ease of applicability. Recently we have introduced a new approach which is based on the compatibility of the symmetry equation (linearized equation) and an eigenvalue equation.[10] Our method can be put into an algorithmic scheme and utilized for two purposes. The first is to test whether a given partial differential equation is integrable. The second is to classify nonlinear partial differential equations according to whether they admit generalized symmetries.

In this work we show that the most general equations of the type $q_{,t}=f(x,t)q_{,3}+H(x,t,q,q_{,1})$, up to coordinate transformations, have the same integrable subclass as the autonomous equations $q_{,t}=q_{,3}+F(q,q_{,1})$. Here $f(x,t)$ is an analytic function of the independent variables x and t, H is a function of the dependent variable q, its x-derivative $q_{,1}$, and also on the independent variables x and t. The function F depends on only q and $q_{,1}$. First we will give an outline of the method.

Consider an evolution equation of the form

$$q_{,t}=K(x,t,q,q_{,1},q_{,2},\ldots,q_{,n})\equiv K(q), \tag{1}$$

where $q_{,i}=(\partial/\partial x)^i q$, $i=0,1,2,\ldots,n$. The order of $K(=n)$ is called the order of the equation. A symmetry $\sigma(x,t,q)$ of Eq. (1) satisfies

$$\sigma_{,t}=K'(\sigma)=\sum_{i=0}^{n}\frac{\partial K}{\partial q_{,i}}\sigma_{,i} \tag{2}$$

such that Eq. (1) is form invariant under the transformation

$$q\to q+\epsilon\sigma, \quad (\epsilon,\ \text{infinitesimal}). \tag{3}$$

Here $\sigma(x,t,q)$ is a differentiable function of $q,q_{,1},q_{,2},\ldots$ and the prime denotes the Fréchet derivative.

In Ref. 10 we conjectured that a nonlinear partial differential equation is integrable if the linearized equation (2) supports an eigenvalue equation. Therefore let us introduce an eigenvalue equation, linear in λ, for σ in the form

$$\sigma_{,n}=\sum_{i=0}^{n-1}(A_i\lambda+B_i)\sigma_{,i}, \tag{4}$$

where A_i and B_i are functions of x, t, and $q_{,i}$. Their dependences on $q_{,i}$ are decided by the order of K. The order of the eigenvalue equation is determined by the order of K. The compatibility of linearized and eigenvalue equations, at all powers of λ, gives

(a) a set of algebraic equations among A_i, B_i, and $\partial K/\partial q_{,i}$'s;

(b) a set of coupled partial differential equations (PDEs) among A_i, B_i, and $\partial K/\partial q_{,i}$'s. Using the definition of total derivatives

$$\frac{df}{dx}=D_xf=\frac{\partial f}{\partial x}+\sum_{i=0}^{\infty}q_{,i+1}\frac{\partial f}{\partial q_{,i}}, \tag{5}$$

$$\frac{df}{dt}=D_tf=\frac{\partial f}{\partial t}+\sum_{i=0}^{\infty}K_{,i}\frac{\partial f}{\partial q_{,i}} \tag{6}$$

for any function f in the set b of coupled PDEs and comparing coefficients of $q_{,i}$'s, we obtain several classes of A_i, B_i along with the explicit forms of K in a self-consistent way. If the integrability is proved for a given class, the eigenvalue equation (4) can always be put in the form

$$M\sigma=\lambda N\sigma, \tag{7}$$

where M and N are local operators and depend on $x,t,q_{,i}$. Equation (7) is nothing but the definition of the recursion operator, provided that N^{-1} exists

$$R=N^{-1}M, \tag{8}$$

which maps symmetries to symmetries

$$R\sigma_n=\sigma_{n+1}, \tag{9}$$

where n is a non-negative integer. Thus the existence of an eigenvalue equation (4) is equivalent to the existence of a recursion operator.

As an illustration let us give the classification of third order autonomous evolution equations of the form

$$q_{,t}=q_{,3}+F(q,q_{,1}). \tag{10}$$

This classification has been investigated by several authors, mainly from the point of view of their integrability.[1,2,4,11] Let us follow the method outline above.

Linearized equation:

$$\sigma_{,t}=\sigma_{,3}+\frac{\partial F}{\partial q_{,1}}\sigma_{,1}+\frac{\partial F}{\partial q}\sigma. \tag{11}$$

Eigenvalue equation:

$$\sigma_{,3}=(A_2\lambda+B_2)\sigma_{,2}+(A_1\lambda+B_1)\sigma_{,1}+(A_0\lambda+B_0)\sigma, \tag{12}$$

where A_i and B_i depend on q, $q_{,1}$, and $q_{,2}$. The compatibility equation of Eqs. (11) and (12) gives the following integrable equations with nonzero eigenvalue coefficients:

$$\text{Case I.} \quad q_{,t}=q_{,3}+\frac{a}{6}\,q_{,1}^3+\frac{b}{2}\,q_{,1}^2+cq_{,1}+d, \tag{13}$$

with

$$B_2=\frac{aq_{,2}}{aq_{,1}+b}, \quad B_1=-\frac{1}{3}\,[aq_{,1}^2+2bq_{,1}+2c], \quad A_1=1,$$
$$B_0=\frac{1}{3(aq_{,1}+b)}\,[q_{,2}(2ac-b^2)], \quad A_0=-\frac{aq_{,2}}{aq_{,1}+b}. \tag{14}$$

Here a, b, c, and d are constants, and

$$\text{Case II.} \quad q_{,t}=q_{,3}+\frac{a}{6}\,q_{,1}^3+b(q)q_{,1}, \quad 3\,\frac{d^3b}{dq^3}+4a\,\frac{db}{dq}=0, \tag{15}$$

with

$$B_0=\frac{1}{3}\left[2\,\frac{bq_{,2}}{q_{,1}}-3\,\frac{db}{dq}\,q_{,1}\right], \quad B_1=-\frac{1}{3}\,(aq_{,1}^2+2b), \quad B_2=\frac{q_{,2}}{q_{,1}},$$
$$A_0=-B_2, \quad A_1=1. \tag{16}$$

Here a is a constant. The basic equations in the classification are the Korteweg–de Vries (KdV) ($a=0$, $b=6q$, in case II), potential Korteweg–de Vries (pKdV) ($a=0$, in case I), modified Korteweg–de Vries (mKdV) ($a=0$, $b=6q^2$, in case II), potential modified Korteweg–de Vries (pmKdV) ($b=0$, in both cases), and Callegero–Degasperis–Fokas (CDF) equation ($a=-\frac{3}{4}$, in case II). The recursion operators for Eqs. (13) and (15) are found by the utility (14) and (16). They are, respectively, given by

$$\text{I.} \quad R=D^2+\frac{2c}{3}+\frac{aq_{,1}^2}{3}+\frac{2bq_{,1}}{3}-\frac{aq_{,1}}{3}\,D^{-1}(q_{,2}\cdot)-\frac{b}{3}\,D^{-1}(q_{,2}\cdot), \tag{17}$$

$$\text{II.} \quad R=D^2+\frac{aq_{,1}^2}{3}+\frac{2b}{3}-\frac{aq_{,1}}{3}\,D^{-1}(q_{,2}\cdot)+\frac{q_{,1}}{3}\,D^{-1}\left(\frac{db}{dq}\cdot\right), \tag{18}$$

where $D^{-1}=\int_{-\infty}^{x}.dx$. We have the following proposition.

Proposition 1:[1–4] Under the symmetry classification the equations of the type $q_{,t}=q_{,3}+F(q,q_{,1})$, up to coordinate transformations, has integrable subclasses given in Eqs. (13) and (15).

Our aim, in this work, is to give a classification of the nonautonomous type of integrable equations (10)

$$q_{,t}=f(x,t)q_{,3}+H(x,t,q,q_{,1}). \tag{19}$$

We divide the classification procedure, for Eq. (19), into the following three cases: (i) f depends only on t, (ii) f depends only on x, and (iii) f depends on both x and t.

(i) f depends only on t: One of the integrable subclass, using our classification scheme, is

$$q_{,t}=v^3q_{,3}+\left(\frac{hw^2q^2}{2}+c_1whvq+hc_2v\right)q_{,1}+\left(\frac{\dot{h}}{2h}-\frac{\dot{v}}{v}\right)xq_{,1}-\frac{\dot{w}q}{w}, \tag{20}$$

where $f(t)=v(t)^3$, h, w depend on t only and c_1, c_2 are constants. The dot appearing over a quantity denotes t derivative. The recursion operator is given by

$$R=\frac{v^2}{h}D^2+\frac{w^2}{3}q^2+\frac{2c_1}{3}q+\frac{2}{3}c_2+\frac{w^2}{3}q_{,1}D^{-1}(q.)+\frac{c_1w}{3}q_{,1}D^{-1}. \tag{21}$$

We observe that Eq. (20) is transformed into an equation which belongs to the case II ($a=0$) in Eq. (15) through the transformation

$$q=w^{-1}u(\xi,\tau),$$
$$\xi=x\beta(t),\quad \beta=\frac{h^{1/2}}{v},\quad \tau=\int^t h^{3/2}\,dt'. \tag{22}$$

In the classification programs, if it is possible, we transform (by coordinate or contact transformations) the given class of PDEs to more simpler ones. To this end in the sequel we shall transform all cases (i), (ii), (iii) to the form

$$q_{,\tau}=q_{,3}+H_2(\xi,\tau,q,q_{,1}) \tag{23}$$

and then classify this type of equation. In this first case (i) we have the following proposition.

Proposition 2: Under the symmetry classification equations of the type

$$q_{,t}=f(t)q_{,3}+F_1(x,t,q,q_{,1}) \tag{24}$$

up to coordinate transformations, give the same integrable subclass as in Eq. (23). Equation (24) reduces to Eq. (23) by the transformation $dt=(1/f)d\tau$ and $x=\xi$.

(ii) f depends only on x: In this case the form of the equation is

$$q_{,t}=f(x)q_{,3}+F_2(x,t,q,q_{,1}) \tag{25}$$

and one integrable class turns out to be simply

$$q_{,t}=q_{,3}+q_{,1}^2+c_1x+c_2. \tag{26}$$

The recursion operator for Eq. (26) is

$$R=D^2+\tfrac{4}{3}q_{,1}-\tfrac{4}{3}c_1t-\tfrac{2}{3}D^{-1}(q_{,2}.). \tag{27}$$

Now, differentiate Eq. (26) with respect to x and substitute $q=z_{,1}$ and use the transformations $x=\xi-\frac{1}{2}c_1t^2$, $t=\tau$ then Eq. (26) belongs to Eq. (13). Before proceeding to the next case, we observe the following:

Proposition 3: Under the symmetry classification the equations of the type (25) up to coordinate transformation

$$q=f^{1/3}u(\xi,\tau),$$
$$\xi=\int^x\frac{1}{f^{1/3}}\,dx',\quad \tau=t \tag{28}$$

give the same integrable subclass as in Eq. (23).

(iii) f depends on both x and t: In this case we have $q_{,t}=f(x,t)q_{,3}+F_3(x,t,q,q_{,1})$ type of equation and its one integrable class turns out to be relatively simple. Below we give this equation and its recursion operator as

$$q_{,t}=u^3q_{,3}+\left[\frac{a}{2u}q^2+\frac{3}{2}u(u_{,1}^2-2uu_{,2})\right]\left(q_{,1}-\frac{u_{,1}}{u}q\right), \tag{29}$$

where a is an arbitrary constant. Here we have set $f(x,t)=u(x,t)^3$ and $u(x,t)$ satisfies the Harry Dym equation

$$u_{,t}=u^3u_{,3}. \tag{30}$$

This means that Eq. (29) is integrable if u satisfies Eq. (30). The recursion operator is given by

$$R=usD^{-1}uD\left(\frac{1}{s}R_1\cdot\right), \tag{31}$$

where

$$\begin{aligned} R_1&=D^2-\frac{u_{,1}q_{,2}}{us}+\frac{u_{,2}q_{,1}}{us}-\frac{u_{,2}}{u}+\frac{aq^2}{3u^4}+sD^{-1}\left(\frac{V}{s}\cdot\right),\\ V&=\frac{a}{3u^4}qq_{,1}+\frac{1}{us}\left[u_{,1}q_{,3}-q_{,1}u_{,3}+\frac{2s_{,1}}{s}(-u_{,1}q_{,3}+q_{,1}u_{,3})\right] \end{aligned} \tag{32}$$

and $s=-q_{,1}+q(u_{,1}/u)$. Equation (29) together with Eq. (30) is equivalent to the mKdV. We will give the proof of this in two steps: Let $f=u^3$ and $q=uz$ then we have

$$z_{,t}=u^3z_{,3}+3u^2u_{,1}z_{,2}+\left(\frac{a}{2}uz^2+\frac{3}{2}uu_{,1}^2\right)z_{,1}, \tag{33}$$

where $z(x,t)$ is the new dynamical variable. Now let us perform the following transformation:

$$\xi=\int^x\frac{dx'}{u(x',t)},\qquad \tau=t. \tag{34}$$

It is straightforward to show that under this transformation Eq. (33) goes to the mKdV. Now we state the following proposition.

Proposition 4: Under the symmetry classification the equations of the type $q_{,t}=f(x,t)q_{,3}+F_3(x,t,q,q_{,1})$, up to the transformations

$$\begin{aligned} q&=f^{1/3}z(\xi,\tau),\\ \xi&=\int^x\frac{1}{f^{1/3}}dx',\qquad \tau=t, \end{aligned} \tag{35}$$

like in the previous example, give the same integrable subclass [for $z(\xi,\tau)$] as in Eq. (23).

Hence whatever the coefficient function $f(x,t)$ we showed that in general the type of equation (19), by coordinate transformations, reduces to the following type of equations:

$$q_{,t}=q_{,3}+P(x,t,q,q_{,1}). \tag{36}$$

We now give the classification of this type of equations.

$$(1)\qquad q_{,t}=q_{,3}+\frac{a}{2}q_{,1}^2+bq_{,1}-\frac{\dot w}{w}q+c, \tag{37}$$

with

$$\dot{b}=b_{,3}+bb_{,1}-ac_{,1}+\frac{\dot{h}}{2h}b+\frac{\dot{d}}{2h}-\frac{\dot{h}}{h^2}d+\left(\frac{\ddot{h}}{2h}-\frac{\dot{h}^2}{h^2}\right)x, \tag{38}$$

where $w=a/\sqrt{h},a,d,h$ depend on t only and b, c depend on x, t.

$$(2)\quad q_{,t}=q_{,3}-\frac{a^2}{8}q_{,1}^3+\left(\frac{h}{w}e^{aq}-whe^{-aq}+b\right)q_{,1}+\frac{\dot{h}}{2h}xq_{,1}-\frac{\dot{a}}{a}q+\frac{\dot{w}}{aw}, \tag{39}$$

where a, w, and b depend on t only.

$$(3)\quad q_{,t}=q_{,3}+\frac{a}{2}q^2q_{,1}+\left(\frac{b}{w}\right)^{1/2}qq_{,1}+\left(\frac{\dot{h}}{2h}x+c\right)q_{,1}+\frac{\dot{w}}{2w}q-\frac{h}{2}\left(\frac{w}{b}\right)^{1/2}\dot{d}, \tag{40}$$

where $b=h^2d$, and all parameters appearing in the equation depend on t.

$$(4)\quad q_{,t}=q_{,3}+\frac{a}{2}q^2q_{,1}+bqq_{,1}-\frac{1}{2}\left(-\frac{\dot{h}}{h}x-\frac{b^2}{a}+\frac{c}{ah}\right)q_{,1}+\frac{b_{,1}}{2}q^2-\frac{1}{2}\left(\frac{\dot{w}}{w}-\frac{2b}{a}b_{,1}\right)q$$

$$-\frac{1}{2a^2}\left(-\dot{a}b-2ab_{,3}-\frac{\dot{h}}{h}b_{,1}ax-b^2b_{,1}+\frac{c}{h}b_{,1}+2\dot{b}a-\frac{\dot{h}}{h}ab\right), \tag{41}$$

where $w=a/h$, a, h,c, depend on t only and b depends on x, t.

$$(5)\quad q_{,t}=q_{,3}+aqq_{,1}+bq_{,1}-\frac{1}{2}\left(\frac{\dot{w}}{w}-2b_{,1}\right)q-\frac{1}{2a}$$

$$\times\left(-2b_{,3}-2bb_{,1}+2\dot{b}-\frac{\dot{c}}{h}-\frac{\ddot{h}}{h}x+2\frac{\dot{h}^2}{h^2}x-\frac{\dot{h}}{h}b+2\frac{\dot{h}}{h^2}c\right), \tag{42}$$

where $w=a^2/h$, a depend on t only and b depends on x, t.

$$(6)\quad q_{,t}=q_{,3}+\frac{a}{6}q_{,1}^3+\frac{ab}{2}q_{,1}^2+\frac{1}{2}\left(\frac{\dot{h}}{h}x+b^2a-\frac{c}{h}\right)q_{,1}-\frac{1}{2}\left(\frac{\dot{a}}{a}q-2d\right), \tag{43}$$

with

$$d_{,1}=\frac{1}{2}\left[2b_{,3}+b_{,1}\left(\frac{\dot{h}}{h}x+ab^2-\frac{c}{h}\right)-2\dot{b}-\frac{b\dot{a}}{a}+\frac{b\dot{h}}{h}\right], \tag{44}$$

where b, d depend on t, x and a, c and h depend on t only.

$$(7)\quad q_{,t}=q_{,3}-\frac{a^2}{8}q_{,1}^3-\frac{3ab_{,1}}{8b}q_{,1}^2+\left(\frac{b}{a}e^{aq}-\frac{ah^2f}{b}e^{-aq}+\frac{\dot{h}}{2h}x+\frac{c}{2h}-\frac{hd}{2}-\frac{3b_{,1}^2}{8b^2}\right)q_{,1}-\frac{\dot{a}}{a}q$$

$$+\left(\frac{b_{,1}}{a^2}e^{aq}-\frac{h^2fb_{,1}}{b^2}e^{-aq}+\frac{\dot{h}b_{,1}}{2abh}x+\frac{b_{,3}}{ab}-\frac{3b_{,1}b_{,2}}{ab^2}+\frac{15b_{,1}^3}{8ab^3}+\frac{cb_{,1}}{2abh}-\frac{dhb_{,1}}{2ab}+\frac{\dot{w}}{aw}\right), \tag{45}$$

where $w=ah/b$, a,c depend on t only, b depends on x,t, and d,f are constants. All these classes are transformable to those given in Eqs. (13) and (15). For this purpose, we first perform the following transformation:

$$q=\alpha(t)z(x,t)+\beta(x,t), \tag{46}$$

where α and β are arbitrary functions, and $z(x,t)$ is the new dynamical variable. By choosing α and β properly we eliminate arbitrary functions of x and t appearing in these equations. Secondly, if the resultant equations contain further arbitrary functions depending upon t, we perform the transformation of the following type:

$$z=v(x,t)+s_0x^2+s_1x+s_2 \tag{47}$$

to eliminate such arbitrary functions as the products of x^2 and x in these equations. Here s_0, s_1, and s_2 depend only on t and $v(x,t)$ is now the new dynamical variable. At this point we transform dependent and independent variables according to

$$\begin{aligned} v(x,t)&=\mu(t)u(\xi,\tau),\\ \xi=x\rho(t)+\gamma(t), &\quad \tau=\nu(t), \end{aligned} \tag{48}$$

which reduces the classes (1)–(7) to one of the type given in Eqs. (13) and (15) exactly. As an example Eq. (41) is transformed into the Eq. (13) through the following transformations:

$$\begin{aligned} q&=\frac{h^{1/2}}{a}\,u(\xi,\tau)-\frac{b}{a},\\ \xi=xh^{1/2}+\gamma(t), &\quad \tau=\int^t h^{3/2}\,dt', \end{aligned} \tag{49}$$

where

$$\gamma=-\int^t\left(\frac{c}{2ah^{1/2}}+h^{2/3}\right)dt'. \tag{50}$$

Finally we have the following proposition.

Proposition 5: Under the symmetry classification the integrable subclass of the type of equations (36) is, up to coordinate transformations, equivalent to Eqs. (13) and (15).

In conclusion this work shows that there are no generic integrable nonautonomous type of equation (19). Any integrable PDE (admitting infinitely many generalized symmetries) containing explicit (x,t) dependencies of the form (19) is transformable into Eq. (10).

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