

J matrix whose entries are all zero. \mathbf{I}_N and \mathbf{J}_N are the $N \times N$ identity and opposite identity matrices, respectively. \mathbf{I}'_N denotes an $N \times N$ diagonal matrix defined as

$$\mathbf{I}'_N = \text{diag}[\sqrt{2}, 1, \dots, 1]$$

Proposed cosine-modulated FIR filter banks: To describe the proposed CMFBs with $2N$ sub-channels, two cases for N , odd or even, are considered separately. We also assume that the length of the prototype filter $h(n)$ is $L = 2NM$, and the type 1 polyphase components of $H(z)$ are $H_q(z)$ ($q = 0, 1, \dots, 2N - 1$). The CMFBs are defined as

(i) for N even ($N = 2m$)

$$\begin{cases} a_k(n) = c_k h(n) \cos\left(\frac{\pi k(n-m+0.5)}{N}\right) & k = 0, 1, \dots, N-1 \\ a_{k+N-1}(n) = c_k h(n-N) \sin\left(\frac{\pi k(n-m+0.5)}{N}\right) & k = 1, 2, \dots, N \end{cases} \quad (1)$$

(ii) for N odd ($N = 2m + 1$)

$$\begin{cases} a_k(n) = c_k h(n) \cos\left(\frac{\pi k(n-m)}{N}\right) & k = 0, 1, \dots, N \\ a_{k+N-1}(n) = c_k h(n-N) \sin\left(\frac{\pi k(n-m)}{N}\right) & k = 1, 2, \dots, N-1 \end{cases} \quad (2)$$

where in eqns. 1 and 2

$$c_k = \begin{cases} \frac{\sqrt{2}}{\sqrt{N}} & \text{if } k \neq 0 \text{ and } k \neq N \\ \frac{1}{\sqrt{N}} & \text{if } k = 0 \text{ or } k = N \end{cases} \quad (3)$$

If $h(n)$ has linear phase, then it can be easily verified that all the modulated filters have linear phase. We assume that $h(n)$ is symmetric. By using the properties of cosine and sine function, the polyphase component matrix can be expressed as

(i) for N even

$$\mathbf{E}(z) = \begin{bmatrix} \mathbf{C}_N^T & \mathbf{C}_N^T \mathbf{J}_N \\ \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(2N-m) \times m} & \mathbf{I}_{2N-m} \\ \mathbf{I}_m & \mathbf{0}_{m \times (2N-m)} \end{bmatrix} \begin{bmatrix} \mathbf{E}_0(z) & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \mathbf{E}_1(z) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} \\ \mathbf{S}_N^T & -\mathbf{S}_N^T \mathbf{J}_N \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(2N-m) \times m} & \mathbf{I}_{2N-m} \\ \mathbf{I}_m & \mathbf{I}_{m \times (2N-m)} \end{bmatrix} \begin{bmatrix} z^{-1} \mathbf{E}_1(z) & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \mathbf{E}_0(z) \end{bmatrix} \quad (4)$$

(ii) for N odd

$$\mathbf{E}(z) = \begin{bmatrix} \mathbf{C}_{N+1}^T & \mathbf{C}_{N+1}^T \mathbf{J}_{N+1} \\ \mathbf{0}_{(N-1) \times (N+1)} & \mathbf{0}_{(N-1) \times (N+1)} \end{bmatrix} \begin{bmatrix} \mathbf{I}'_N & \mathbf{0}_{N \times N} \\ \mathbf{0}_{1 \times N} & \mathbf{0}_{1 \times N} \\ \mathbf{0}_{N \times N} & \mathbf{I}'_N \\ \mathbf{0}_{1 \times N} & \mathbf{0}_{1 \times N} \end{bmatrix} \times \begin{bmatrix} \mathbf{0}_{(2N-m) \times m} & \mathbf{I}_{2N-m} \\ \mathbf{I}_m & \mathbf{0}_{m \times (2N-m)} \end{bmatrix} \begin{bmatrix} \mathbf{E}_0(z) & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \mathbf{E}_1(z) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(N+1) \times 1} & \mathbf{0}_{(N+1) \times (N-1)} & \mathbf{0}_{(N+1) \times 1} & \mathbf{0}_{(N+1) \times (N-1)} \\ \mathbf{0}_{(N-1) \times 1} & \mathbf{S}_{N-1}^T & \mathbf{0}_{(N-1) \times 1} & -\mathbf{S}_{N-1}^T \mathbf{J}_{N-1} \end{bmatrix} \times \begin{bmatrix} \mathbf{0}_{(2N-m) \times m} & \mathbf{I}_{2N-m} \\ \mathbf{I}_m & \mathbf{0}_{m \times (2N-m)} \end{bmatrix} \begin{bmatrix} z^{-1} \mathbf{E}_1(z) & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \mathbf{E}_0(z) \end{bmatrix} \quad (5)$$

where $\mathbf{E}_0(z)$, $\mathbf{E}_1(z)$ are $N \times N$ diagonal matrices defined as

$$\mathbf{E}_0(z) = \text{diag}[H_0(z), H_1(z), \dots, H_{N-1}(z)] \quad (6)$$

$$\mathbf{E}_1(z) = \text{diag}[H_N(z), H_{N+1}(z), \dots, H_{2N-1}(z)] \quad (7)$$

From eqns. 4 and 5 the CMFBs defined in eqns. 1 and 2 can be efficiently implemented by using the DCT and DST. Based on these equations, and by using the symmetry of $h(n)$, it can be proved that $\mathbf{E}(z)$ is paraunitary if and only if

$$\tilde{\mathbf{E}}_0(z) \mathbf{E}_0(z) + \tilde{\mathbf{E}}_1(z) \mathbf{E}_1(z) = \mathbf{I}_N \quad (8)$$

This means that appropriate pairs of polyphase components of $H(z)$ are power complementary.

Design of prototype filter: As in [1], the design of the prototype filter can be performed via an unconstrained optimisation routine based on the two-channel lossless lattice. The cost function is as follows:

$$J = \int_{(\pi/2N+\delta)}^{\pi} |H(e^{j\omega})|^2 d\omega = \mathbf{b}^T \mathbf{W} \mathbf{b} \quad (9)$$

where $\delta < \pi/2N$, $\mathbf{b} = [2h(0), 2h(1), \dots, 2h(L/2 - 1)]^T$, and \mathbf{W} is a positive definite matrix whose entries are given by

$$[\mathbf{W}]_{ij} = \int_{(\pi/2N+\delta)}^{\pi} \cos(\omega((L-1)/2-i)) \cos(\omega((L-1)/2-j)) d\omega \quad (10)$$

In the optimisation procedure, $h(n)$ are calculated from m lattices whose parameters are to be optimised. As an example, an eight-channel CMFB was designed. Table 1 shows the impulse response coefficients of the prototype filter with the length $L = 24$.

Table 1: Impulse response coefficients of prototype filter ($L=24$) of an eight-channel cosine-modulated filter bank ($N=4$)

n	$h(n), h(23-n)$	n	$h(n), h(23-n)$
0	-8.4035×10^{-3}	6	4.3684×10^{-2}
1	-2.7719×10^{-3}	7	1.4325×10^{-1}
2	-2.8891×10^{-2}	8	2.1871×10^{-1}
3	-3.8018×10^{-2}	9	3.1264×10^{-1}
4	-3.1664×10^{-2}	10	3.8664×10^{-1}
5	-4.1911×10^{-3}	11	4.2324×10^{-1}

Conclusion: We have proposed a new type of cosine-modulated filter bank from which linear phase paraunitary cosine-modulated FIR filter banks can be constructed. When the prototype filter has linear phase, the necessary and sufficient condition for paraunitary is that appropriate pairs of polyphase components of the prototype filter are power complementary. Based on this condition, the prototype filter can be designed by optimising its stopband energy on the two-channel lattice parameters. Also, the filter banks can be implemented efficiently using DCT and DST.

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Frequency band characteristics of tree-structured filter banks

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Indexing terms: Signal processing, Digital filters, Gray codes

A sub-band decomposition filter bank can be recursively used in a tree structure to divide the frequency domain into various subfrequency bands. The frequency bands of the sub-band signals have a counter intuitive order in such a decomposition. The authors show that the relationship between the frequency content and the index of a sub-band signal can be expressed by an extension of the Gray code.

Introduction: Sub-band decomposition is widely used in signal analysis and coding [1 – 5]. A large order multi-channel filter bank can be constructed by cascading filter banks with smaller orders; e.g. the tree-structured 4 channel filter bank shown in Fig. 1a is constructed from 2-channel filter banks in two stages. A tree can be formed from binary (2-channel) or M -ary perfect reconstruction (PR) filter banks. The frequency contents of the resultant sub-band signals do not have a natural increasing order for tree structured sub-band decomposition. The counter intuitive ordering is not emphasised in most of the books on this subject. Also, some of the band partitioning methods in the literature [1 – 3] appear to be erroneous. In this Letter, the correct frequency order of the sub-band signals is described and an efficient method for calculating the order for a tree-structured sub-band filter bank with arbitrary number of stages is given.

Binary tree-structured sub-band filtering: In the 4-channel filter bank structure of Fig. 1a, $H_0(\omega)$ and $H_1(\omega)$ stand for the low pass and the high pass filters, respectively. An equivalent structure is shown in Fig. 1b. The two-stage operation in Fig. 1a is equivalent to a single stage operation consisting of filtering $x[n]$ by the convolution of filters on the signal path and down-sampling the filtered signal by $2^2 = 4$. The frequency response of the equivalent filter in the first branch is given by

$$F^0(\omega) = H_0(\omega)H_0(2\omega) \quad (1)$$

In general the frequency response of F^k in an N stage partition is given by

$$F^k(\omega) = H_{i_0}(\omega)H_{i_1}(2\omega)\dots H_{i_{N-1}}(2^{N-1}\omega) \quad (2)$$

$$k = 0, 1, \dots, N - 1$$

where $k = 2^{l-1}i_0 + 2^{l-2}i_1 + \dots + i_{N-1}$ and $i_j = 0, 1$.

In Figs. 1 and 2, the third branch filter $F^2(\omega)$ covers the frequency range $[3\pi/4, \pi]$ whereas the fourth branch filter $F^3(\omega)$ covers the frequency range $[\pi/2, 3\pi/4]$. The output signals $X_0(\omega) - X_3(\omega)$ split the frequency band into four regions, but the index ordering from the lowest to the highest frequency is : 0, 1, 3, 2. This sequence represents the Gray code (or minimum change code) ordering [6] where only one bit in the binary representation of the number changes between two consecutive numbers. This rule is also valid for higher levels of decomposition.

Consider the generation of binary Gray code in three digits. We start with a binary pair in a column $[0 \ 1]^T$. Next, we construct the second digit column by inserting zeros and ones as many times as the number of digits on the right column i.e. two zeros and two ones. After this stage, the rest of the right column is constructed by warping around the first elements i.e. the digits are written in reverse order. The left most column is now generated by inserting zeros and ones as many times as the number of digits in the right columns i.e. four zeros and four ones. In the last stage, the right columns are filled by periodically warping around the already filled columns

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \\ 6 \\ 7 \\ 5 \\ 4 \end{bmatrix} \quad (3)$$

Notice that the resultant sequence of numbers corresponds to the frequency regions of the sub-band signals (Fig. 2).

Arbitrary tree-structured sub-band filtering: A PR sub-band tree can have a different number of branches at each stage. In the case of trees with M -ary branches, the frequency contents of the sub-signals can be determined by an extension of the Gray code. In this case, the M -ary minimum change code can be constructed by changing only one digit in the M -ary representation of the number.

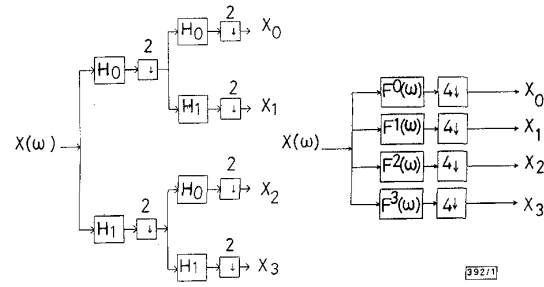


Fig. 1 Binary tree-structured 2 level sub-band decomposition

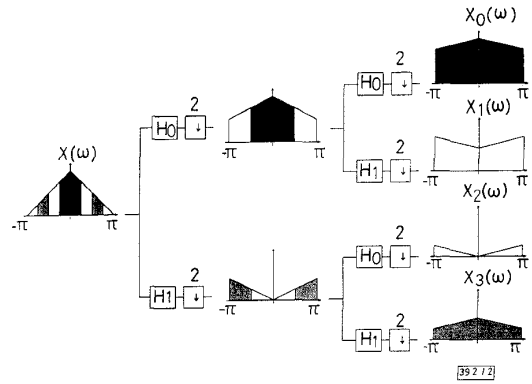


Fig. 2 4 channel decomposition of signal

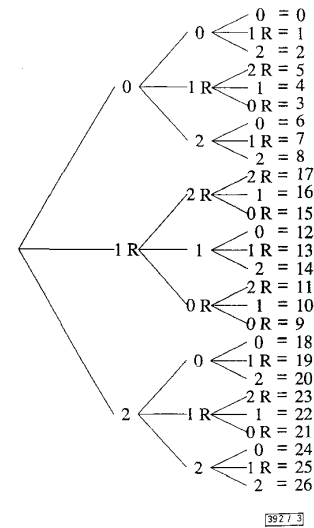


Fig. 3 Tree structure generation of ternary Gray code

In the construction of the minimum change codes, the sequence of the right digits in the M -ary representation gets periodically warped. This indicates that the ordering gets shuffled after two stages in the tree. In a tree-structured sub-band decomposition, the same frequency warping can be observed (Fig. 2).

In the most general case the hybrid minimum change codes should be used. These codes are constructed by changing only one digit of the hybrid M -ary representation of the number. For instance, if the first stage is binary and the second stage is ternary, then the new code is generated by one digit change of mod-2 in the most significant digit and one digit change of mod-3 in the least significant digit. This algorithm can be visualised in a tree-structured manner as in Fig. 3. In this Figure, the tree is generated by recursively attaching ternary branches. When a node is labelled with an 'R', this indicates that the labels of the branch emitted from that node are in reverse order. The final index is determined by tracing the digits, starting from the top to the bottom of the tree.

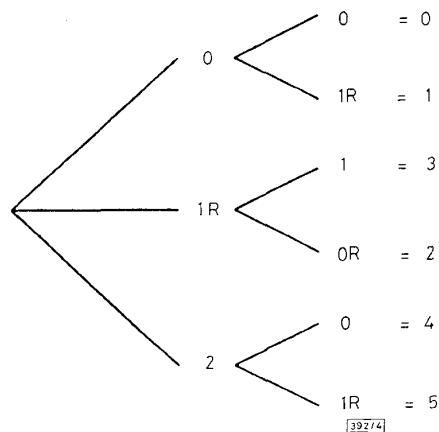


Fig. 4 Hybrid tree structure

If the sub-band tree is composed of layers of M -ary and N -ary branches, the codeword generation works in a similar way. Each digit for the codeword is in modulo- m for the layer corresponding to the M -ary branch and it is in modulo- n for the layer corresponding to the N -ary branch. Suppose the first branch is ternary, and it is followed by binary branches at each node. This time, the code generation is as follows:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & | & 0 \\ 0 & | & 1 \\ \hline 1 & | & \\ 1 & | & \\ \hline 2 & | & \\ 2 & | & \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & | & 0 \\ 0 & | & 1 \\ \hline 1 & | & 1 \\ 1 & | & 0 \\ \hline 2 & | & 0 \\ 2 & | & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix} \quad (4)$$

Visually, this corresponds to the tree in Fig. 4. When different number of branches exist at the same layer in the decomposition, the calculation of the frequency order is carried out separately for each branch.

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Least-squares multirate FIR filters

R. Manduchi and P. Perona

Indexing terms: FIR filters, Digital filters

The authors propose a new least-squares design procedure for multirate FIR filters with any desired shape of the (band-limited) frequency response. The aliasing, inherent in such systems, is implicitly taken into account in the approximation criterion.

Introduction: The multirate implementation of FIR filters (see Fig. 1), introduced by Rabiner and Crochiere [1], leads to reduced computational complexity. In fact, the samples at the output of the FIR filter $g(n)$ that are deleted by the M -fold sampler do not need to be computed, and the null-valued samples introduced by the M -fold interpolator do not contribute to the convolution operated by the FIR filter $h(n)$. Only the case of brick-wall frequency response was considered in [1], and the design technique was inspired by minimax criteria.

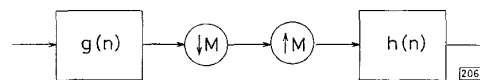


Fig. 1 Multirate implementation of filter

We propose a least-squares criterion for the design of multirate FIR filters, to approximate the spectral shape of any desired prototype $d(n)$ (assuming that the necessary band-limiting conditions are met i.e. that the spectral support of $d(n)$ has a length $< 2\pi/M$). The resulting system is linear periodically time-invariant (LPTV [2]), and it is characterised by the M impulse responses $\{t^{(i)}(n+i), 0 \leq i < M\}$, corresponding to the M inputs $\{\delta(n+i), 0 \leq i < M\}$. The fact that the impulse responses differ from each other is usually referred to as the aliasing effect. The least-squares criterion introduced in this Letter makes for the joint reduction of the approximation error and of the inherent aliasing.

Theory: We only consider the case $M = 2$ (definitions and results are extended straightforwardly to the case of higher M). Define the polyphase components [2] of $g(n)$ as

$$g_0(n) = g(2n) \quad g_1(n) = g(2n+1)$$

It is easily shown [2] that

$$t^{(0)}(n) = h * \bar{g}(n) \quad t^{(1)}(n+1) = h * \bar{g}_1(n)$$

where $\bar{g}_0(n)$ and $\bar{g}_1(n)$ are obtained by interleaving $g_0(n)$ and $g_1(n)$ with null-valued samples.

We propose the following design criterion: given the kernel $d(n)$ to be approximated, find the filters $g(n)$ and $h(n)$ with given length N_g and N_h respectively, which minimise the approximation error ϵ^2 , defined as

$$\epsilon^2 = \frac{\|t^{(0)}(n) - d(n)\|^2 + \|t^{(1)}(n) - d(n)\|^2}{2} \quad (1)$$

The term ϵ^2 implicitly accounts for both the approximation quality and the aliasing. In fact, if ϵ^2 is small, we may expect both impulse responses of the system to be 'close' to $d(n)$, and therefore 'close' to each other. More precisely, the following upper bound holds:

$$\|t^{(0)}(n) - t^{(1)}(n)\|^2 \leq 2(\epsilon^2 + \|t^{(0)}(n) - d(n)\| \|t^{(1)}(n) - d(n)\|)$$

No simple closed form solution can be found to the minimisation problem, since the error ϵ^2 in eqn. 1 is composed of quadratic forms of bilinear expressions in $g(n)$ and $h(n)$. A standard procedure in such cases is based on iterative minimisation [3]. Our iterative algorithm is briefly outlined in the remainder. Vectorial notation is used for sequences: a sequence $x(n)$ is represented by a column vector \mathbf{x} whose entries are the samples of $x(n)$. The symbol T stands for vector/matrix transposition. We start from an initial guess of $g_0(n)$ and $g_1(n)$, and then iterate through the following two steps:

Optimisation of $h(n)$ for fixed $g_0(n), g_1(n)$: Let $\bar{\mathbf{G}}_0$ and $\bar{\mathbf{G}}_1$ be the Toeplitz matrices representing the filtering with $\bar{g}_0(n)$ and $\bar{g}_1(n)$, respectively. Then

$$\epsilon^2 = \frac{(\bar{\mathbf{G}}_0 \mathbf{h} - \mathbf{d})^T (\bar{\mathbf{G}}_0 \mathbf{h} - \mathbf{d}) + (\bar{\mathbf{G}}_1 \mathbf{h} - \mathbf{d}_+)^T (\bar{\mathbf{G}}_1 \mathbf{h} - \mathbf{d}_+)}{2}$$

where \mathbf{d}_+ is the vector representing $d(n+1)$. Hence, ϵ^2 is minimised for

$$\mathbf{h} = (\bar{\mathbf{G}}_0^T \bar{\mathbf{G}}_0)^{-1} (\bar{\mathbf{G}}_0^T \mathbf{d} + \bar{\mathbf{G}}_1^T \mathbf{d}_+)$$

Optimisation of $g_0(n)$ and $g_1(n)$ for fixed $h(n)$: Let \mathbf{H} be the Toeplitz matrix representing the convolution with $h(n)$, and let \mathbf{U} be a matrix obtained by interleaving the rows of a suitably sized identity matrix with null-valued rows. Then