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On Possible Deterioration of Smoothness under the Operation of Convolution

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We give some sufficient conditions of deterioration of smoothness under the operation of convolution. We show that the convolution of two probability densities which are restrictions to $\mathbb{R}$ of entire functions can possess infinite essential supremum on each interval.

1. INTRODUCTION

It is known that, as a rule, the operation of convolution improves smoothness. This rule was mentioned by Paul Lévy in his book [1, p. 91]. In order to elaborate the domain of applicability of this rule, D. Raikov [2] constructed two probability densities p_1, p_2 on $\mathbb{R}$ which are restrictions to $\mathbb{R}$ of entire functions, but their convolution

$$p(x) = (p_1 * p_2)(x) = \int_{-\infty}^{\infty} p_1(x-s)\, p_2(s)\, ds, \qquad x \in \mathbb{R},$$

although infinitely differentiable, is not analytic everywhere on $\mathbb{R}$. We show that the deterioration of smoothness under convolution can be much greater than in Raikov's example. We prove this by a method different from Raikov's. Nevertheless, Raikov's method permits us to obtain some conditions of deterioration presented in this article.

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2. NOTATION

We shall adopt the following notation for some subsets of $L_1(\mathbb{R})$:

- L_1^+ is the set of all nonnegative functions on $\mathbb{R}$ belonging to $L_1(\mathbb{R})$ and not equivalent to 0;
- EL_1^+ is the set of all functions of L_1^+ which are restrictions to $\mathbb{R}$ of entire functions;
- $E^\infty L_1^+$ is the subset of EL_1^+ consisting of the restrictions to $\mathbb{R}$ of entire functions bounded in each strip $\{z: |\text{Im } z| \le r\}$, $r > 0$;
- $E_\rho L_1^+$ is the subset of EL_1^+ consisting of the functions which are restrictions to $\mathbb{R}$ of entire functions of order not exceeding ρ;
- UL_1^+ is the set of all functions $f \in L_1^+$ possessing the following property: for any nonempty interval $]\alpha, \beta[$, the equality

$$\operatorname*{ess\,sup}_{x \in]\alpha, \beta[} f(x) = \infty \tag{1}$$

is valid.

The set L_1^+ consists of functions equal to a probability density up to a positive constant factor. The sets $EL_1^+, E^\infty L_1^+, E_\rho L_1^+$ can be viewed as subsets of L_1^+ consisting of functions with "extremely good smoothness." The set UL_1^+ can be viewed as a subset of L_1^+ consisting of functions with "extremely bad smoothness."

We define the operators $S: L_1^+ \to L_1^+$ by the equality

$$(Sf)(x) = \int_{-\infty}^{\infty} f(x+t) f(t)\, dt, \qquad x \in \mathbb{R}. \tag{2}$$

We accept the agreement that S is defined by (2) *everywhere* on $\mathbb{R}$. Note that Sf is an even function, and S is the operator of convolution of $f(x)$ and $f(-x)$. We call Sf the *symmetrization* of f.

A standard characterization of growth of a function analytic in the disc $\{z: |z| < R\}$, $R \le \infty$, is

$$M(r, f) := \max_{|z| \le r} |f(z)|, \qquad 0 \le r < R.$$

If f is analytic in the strip $\{z: |\text{Im } z| < R\}$, we shall use, besides $M(r, f)$, the characteristic

$$H(r, f) := \sup_{|\text{Im } z| \le r} |f(z)|, \qquad 0 \le r < R.$$

Evidently, $M(r, f) \le H(r, f)$, for $0 \le r < R$. If $R = \infty$, i.e., f is an entire function, then, besides its order, defined by

$$\rho[f] := \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r},$$

we shall consider another characteristic $\kappa[f]$, defined by

$$\kappa[f] := \limsup_{r \to \infty} \frac{\log \log H(r, f)}{\log r}.$$

Evidently, $0 \le \rho[f] \le \kappa[f] \le \infty$.

If $f \in L_1^+$, we define the quantity $h[f]$ as

$$\begin{aligned} h[f] := \sup\{r > 0 \colon & f \text{ is the restriction to } \mathbb{R} \text{ of a function} \\ & \text{analytic and bounded in the strip } \{z \colon |\mathrm{Im}\, z| \le r\}\}. \end{aligned}$$

If $f \in E^\infty L_1^+$, we define $h[f] = \infty$. If there is no function whose restriction to $\mathbb{R}$ is f and analytic and bounded in some strip $\{z\colon |\mathrm{Im}\, z| \le r\}$, we define $h[f] = 0$.

3. STATEMENT OF RESULTS

As we have mentioned, the set UL_1^+ consists of functions with extremely bad smoothness. For example, if $f \in UL_1^+$, then f cannot coincide almost everywhere with a continuous function in any interval.

THEOREM 1. *There exists $f \in EL_1^+$ such that $Sf \in UL_1^+$, i.e., $S(EL_1^+) \cap UL_1^+ \neq \varnothing$.*

In the proof of this theorem, we use a theorem of T. Carleman on "touching" approximation by entire functions on $\mathbb{R}$. By the help of the generalization of this theorem due to Keldysh, it is possible to prove the following refinement of Theorem 1:

THEOREM 2. *There is a function $f \in E_3 L_1^+$ such that $Sf \in UL_1^+$, i.e., $S(E_3 L_1^+) \cap UL_1^+ \neq \varnothing$.*

Now we give some conditions of deterioration of smoothness obtained by use of Raikov's method. The basic result in this direction is the next theorem.

THEOREM 3. *Let $f \in L_1^+$. Then*

(i) *$h[Sf] \geq h[f]$, and $M(r, Sf) = H(r, Sf) \leq \|f\|_1 H(r, f)$ for $r < h[f]$.*

(ii) *If Sf is analytic in the disc $\{z: |z| < R\}$, then $h[Sf] \geq R$, $h[f] \geq R/2$, and the following inequality is valid:*

$$M(r, Sf) = H(r, Sf) \leq \|f\|_1 H(r, f) \leq \|f\|_1 \left\{ \frac{1}{\pi h} M(2(r + h), Sf) \right\}^{1/2},$$
$$r > 0, h > 0, 2(r + h) \leq R. \quad (3)$$

Since $f \in E^\infty L_1^+ \Leftrightarrow h[f] = \infty$, the following corollary is immediate:

COROLLARY 1. *In order that $f \in E^\infty L_1^+$ it is necessary and sufficient that $Sf \in EL_1^+$. Moreover, $Sf \in EL_1^+$ implies $Sf \in E^\infty L_1^+$. If $f \in E^\infty L_1^+$ then the relation $\rho[f] \leq \kappa[f] = \rho[Sf] = \kappa[Sf]$ is valid.*

Now we describe the possible pairs $(\rho[f], \kappa[f])$ for $f \in E^\infty L_1^+$:

THEOREM 4. *Let (ρ, κ) be a pair of numbers such that $1 \leq \rho \leq \kappa \leq \infty$. There exists a function $f \in E^\infty L_1^+$ such that $\rho[f] = \rho$, $\kappa[f] = \kappa$.*

Therefore, if $f \in E^\infty L_1^+$ is of fixed order $\rho[f] = \rho$, then the order $\rho[Sf]$ of Sf can be arbitrarily large. Now let $f, g \in E^\infty L_1^+$. If $\rho[f] < \rho[g]$, then it is natural to consider f as "smoother" than g. Since $\rho[Sf] = \kappa[f]$, the functions constructed in Theorem 4 can be interpreted as examples of deterioration of smoothness under convolution.

From Theorem 3(ii), it also follows that if $h[f] = 0$, then Sf cannot be analytic at the origin. This is Raikov's result [2]. To show that convolution can deteriorate smoothness, he then considered the function $f(x) = d/dx \exp\{1 - \exp[e^{-x}]\}$, which belongs to EL_1^+, but $h[f] = 0$. However, although not analytic at the origin, Sf is infinitely differentiable on $\mathbb{R}$, and $\rho[f] = \infty$. We construct the following examples:

THEOREM 5. *There exists $f \in E_1 L_1^+$ with $h[f] = 0$, i.e., Sf is not analytic at the origin.*

THEOREM 6. *For each n, there exists an $f \in E_{1+1/n} L_1^+$ such that Sf is not $(2n + 2)$-times differentiable at the origin.*

Theorem 6 is proved by the help of the following theorem, which is obtained by a refinement of Raikov's method.

THEOREM 7. *If $f \in L_1^+$ is not n-times differentiable, or if it is but not all of the n derivatives are bounded, then Sf is not $(2n + 2)$-times differentiable at the origin.*

Note that, by Theorem 7, for any function $f \in EL_1^+$ unbounded on $\mathbb{R}$, Sf is not twice differentiable at the origin.

4. PROOF OF THEOREM 1

We begin by the construction of a continuous function $g \in L_1^+$ such that $(Sg)(x) = \infty$ for any $x \in \mathbb{Q}$. Note that $(Sg)(0) = \|g\|_2$. However, there are continuous functions $g \in L_1^+$ such that $g \notin L_2(\mathbb{R})$. This is the basic fact in this construction.

For $2 \le n \in \mathbb{N}$ denote by s_n the function continuous on $\mathbb{R}$, equal to zero for $x \notin]n - n^{-3}, n + 2n^{-3}[$, equal to n for $x \in [n, n + n^{-3}]$, and linear for $x \in [n - n^{-3}, n]$, and for $x \in [n + n^{-3}, n + 2n^{-3}]$. Define

$$q = \sum_{n=2}^{\infty} s_n. \tag{4}$$

Since the supports of s_ns do not overlap, and $\|s_n\|_1 = 2n^{-2}$, it is easy to verify that q is continuous on $\mathbb{R}$ and belongs to L_1^+. For any nonnegative integer a, we have

$$\begin{aligned}(Sq)(a) &= \int_{-\infty}^{\infty} \left\{ \sum_{n=2}^{\infty} s_n(t) \right\} \left\{ \sum_{n=2}^{\infty} s_n(t+a) \right\} dt \\ &= \int_{-\infty}^{\infty} \sum_{n=2}^{\infty} s_n(t) s_{n+a}(a+t)\, dt \\ &\ge \sum_{n=2}^{\infty} \int_{n}^{n+(n+a)^{-3}} n(n+a)\, dt \ge \sum_{n=2}^{\infty} \frac{n}{(n+2)^2} = \infty.\end{aligned}$$

Set

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{k} q(kx - k^2). \tag{5}$$

Since each summand of (5) is continuous on $\mathbb{R}$ and, moreover, the support of the kth summand is contained in $[k, \infty[$, the series converges everywhere and g is continuous on $\mathbb{R}$. Since

$$\|g\|_1 \le \sum_{k=1}^{\infty} \frac{1}{k} \left\| q(kx - k^2) \right\|_1 = \|q\|_1 \sum_{k=1}^{\infty} \frac{1}{k^2},$$

we have $g \in L_1^+$. Let x be a non-negative rational number; set $x = a/b$, where $a \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$. We have

$$\begin{aligned}
(Sg)(x) &= \int_{-\infty}^{\infty} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} q(kt - k^2) \right\} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} q(kt + kx - k^2) \right\} dt \\
&\geq \int_{-\infty}^{\infty} \left\{ \frac{1}{b} q(bt - b^2) \right\} \left\{ \frac{1}{b} q(bt + a - b^2) \right\} dt \\
&= \frac{1}{b^3} \int_{-\infty}^{\infty} q(s) q(s + a)\, ds \\
&= \frac{1}{b^3} (Sq)(a) = \infty.
\end{aligned}$$

Since Sg is an even function, we conclude that $(Sg)(x) = \infty$ for any $x \in \mathbb{Q}$. Thus, the function g with the properties mentioned at the beginning of the proof has been constructed. In order to construct the desired function $f \in EL_1^+$, we need the following theorem by Carleman [4].

THEOREM (Carleman). *Let g be a (complex-valued) continuous function on $\mathbb{R}$. Let $\epsilon = \epsilon(r)$ be a positive decreasing continuous function on* $[0, \infty[$. *There exists an entire function f such that*

$$|g(x) - f(x)| < \epsilon(|x|), \qquad x \in \mathbb{R}. \tag{6}$$

We shall use the following corollary to this theorem.

COROLLARY 2. *If g is assumed to be real valued on $\mathbb{R}$, then f can be chosen real valued and such that $f(x) > g(x)$ on $\mathbb{R}$.*

To derive the corollary, note that, by Carleman's theorem, there exists an entire function f_1 such that

$$\left| g(x) + \tfrac{1}{2}\epsilon(|x|) - f_1(x) \right| < \tfrac{1}{4}\epsilon(|x|), \qquad x \in \mathbb{R}.$$

It is easy to see that the function $f(z) = \frac{1}{2}\{f_1(z) + \overline{f_1(\bar{z})}\}$ is entire, satisfies $f(x) > g(x)$ on $\mathbb{R}$, and (6) is valid. Now we can construct the function $f \in EL_1^+$ such that $Sf \in UL_1^+$. Let g be the function defined by (5). By the corollary to Carleman's theorem, there exists an entire function f positive on $\mathbb{R}$ and satisfying the condition

$$|g(x) - f(x)| < e^{-|x|}, \qquad x \in \mathbb{R}.$$

Hence, $f \in EL_1^+$. It remains to show that $Sf \in UL_1^+$. From $f(x) > g(x) \geq 0$, it follows that $(Sf)(x) \geq (Sg)(x)$. Since $(Sg)(x) = \infty$ for $x \in \mathbb{Q}$, we conclude that

$$(Sf)(x) = \infty, \qquad x \in \mathbb{Q}. \tag{7}$$

In order to derive from (7) that $Sf \in UL_1^+$, we shall use the following two lemmas.

LEMMA 1. *If f, g are continuous nonnegative functions, then the convolution $f * g$ is lower semicontinuous.*

Proof. The function $f * g$ can be represented as the pointwise limit of the nondecreasing sequence of continuous functions

$$\left\{ \int_{-n}^{n} f(x-t) g(t)\, dt \right\}_{n=1}^{\infty}.$$

Since the limit of a nondecreasing sequence of continuous functions is lower semicontinuous, so is $f * g$. ∎

LEMMA 2. *If f is a lower semicontinuous function such that $f(x) = +\infty$ for x in a dense subset M of $\mathbb{R}$, then f possesses infinite essential supremum in any interval.*

Proof. By the lower semicontinuity of f, the set $\{x\colon f(x) > C\} \cap]\alpha, \beta[\}$ is open for any $C > 0$ and for any interval $]\alpha, \beta[$. By the condition of the lemma, this set is nonempty. Since any nonempty open set has a positive Lebesgue measure, we obtain, for any set E with $\operatorname{meas} E = 0$, $\sup_{x \in]\alpha, \beta[\setminus E} f(x) > C$. Hence $\operatorname{ess\,sup}_{x \in]\alpha, \beta[} f(x) > C$. Using the arbitrariness of C, we get the desired result. ∎

We are now ready to complete the proof of Theorem 1. By Lemma 1, Sf is a lower semicontinuous function. Since $(Sf)(x) = \infty$ for $x \in \mathbb{Q}$, $Sf \in UL_1^+$ according to Lemma 2. ∎

5. PROOF OF THEOREM 2

Now we proceed to show that there exists an entire function f of order ≤ 3 such that $Sf \in UL_1^+$. In order to construct the function f, we shall use a refinement of the Carleman theorem due to Keldysh [5]. For a detailed exposition of this theorem see [6].

THEOREM (Keldysh). *Let g be a (complex-valued) differentiable function on $\mathbb{R}$. Put*

$$\mu(r) := \max_{|x| \leq r} |g'(x)| \quad \textit{and} \quad \nu[g] := \limsup_{r \to \infty} \frac{\log \mu(r)}{\log r}.$$

Then for each $\epsilon \geq 0$ there exists an entire function f whose order does not exceed $\nu[g] + 1$ and satisfying $|f(x) - g(x)| < \epsilon$ for all $x \in \mathbb{R}$.

The following corollary is easily derived by imitating the proof of Corollary 2.

COROLLARY 3. *Let g be a real-valued differentiable function on $\mathbb{R}$. Then there exists an entire function f whose order does not exceed $\nu[g]+1$ which is real valued on $\mathbb{R}$ and satisfies* $0 < f(x) - g(x) < 1$.

Now we start with the construction of the function f whose existence is asserted by Theorem 2.

Step 1. For $3 \le n \in \mathbb{N}$ denote by s_n the function continuous on $\mathbb{R}$, equal to zero for $x \notin]n - n^{-1}\log^{-3} n, n + 2n^{-1}\log^{-3} n[$, equal to $n\log^3 n$ in the interval $[n, n + n^{-1}\log^{-3} n]$, and linear for $[n - n^{-1}\log^{-3} n, n]$ and for $[n + n^{-1}\log^{-3} n, n + 2n^{-1}\log^{-3} n]$. One can make the edges of s_n smoother, so that it becomes a differentiable function. Define

$$q(x) = \sum_{n=3}^{\infty} s_n(x) \tag{8}$$

and set

$$g(x) = \sum_{k=3}^{\infty} \frac{q(kx - k!)}{k^2}. \tag{9}$$

Since the supports of s_ns do not overlap, q is differentiable on $\mathbb{R}$. Likewise, the support of the function $q(kx - k!)$ is contained in $[k-1, \infty[$, so that the series defining g converges everywhere and g is also differentiable. We will approximate this function by an entire function according to the corollary to Keldysh's theorem. Let us first calculate $\mu(r)$ for the function g. If $x < 1$, then $g'(x)$ is identically 0; so it suffices to consider $x > 1$ only. So assume that $1 < x < r$. Then, since the function $q(kx - k!)$ vanishes for $kx - k! < 1$, only the finite number $n(r) := \#\{k\colon (k-1)! < r\}$ of terms contributes to g. Note that by Stirling's formula $n(r) = O(\log r)$ as $r \to \infty$. Hence

$$g(x) = \sum_{k=3}^{n(r)} \frac{q(kx - k!)}{k^2}, \qquad 1 \le x \le r,$$

$$|g'(x)| \le \sum_{k=3}^{n(r)} \left| \frac{q'(kx - k!)}{k} \right| \le \sum_{k=3}^{n(r)} |q'(kx - k!)|, \qquad 1 \le x \le r.$$

Now clearly we have $|q'(x)| \le (x+1)^2 \log^6(x+1)$. Inserting this in the above inequality, we get, as $r \to \infty$,

$$|g'(x)| \le \sum_{k=3}^{n(r)} (kx - k! + 1)^2 \log^6(kx - k! + 1)$$
$$\le n(r) x^2 \log^6 x = O\big(r^2 \log^7 r\big).$$

Therefore $\mu(r) = O(r^2 \log^7 r)$ as $r \to \infty$, and hence

$$\nu[g] = \limsup_{r\to\infty} \frac{\log \mu(r)}{\log r} \le 2.$$

By Corollary 3, we conclude that there is an entire function f_0, real valued and nonnegative on $\mathbb{R}$ with $\rho[f_0] \le 3$, satisfying $f_0(x) = g(x) + \delta(x)$, where $0 < \delta(x) < 1$. The desired function will be obtained by "shrinking" f_0 by multiplying with the function h we describe in the lemma below, whose proof is rather technical and will be given at the end of this section.

LEMMA 3. *There exists a function $h \in EL_1^+$ such that $\rho[h] = 1$ and $h(x) = 1/(x \log^2 x) + O(x^{-3/2})$ as $x \to \infty$ in $\mathbb{R}$.*

Put $f(z) := f_0(z)h(z)$. We claim that f is a function with desired properties.

Step 2. Now we shall prove that $f \in E_3 L_1^+$. Since $\rho[h] = 1$, $\rho[f_0] \le 3$, f is entire and $\rho[f] \le 3$. Clearly f is nonnegative on $\mathbb{R}$, and it remains only to show that it is integrable. Put $\delta := f_0 - g$. Then we have

$$\|f\|_1 = \|(g+\delta)h\|_1 \le \|\delta h\|_1 + \|gh\|_1.$$

Since δ is bounded and h is integrable, $\|\delta h\|_1 < \infty$. On the other hand, by (9) we have

$$\|gh\|_1 \le \sum_{k=3}^{\infty} \frac{1}{k^2} \|q(kx - k!)h(x)\|_1.$$

By the change of variable $kx - k! = y$ we obtain

$$\|q(kx - k!)h(x)\|_1 = \frac{1}{k} \left\| q(y) h\left(\frac{y + k!}{k}\right)\right\|_1.$$

Hence, in order to show that $\|gh\|_1 < \infty$, it suffices to show that $\|q(y)h((y + k!)/k)\|_1 = O(k)$. Indeed, by (9) we have

$$\left\| q(y) h\left(\frac{y+k!}{k}\right)\right\|_1 \le \sum_{n=3}^{\infty} \left\| s_n(y) h\left(\frac{y+k!}{k}\right)\right\|_1.$$

Since the support of s_n is contained in $]n - 2n^{-1}\log^{-3} n, n + 2n^{-1}\log^{-3} n[$, and $s_n(x) \le n \log^3 n$, we obtain

$$\left\| s_n(y) h\left(\frac{y+k!}{k}\right)\right\|_1 \le \int_{n-2/(n\log^3 n)}^{n+2/(n\log^3 n)} h\left(\frac{y+k!}{k}\right) dy \le 4 \max_{y\in[n-1,n+1]} h\left(\frac{y+k!}{k}\right).$$

Now define $r(x) := 1/(\log^2 x)$. From $h(x) = r(x) + O(x^{-3/2})$ as $x \to \infty$ it follows that $h(x) \le Cr(x)$, $x \ge 2$ with some positive constant C. The function r is decreasing, so we have, for $k \ge 3$

$$\max_{y\in[n-1,n+1]} h\left(\frac{y+k!}{k}\right) \le r\left(\frac{n-1+k!}{k}\right).$$

On the other hand, recall the formula

$$\sum_{k=k_0}^{\infty} f(k) \le f(k_0) + \int_{k_0}^{\infty} f(y)\, dy,$$

which is valid if f is a decreasing function. Using this formula, we get

$$\begin{aligned}\left\| q(y) h\left(\frac{y+k!}{k}\right)\right\|_1 &\le \sum_{n=3}^{\infty} 4Cr\left(\frac{n-1+k!}{k}\right) \\ &\le 4Cr\left(\frac{2+k!}{k}\right) + 4C\int_3^{\infty} r\left(\frac{y-1+k!}{k}\right) dy.\end{aligned}$$

Substitute $x = (y-1+k!)/k$ in the integral to get, for some C,

$$\begin{aligned}\left\| q(y) h\left(\frac{y+k!}{k}\right)\right\|_1 &\le Cr\left(\frac{2+k!}{k}\right) + Ck\int_{(2+k!)/k}^{\infty} r(x)\, dx \\ &= Cr\left(\frac{2+k!}{k}\right) - \frac{Ck}{\log x}\bigg|_{(2+k!)/k}^{\infty} = O(k) \qquad \text{as } k \to \infty.\end{aligned}$$

We conclude that f is integrable.

Step 3. We shall prove $Sf \in UL_1^+$. Let us first show $(Sf)(x) = \infty$ for $x \in \mathbb{Q}$. We shall, for $b \in \mathbb{N}$,

$$Sf = S(hg + h\delta) \ge S(hg) = S\left(h \sum_{k=3}^{\infty} \frac{q(kt-k!)}{k^2}\right) \ge S\left(h \frac{q(bt-b!)}{b^2}\right).$$

Therefore,

$$(Sf)(x) \geq \frac{1}{b^4}\int_{-\infty}^{\infty} h(t)h(x+t)q(bt-b!)q(bx+bt-b!)\,dt.$$

Now let x be a nonnegative rational number; set $x = a/b$, where $a \in \mathbb{N}\cup\{0\}$, $b \in \mathbb{N}$. Upon the change of variable $bt - b! = y$ and recalling that $bx = a$, the above inequality becomes

$$(Sf)(x) \geq \frac{1}{b^5}\int_{-\infty}^{\infty} h\left(\frac{y+b!}{b}\right)h\left(\frac{y+a+b!}{b}\right)q(y)q(a+y)\,dy.$$

Recall that $h(x) = r(x) + O(x^{-3/2})$ as $x \to +\infty$, so that $h(y) \geq r(y)/2$ for $y > y_0$. If we increase the lower limit of the above integral to by_0, the inequality will be preserved:

$$(Sf)(x) \geq \frac{1}{4b^5}\int_{by_0}^{\infty} r\left(\frac{y+b!}{b}\right)r\left(\frac{y+a+b!}{b}\right)q(y)q(a+y)\,dy.$$

Since r is a decreasing function we have

$$(Sf)(x) \geq \frac{1}{4b^5}\int_{by_0}^{\infty} r^2\left(\frac{y+a+b!}{b}\right)q(y)q(a+y)\,dy.$$

Now we insert the series defining q into the last inequality:

$$\begin{aligned}(Sf)(x) &\geq \frac{1}{4b^5}\int_{by_0}^{\infty} r^2\left(\frac{y+a+b!}{b}\right)\left\{\sum_{n=3}^{\infty} s_n(y)\right\}\left\{\sum_{n=3}^{\infty} s_n(a+y)\right\}dy \\ &\geq \frac{1}{4b^5}\int_{by_0}^{\infty} r^2\left(\frac{y+a+b!}{b}\right)\sum_{n=3}^{\infty} s_n(y)s_{a+n}(a+y)\,dy.\end{aligned}$$

Put $n_0 := a + [by_0] + 1$. Simply by eliminating the terms for which $n < n_0$ above, we obtain

$$\begin{aligned}(Sf)(x) &\geq \frac{1}{4b^5}\int_{by_0}^{\infty} r^2\left(\frac{y+a+b!}{b}\right)\sum_{n=n_0}^{\infty} s_n(y)s_{a+n}(a+y)\,dy \\ &\geq \sum_{n=n_0}^{\infty}\frac{1}{4b^5}\int_{n}^{n+1/(n+a)\log^3(n+a)} r^2\left(\frac{y+a+b!}{b}\right) \\ &\qquad\times\sum_{n=n_0}^{\infty} s_n(y)s_{a+n}(a+y)\,dy.\end{aligned}$$

The last inequality follows from the fact that $\bigcup_{n=n_0}^{\infty}[n, n+1/\{(n+a)\log^3(n+a)\}] \subset [by_0, \infty]$. On the other hand, note that $s_n(y) = n\log^3 n$, $s_{a+n}(a+y) = (a+n)\log^3(a+n)$, and $r^2(\{y+a+b!\}/b) \geq r^2(\{2n+a+b!\}/b)$ for $y \in [n, n+1/\{(n+a)\log^3(n+a)\}]$. Using these, we obtain

$$
\begin{aligned}
(Sf)(x) &\geq \frac{1}{b^5}\sum_{n=n_0}^{\infty} n\log^3 n\left(\frac{2n+b+b!}{b}\right)^{-2}\log^{-4}\left(\frac{2n+b+b!}{b}\right) \\
&= \infty.
\end{aligned}
$$

Hence, the result $(Sf)(x) = \infty$ for all $x \in \mathbb{Q}^+$ is proved. Since Sf is an even function, we conclude that $(Sf)(x) = \infty$ for all $x \in \mathbb{Q}$. Since Sf is lower semicontinuous to Lemma 1, $Sf \in UL_1^+$ by Lemma 2. ∎

We believe that $S(E_1L_1^+) \cap UL_1^+ \neq \varnothing$, but we have failed to prove it.

The latter condition cannot be improved since $E_\rho L_1^+ = \varnothing$ for $\rho < 1$ by the Phragmén–Lindelöf theorem.

Proof of Lemma 3. For the construction of a function with properties described in Lemma 3, we will use the following theorem based on an idea first used by Mittag–Leffler in 1903 [7]. This theorem will be used extensively throughout this paper.

THEOREM 8 (Mittag-Leffler). *Denote by G_θ the angle $\{z\colon |\arg z| < \theta\}$, $0 < \theta < \pi$, and let g be a function analytic in G_γ for some γ, satisfying*

$$g(z) = O(|z|^{-2}) \qquad \text{as } |z| \to \infty \text{ in } G_\gamma \setminus G_\alpha, \tag{10}$$

where $0 < \alpha < \gamma$. For $z \in \mathbf{int}(\mathbb{C} \setminus G_\delta)$, define

$$f(z) := -\frac{1}{2\pi i}\int_{\partial G_\delta}\frac{g(\zeta)\,d\zeta}{\zeta - z} \tag{11}$$

for some δ such that $\alpha < \delta < \gamma$. Then

(i) *The function f does not depend on $\delta \in (\alpha, \gamma)$ and can be continued to $\mathbb{C}$ as an entire function.*

(ii) *The following asymptotic formulas are valid for any $n = 0, 1, 2, \ldots$:*

$$f^{(n)} = \begin{cases} O(|z|^{-n-1}), & \text{as } |z| \to \infty \text{ in } \mathbb{C} \setminus G_\delta, \\ g^{(n)}(z) + O(|z|^{-n-1}), & \text{as } |z| \to \infty \text{ in } G_\delta. \end{cases} \tag{12}$$

Proof. (i) By virtue of (10), integral (11) converges uniformly on every compact subset of $\mathbf{int}(\mathbb{C} \setminus G_\delta)$. Hence f is analytic in $\mathbf{int}(\mathbb{C} - G_\delta)$. Now put $G_{\delta,R} := \{z \in G_\delta\colon |z| \geq R\}$. If $z \in \mathbf{int}(\mathbb{C} \setminus G_\delta)$, then, by the Cauchy theorem, the integral (11) does not change if we replace ∂G_δ by $\partial G_{\delta,R}$ for any $R > 0$. Since the integral along $\partial G_{\delta,R}$ is analytic in $z \in \mathbf{int}(\mathbb{C} - G_{\delta,R})$, and, moreover, R is arbitrary, we conclude that f can be analytically continued into $\mathbb{C}$. According to this rule of continuation, for $z \in G_\delta$ we have the representation

$$f(z) = -\frac{1}{2\pi i}\int_{\partial G_{\delta,R}} \frac{g(\zeta)\,d\zeta}{\zeta - z}, \tag{13}$$

where $R > |z|$. The function f does not depend on $\delta \in (\alpha, \gamma)$ since, for $z \in \mathbf{int}(\mathbb{C} \setminus G_\gamma)$, the integral (11) does not change if we replace ∂G_δ by $\partial G_{\delta'}$ for any $\delta' \in (\alpha, \gamma)$: This follows from the Cauchy theorem and condition (10).

(ii) For $z \in \mathbf{int}(\mathbb{C} \setminus G_\delta)$, we choose $\delta' \in (\alpha, \delta)$ and represent $f^{(n)}$ in the form

$$f^{(n)} = -\frac{n!}{2\pi i}\int_{\partial G_{\delta'}} \frac{g(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}. \tag{14}$$

Since

$$|\zeta - z| \geq |z| \sin(\delta - \delta') \qquad \text{for } \zeta \in \partial G_{\delta'} \text{ and } z \in \mathbf{int}(\mathbb{C} \setminus G_\delta), \tag{15}$$

by using (14) and (10), we obtain the first part of (12). On the other hand, for $z \in \partial G_\delta$, by the Cauchy theorem, the representation (13) gives

$$-\int_{\partial G_{\delta,R}} \frac{g(\zeta)\,d\zeta}{\zeta - z} + \int_{\partial G_\delta} \frac{g(\zeta)\,d\zeta}{\zeta - z} = -\int_{\partial(G_\delta \setminus G_{\delta,R})} \frac{g(\zeta)\,d\zeta}{\zeta - z} = 2\pi i g(z).$$

Hence, for $z \in \mathbf{int}(G_\delta)$ we have the representation

$$f(z) = g(z) - \frac{1}{2\pi i}\int_{\partial G_{\delta'}} \frac{g(\zeta)\,d\zeta}{\zeta - z}. \tag{16}$$

Now choose $\delta' \in (\delta, \gamma)$. By (16), for $z \in G_\delta$ we have

$$f^{(n)}(z) = g^{(n)}(z) - \frac{n!}{2\pi i}\int_{\partial G_{\delta'}} \frac{g(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}. \tag{17}$$

Using (15) and (10), we obtain the second part of (12). ∎

Now we are ready to prove Lemma 3. We apply the Mittag-Leffler theorem to the function $g(z) := e^{-iz}/(\sqrt{z}\,\log z)$ which is analytic in the region $G_0 := \{z = re^{i\theta}:\ -\pi/2 < \theta < 3\pi/2;\ z \neq 1, 0\}$, with the branch of the logarithm real on $\mathbb{R}^+$.

1. Let us first look at the asymptotic behaviour of g in the subset of G_0 lying in the lower half-plane. Let δ be such that $0 < \delta < \pi/2$. Then if $\arg z < -\delta$ or $\arg z > \pi + \delta$, we have

$$|g(z)| = \frac{e^{-|z|\sin\delta}}{|\sqrt{z}\,\log z|} \leq e^{-|z|\sin\delta} \qquad \text{for } |z| > 3.$$

Denote $G := \{z:\ -\pi/4 < \arg z < 5\pi/4\}$. By Theorem 8, the function given in $\mathbb{C}\setminus G$ by

$$f(z) := -\frac{1}{2\pi i}\int_{\partial G}\frac{g(\zeta)\,d\zeta}{\zeta - z}$$

can be continued analytically to $\mathbb{C}$ and satisfies

$$f(z) = \begin{cases} O\big(|z|^{-1}\big), & \text{as } |z| \to \infty \text{ in } \mathbb{C}\setminus G, \\ g(z) + O\big(|z|^{-1}\big), & \text{as } |z| \to \infty \text{ in } G. \end{cases}$$

Clearly, $\rho[f] = 1$ and $f(x) = g(x) + O(1/|x|)$ as $|x| \to \infty$ in $\mathbb{R}$. Hence we have

$$f(x) = \begin{cases} \dfrac{e^{-ix}}{\sqrt{x\log x}} + O(x^{-1}), & \text{as } x \to +\infty, \\[2ex] \dfrac{e^{-ix}}{i\sqrt{|x|}\,(\log x + i\pi)} + O\big(|x|^{-1}\big), & \text{as } x \to -\infty. \end{cases} \tag{18}$$

2. Consider the function $t(z) := \{f(z) + \overline{f(\bar z)}\}^2$, which is also entire, and nonnegative on $\mathbb{R}$. From the estimate for f we obtain $|t(x)| \leq 2/(|x|\log^2|x|)$. Therefore t is integrable. On the other hand, as $x \to \infty$, by (18) we have

$$t(x) = \left\{\frac{e^{-ix}}{\sqrt{x}\,\log x} + \frac{e^{ix}}{\sqrt{x}\,\log x} + O(x^{-1})\right\}^2 = \frac{4\cos^2 x}{x\log^2 x} + O(x^{-3/2}).$$

3. Finally, define the entire function h of order 1 as $h(z) := \{t(z) + t(z + \pi/2)\}/4$. Then h is the desired function. Clearly, $h \in EL_1^+$. As $x \to +\infty$ we have

$$\begin{aligned} h(x) &= \frac{\cos^2 x}{x\log^2 x} + \frac{\sin^2 x}{(x + \pi/2)\log^2(x + \pi/2)} + O(x^{-3/2}) \\ &= \frac{1}{x\log^2 x} + O(x^{-3/2}). \end{aligned}$$

∎

6. PROOF OF THEOREM 3

Before proving Theorem 3, we recall some of Raikov's results cited in [3].

THEOREM 9 (Raikov). *Let $\hat{g}$ be the Fourier transform of the function $g \in L_1^+$. If $\hat{g}$ is analytic in the disc $\{z\colon |z| < R\}$ then*

$$\int_{-\infty}^{\infty} e^{rx} g(x)\, dx \le \infty, \qquad -R < r < R.$$

Moreover, $h[\hat{g}] \ge R$ and the following representation is valid in the strip $\{z\colon |\mathrm{Im}\, z| < R\}$:

$$\hat{g}(z) = \int_{-\infty}^{\infty} e^{izx} g(x)\, dx.$$

Now note that $\widehat{f(-x)}(t) = \overline{\hat{f}(t)}$. Thus, we have

$$\widehat{(Sf)}(t) = \hat{f}(t)\overline{\hat{f}(t)} = |\hat{f}(t)|^2 \ge 0.$$

Hence, the transform of Sf is always nonnegative. For such functions, the following fact is valid:

THEOREM 10 (Raikov). *Let $f \in L_1^+$ be continuous at 0. If $\hat{f}(t) \ge 0$, then $\hat{f} \in L_1^+$ and*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \hat{f}(t)\, dt.$$

COROLLARY 4. *If Sf is continuous at 0, then Theorem 9 is applicable for $\hat{g} = Sf$ and $g = \widehat{Sf}/(2\pi)$.*

Now we pass to the proof of Theorem 3.

(i) First, we shall prove the following lemma:

LEMMA 4. *Let f, g be functions such that $g \in L_1$. Then $h[f * g] \ge h[f]$, and the inequality $H(r, f * g) \le \|g\|_1 H(r, f)$ is satisfied for $r < h[f]$.*

Proof. Clearly, for $|\mathrm{Im}\, z| < h[f]$ we have $|f(z - t)| \le H(|\mathrm{Im}\, z|, f)$. Hence the convolution integral converges uniformly in the strip $\{z\colon |\mathrm{Im}\, z| \le r < h[f]\}$, and $f * g$ is analytic in this strip. Moreover,

$$H(r, f * g) = \sup_{|\mathrm{Im}\, z| < r} \left| \int_{-\infty}^{\infty} f(z - t) g(t)\, dt \right| \le \|g\|_1 H(r, f),$$

which also shows that $h[f * g] \ge r$. ∎

Now, by Lemma 4, $h[Sf] \geq h[f]$ and $H(r, Sf) \leq \|f\|_1 H(r, f)$. On the other hand, by Corollary 4 we have, for $|\mathrm{Im}\, z| < h[f]$,

$$|(Sf)(z)| = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izt} |\hat{f}(t)|^2 \, dt \right|$$
$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{Im}(zt)} |\hat{f}(t)|^2 \, dt = (Sf)(i \,\mathrm{Im}\, z),$$

which shows that $H(r, Sf) = M(r, Sf)$.

(ii) We want to show that the integral

$$\int_{-\infty}^{\infty} e^{r|t|} |\hat{f}(t)| \, dt$$

is finite for $0 \leq r < R/2$. Let $r < r' < R/2$. Then

$$\int_{-\infty}^{\infty} e^{rt} |\hat{f}(t)| \, dt \leq \int_{-\infty}^{\infty} e^{r|t|} |\hat{f}(t)| \, dt = \int_{-\infty}^{\infty} e^{(r-r')|t|} e^{r'|t|} |\hat{f}(t)| \, dt.$$

By Schwarz's inequality, it follows that

$$\int_{-\infty}^{\infty} e^{(r-r')|t|} e^{r'|t|} |\hat{f}(t)| \, dt$$
$$\leq \left\{ \int_{-\infty}^{\infty} e^{2(r-r')|t|} \, dt \int_{-\infty}^{\infty} e^{2r'|t|} |\hat{f}(t)|^2 \, dt \right\}^{1/2}. \tag{19}$$

For the first integral in the right-hand side of (19) we have

$$\int_{-\infty}^{\infty} e^{2(r-r')|t|} \, dt = \frac{1}{r' - r}.$$

For the second integral,

$$\int_{-\infty}^{\infty} e^{2r'|t|} |\hat{f}(t)|^2 \, dt \leq \int_{-\infty}^{\infty} e^{2r't} |\hat{f}(t)|^2 \, dt + \int_{-\infty}^{\infty} e^{-2r't} |\hat{f}(t)|^2 \, dt.$$

Now assume that Sf is analytic in the disc $\{z\colon |z| < R\}$. Then, both of the last two integrals are finite by Corollary 4, and

$$\int_{-\infty}^{\infty} e^{2r't} |\hat{f}(t)|^2 \, dt = 2\pi (Sf)(2ir') \leq 2\pi M(2r', Sf),$$

$$\int_{-\infty}^{\infty} e^{-2r't} |\hat{f}(t)|^2 \, dt = 2\pi (Sf)(-2ir') \leq 2\pi M(2r', Sf).$$

Hence

$$\int_{-\infty}^{\infty} e^{2r'|t|}|\hat{f}(t)|^2\,dt \le 4\pi M(2r', Sf),$$

and we finally have

$$\int_{-\infty}^{\infty} e^{r|t|}|\hat{f}(t)|\,dt \le \left\{\frac{4\pi M(2r', Sf)}{r'-r}\right\}^{1/2} \le \infty.$$

It follows that the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-izt}\hat{f}(t)\,dt$$

converges uniformly in the strips $\{z\colon |\mathrm{Im}\, z| \le r\}$ for $r < R/2$, and f is analytic in the strip $\{z\colon |\mathrm{Im}\, z| < R/2\}$. On the other hand

$$\begin{aligned} H(r,f) = \sup_{|\mathrm{Im}\, z|<r} |f(z)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{r|t|}|\hat{f}(t)|\,dt \\ &\le \frac{1}{2\pi}\left\{\frac{4\pi M(2r', Sf)}{r'-r}\right\}^{1/2}, \end{aligned} \tag{20}$$

which means that $H(r,f) < \infty$ for $r < R/2$; i.e., $h[f] \ge r < R/2$. Now put $r' - r = h$ and substitute in (20) to get

$$H(r,f) \le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{r|t|}|\hat{f}(t)|\,dt \le \left\{\frac{M(2(r+h), Sf)}{\pi h}\right\}^{1/2}.$$

By part (i), $M(r, Sf) = H(r, Sf) \le \|f\|_1 H(r,f)$. Joining this with the above inequality, we obtain the desired result.

Proof of Corollary 1. If $f \in E^\infty L_1^+$, then $h[f] = \infty$ and by Theorem 3(i) it follows that $h[Sf] = \infty$, i.e., $Sf \in E^\infty L_1^+ \subset EL_1^+$. Similarly if $Sf \in EL_1^+$, then by Theorem 3(ii) it follows that both $h[Sf] = \infty$ and $h[f] = \infty$, i.e., $Sf \in E^\infty L_1^+$, $f \in E^\infty L_1^+$. Substituting $h = 1$ in inequality (3) one has

$$\begin{aligned} \rho[Sf] = \kappa[Sf] &\le \kappa[f] \\ &\le \limsup_{r\to\infty} \frac{\log\log\{M(2(1+r), Sf)\}^{1/2}}{\log r} = \rho[Sf]. \end{aligned}$$

Hence $\rho[Sf] = \kappa[Sf] = \kappa[f]$. Since the inequality $\rho[f] \le \kappa[f]$ is always valid, we get the desired result.

7. PROOF OF THEOREM 4

To calculate the quantity $\kappa[f]$ for the functions we shall construct, the following lemma will be helpful.

LEMMA 5. *For $0 < \beta < \pi/2$, define the set $A_{r,\beta} := \{z\colon |\mathrm{Im}\, z| < r, |\arg z| < \beta$ or $|\arg z - \pi| < \beta\}$, and for the entire function f put $H_\beta(r, f) := \sup_{z \in A_{r,\beta}} |f(z)|$. Then the inequality*

$$\kappa[f] = \max(\rho[f], \kappa_\beta[f])$$

is valid, where

$$\kappa_\beta[f] := \limsup_{r\to\infty} \frac{\log\log H_\beta(r, f)}{\log r}.$$

Proof. Evidently, $\kappa[f] \geq \max(\rho[f], \kappa_\beta[f])$. On the other hand, put $B_{r,\beta} := \{z\colon |\mathrm{Im}\, z| < r\} \setminus A_{r,\beta}$, and let $B_\beta(r, f) := \sup_{z\in B_{r,\beta}} |f(z)|$. Define

$$b_\beta[f] := \limsup_{r\to\infty} \frac{\log\log B_\beta(r, f)}{\log r}.$$

Since $H(r, f) = \max(B_\beta(r, f), H_\beta(r, f))$, we have $\kappa[f] \leq \max(b_\beta[f], \kappa_\beta[f])$. Finally, $b_\beta[f] \leq \rho[f]$ since

$$B_\beta(r, f) \leq \sup_{|z|\leq r/\sin\beta} |f(z)| = M(r/\sin\beta, f),$$

so that $\kappa[f] \leq \max(\rho[f], \kappa_\beta[f])$. ∎

Now we pass to the proof of the theorem.

Case 1. $1 < \rho < \kappa < \infty$. For $0 < \rho < \sigma - 1 < \rho$, consider the function

$$g(z) := \exp(-iz^\rho - z^\sigma),$$

where the branches of the power functions are taken to be positive on $\mathbb{R}^+$. It is analytic in the region $\{z = re^{i\theta}\colon -\pi < \theta < \pi\}$. For $-\pi < \theta_1 < \theta_2 < \pi$ denote by $G(\theta_1, \theta_2)$ the angle $\{z = re^{i\theta}\colon \theta_1 < \theta < \theta_2\}$; and put $\gamma := \min(2\pi/\rho, \pi)$.

1. We have

$$\log|g(re^{i\theta})| = r^\rho \sin\rho\theta - r^\sigma \cos\sigma\theta, \tag{21}$$

which is majorized by the term $r^\rho \sin\rho\theta$ since $\rho > \sigma$. Hence, $g(z) = O(\exp(-K_\delta r^\rho))$ as $r \to \infty$, and $z = re^{i\theta} \in G(\pi/\rho + \delta, \gamma - \delta) \cup G(-\pi/$

$\rho + \delta, -\delta)$, δ being sufficiently small. Note that, for $x \in \mathbb{R}^+$, one has $g(x) = O(1/x)$.

Fix an α satisfying $\pi/\rho < \alpha < \gamma$. By Theorem 8, the function f given in $\mathbf{int}(\mathbb{C} \setminus G(-\pi/2\rho, \alpha))$ by

$$f(z) := -\int_{\partial G(-\pi/2\rho,\alpha)} \frac{g(\zeta)\, d\zeta}{\zeta - z}$$

can be continued analytically to $\mathbb{C}$ and satisfies

$$f^{(n)} = \begin{cases} O\left(|z|^{-1}\right), & \text{as } |z| \to \infty \text{ in } \mathbb{C} \setminus G(-\pi/2\rho, \alpha), \\ g^{(n)}(z) + O\left(|z|^{-n-1}\right), & \text{as } |z| \to \infty \text{ in } G(-\pi/2\rho, \alpha). \end{cases} \tag{22}$$

2. Clearly, $\rho[f] = \rho$. Now let us show that $\kappa[f] = \sigma/(\sigma - \rho + 1)$. There exists an angle $\beta < \pi/(3\rho)$ such that f is bounded in the angle $\{z: \pi - \arg z| < \beta\}$. Moreover, f is bounded on the lower half-plane. By Lemma 5 we want to estimate $H_\beta(r, f)$. It suffices to consider the angle $\{z: 0 < \arg z < \beta\}$ only. In order to estimate $H_\beta(r, f)$, we shall find the supremum of $|f|$ on the lines $l_y^+ := \{z = x + iy\colon 0 < \arg z < \beta\}$. By the construction of f we have

$$f(z) = g(z) + O\left(|z|^{-1}\right) \qquad \text{as } |z| \to \infty,\ z \in l_y^+,$$

so that by (21) it follows that

$$\log|f(z)| = r^\rho \sin \rho\theta - r^\sigma \cos \sigma\theta + O(r^{-1}) \qquad \text{as } r \to \infty,\ z = re^{i\theta} \in l_y^+.$$

Substitute $\sin\theta = y/r$, and use the estimates

$$\frac{2\rho\theta}{\pi} \le \sin \rho\theta \le \frac{\pi\rho\theta}{2} \qquad \text{for } 0 \le \theta \le \frac{\pi}{2\rho},$$

$$\frac{1}{2} \le \cos \rho\theta \le 1, \qquad \text{for } 0 \le \theta \le \frac{\pi}{3\rho},$$

to get, for $z = x + iy = re^{i\theta} \in l_y^+$, as $r \to \infty$,

$$\frac{2\rho y}{\pi} r^{\rho - 1} - r^\sigma + O(r^{-1}) \le \log|f(z)| \le \frac{\pi\rho y}{2} r^{\rho-1} - \frac{1}{2} r^\sigma + O(r^{-1}).$$

Hence, for r large enough, we have

$$\frac{2\rho y}{\pi} r^{\rho-1} - r^{\sigma} - 1 \le \log|f(z)| \le \frac{\pi\rho y}{2} r^{\rho-1} - \frac{1}{2} r^{\sigma} + 1. \qquad (23)$$

For any fixed $y_0 > 0$, the largeness of r can be taken uniformly in y with $0 < y \le y_0$. In the estimation of f from above, we see that the dominant term as $r \to \infty$ for fixed y is $-r^{\sigma}/2$ (since $\sigma \ge \rho - 1$), so that the function is bounded on l_y^+ and $H_\beta(y, f) < \infty$ for all $y > 0$. Calculation of the maximum of both sides of (23) by the usual method of differentiation gives

$$K_1 y^{\sigma/(\sigma-\rho+1)} \le \sup_{z\in l_y^+} \log|f(z)| \le \log H_\beta(y, f) \le K_2 y^{\sigma/(\sigma-\rho+1)} \qquad (24)$$

for y large enough with some positive constants K_1, K_2. On the other hand, (23) tells us that $\sup_{z\in l_y^+} \log|f(z)|$ is bounded in $0 < y < y_0$ for any y_0. Hence $\log H_\beta(y, f) \le K_2 y^{\sigma/(\sigma-\rho+1)}$ for large y. We conclude that $\kappa[f] = \sigma/(\sigma - \rho + 1)$. Hence, for given κ, if we substitute $\sigma = \kappa(\rho - 1)/(\kappa - 1)$, then $\kappa[f] = \kappa$.

3. Put $h(z) := \{f(z) + \overline{f(\bar z)}\}^2$. Then $h \in E^\infty L_1^+$, $\rho[h] = \rho$, and $\kappa[h] = \kappa$. Indeed, $f(x) = O(1/|x|)$ as $|x| \to \infty$ in $\mathbb{R}$ by construction, so $h(x) = O(1/x^2)$ as $|x| \to \infty$ on $\mathbb{R}$, and h is an integrable function. Being nonnegative on $\mathbb{R}$, we conclude that $h \in EL_1^+$. On the other hand, f is bounded on the lower half-plane, say by the constant C, therefore $(|f(z)| - C)^2 \le |h(z)| \le (|f(z)| + C)^2$ if $\operatorname{Im} z \ge 0$. Applying the same argument for $\operatorname{Im} z \le 0$ we obtain

$$(M(r, f) - C)^2 \le M(r, h) \le (M(r, f) + C)^2,$$
$$(H(r, f) - C)^2 \le H(r, h) \le (H(r, f) + C)^2.$$

Hence, $\rho[h] = \rho[f] = \rho$ and $\kappa[h] = \kappa[f] = \kappa$.

For the remaining cases, we shall give a sketch of proof.

Case 2. $1 < \rho < \kappa = \infty$. For $\rho > 1$, apply the procedure in Case 1 to the function $g(z) := \exp(-iz^{\rho}\log^2 z - z^{\rho-1}\log^3 z)$.

Case 3. $1 = \rho < \kappa < \infty$. Put $\sigma = (2\kappa)/(\kappa - 1)$, and consider the function $g(z) := \exp(-iz\log^2 z - \log^{\sigma} z)$, $\pi/2 < \arg z < 3\pi/2$; $|z| > 1$. Define the entire function by the Cauchy-type integral along the contour $L := \{z: |z| = 2, -\pi/4 < \arg z < 5\pi/4\} \cup \{z: |z| \ge 3, \arg z = -\pi/4$ or $5\pi/4\}$. Then apply the same procedure in Case 1.

Case 4. $1 = \rho < \kappa = \infty$. Consider the function $g(z) := \exp(-iz\log^2 z - \log^2 z \log\log z)$, which is analytic in the region $G = \{z = re^{i\theta}: r > 1; -\pi/2 < \theta < \pi/2\}$. Then apply the procedure in Case 3.

Case 5. $1 \le \rho = \kappa \le \infty$. The desired functions are Sf, where f is one of the functions constructed in previous cases. Indeed, we have exhibited functions f with given $\kappa[f] \ge 1$. By Corollary 1 for each of these functions we have $Sf \in E^\infty L_1^+$ and $\kappa[f] = \rho[Sf] = \kappa[Sf]$. ∎

Proof of Theorem 5. Theorem 3 can be considered as a test for analyticity of Sf. For example, it immediately implies that if $h[f] < \infty$, then $Sf \ne EL_1^+$. For each $h > 0$, there exists a function $f \in EL_1^+$ such that $h[f] = h$; for example, consider the function $f(x) = \exp[-\cosh(\pi x/(2h))]$. Hence, symmetrizations Sf of these functions cannot be entire. As we have already mentioned, another immediate consequence of Theorem 3 is that if $h[f] = 0$, then Sf cannot be analytic at 0. Now we shall show the existence of a function $E_1 L_1^+$ with $h[f] = 0$. From the function

$$g(z) := \frac{e^{-iz\log^2 z}}{z}, \qquad \text{where } -\frac{\pi}{2} < \arg z < \frac{3\pi}{2},$$

construct the entire function f as in the proof of Theorem 4, Case 3, and put $\tilde{f} := \{f(z) + \overline{f(\bar{z})}\}^2$. Then $\tilde{f}$ is the desired function. Clearly, $\tilde{f} \in EL_1^+$ and $\rho[\tilde{f}] = 1$. Now let us show $h[\tilde{f}] = 0$. We have

$$\log|g(re^{i\theta})| = r\log^2 r \sin\theta + 2r\theta \log r \cos\theta - r\theta^2 \sin\theta - \log r,$$

so that for $0 \le \theta \le \pi/2$ and for sufficiently large r one has $\log|g(re^{i\theta})| \ge r\log^2 r \sin\theta - r\theta^2 \sin\theta - \log r$. Now assume $y > 0$. For $z = re^{i\theta} \in l_y := \{z = x + iy \colon x > 0\}$, one has $\sin\theta = y/r$. Hence, for $z = re^{i\theta} \in l_y$ we have

$$\log|g(re^{i\theta})| \ge y\log^2 y - \frac{\pi y}{2} - \log r \to \infty \qquad \text{as } r \to \infty.$$

Since $f(z) = g(z) + O(1/|z|)$ in the upper half-plane, we get $H(y, f) = \infty$. On the other hand, note that $H_1(\overline{f(\bar{z})}) < \infty$ since $\overline{f(\bar{z})}$ is bounded in the upper half-plane. Hence

$$H(y, \tilde{f}) \ge \left\{H(y, f) - H\left(y, \overline{f(\bar{z})}\right)\right\}^2 \ge \left\{H(y, f) - H\left(1, \overline{f(\bar{z})}\right)\right\}^2 = \infty.$$

Since $y > 0$ is arbitrary, we conclude that $h[\tilde{f}] = 0$. ∎

However, symmetrization of the function constructed in Theorem 5, and the symmetrization of Raikov's example, $f(x) = d/dx \exp\{1 - \exp[e^{-x}]\}$, are infinitely differentiable by the following theorem.

THEOREM 11. *Assume $f \in L_1^+$ is a bounded function with continuous, bounded derivatives up to the order n. Then Sf has continuous, bounded derivatives up to the order n.*

Proof. For $k \leq n$ we have

$$|(Sf)^{(k)}(x)| = |f^{(k)} * \bar{f}| \leq \|f^{(k)}\|_\infty \|\bar{f}\|_1 < \infty.$$

It follows that the convolution integrals

$$\int_{-\infty}^{\infty} f^{(k)}(x+t) f(t)\, dt$$

are uniformly convergent, and Sf has n bounded derivatives. ∎

By the refinement of Raikov's method, a statement of the converse type can also be proved. This is the content of Theorem 7.

8. PROOF OF THEOREM 7

We begin with a theorem of Raikov (cited in [3]), which is actually the first step of the proof of Theorem 9.

THEOREM 12 (Raikov). *Let $\hat{g}$ be the transform of $g \in L_1^+$. If $\hat{g}$ is $2n$-times differentiable on $\mathbb{R}$, then*

$$\int_{-\infty}^{\infty} |x|^m g(x)\, dx < \infty, \qquad m = 0, 1, \ldots, 2n.$$

Moreover, $\hat{g}$ is $2n$-times differentiable on $\mathbb{R}$, and these derivatives can be represented by the integrals

$$\hat{g}^{(m)}(t) = i^m \int_{-\infty}^{\infty} x^m e^{itx} g(x)\, dx, \qquad m = 0, 1, \ldots, 2n.$$

Since the transform of Sf is nonnegative, by Theorem 10 we have the following corollary:

COROLLARY 5. *If Sf is continuous at 0, then Theorem 11 is applicable to $\hat{g} = Sf$ and $g = (\widehat{Sf})/(2\pi)$.*

Now we are ready to prove Theorem 7. We have

$$\int_{-\infty}^{\infty} |\hat{f}(t)|\, |t|^\alpha\, dt = \int_{-\infty}^{\infty} \frac{|\hat{f}(t)|}{1 + |t|^\beta} (1 + |t|^\beta) |t|^\alpha\, dt.$$

By Schwarz's inequality, this is bounded above by

$$\left\{\int_{-\infty}^{\infty}|\hat{f}(t)|^2\left(1+|t|^{\beta}\right)^2|t|^{2\alpha}\,dt\int_{-\infty}^{\infty}\frac{1}{\left(1+|t|^{\beta}\right)^2}\,dt\right\}^{1/2}.$$

The second integral above is finite whenever $\beta > 1/2$. For the first integral, we can consider $|\hat{f}|^2$ as the transform of Sf. Suppose that Sf is $2n$-times differentiable at the origin. By Corollary 5 the first integral above is finite whenever $2\alpha + 2\beta \le 2n$, that is, when $\alpha < n - 1/2$. Hence the integral $\int_{-\infty}^{\infty}\hat{f}(t)e^{-itx}(it)^k\,dt$ converges uniformly for $k \le n-1$. Therefore f is $(n-1)$-times differentiable on $\mathbb{R}$, and since these derivatives tends to 0 at infinity by the Riemann–Lebesgue theorem, they are bounded. ∎

Proof of Theorem 6. From the function

$$g(z) = \frac{e^{-iz^{\rho}\log^3 z}}{\sqrt{z}\,\log z},$$

construct the entire function f as in the proof of Theorem 4, Case 1, and put $h(z) := \{f(z) + \overline{f(\bar{z})}\}$ again. Clearly, h is entire, $\rho[h] = \rho$, and it is nonnegative on $\mathbb{R}$. Since $h(x) = O(1/|x|\log^2|x|)$ as $|x| \to \infty$ in $\mathbb{R}$, h is integrable and $h \in E_{\rho}L_1^{+}$. Now let us show that h' is unbounded on $\mathbb{R}$. Indeed, by Theorem 8(ii), we have $f'(x) = g'(x) + O(1/x^2)$ as $x \to +\infty$ on $\mathbb{R}$. Note that, as $x \to +\infty$, we have

$$f(x) + \overline{f(x)} = \frac{2\cos\left(x^{\rho}\log^3 x\right)}{\sqrt{x}\,\log x} + O(x^{-1})$$

and

$$f'(x) + \overline{f'(x)} = -2\rho x^{\rho-3/2}\log^2 x\,\sin\left(x^{\rho}\log^3 x\right) + O\left(x^{\rho-3/2}\log x\right).$$

Hence

$$h'(x) = -4\rho x^{\rho-2}\log x\,\sin\left(2x^{\rho}\log^3 x\right) + O(x^{\rho-2}),$$

so that h' is unbounded if $\rho \ge 2$.

Note that, if $\rho \ge 1 + 1/n$ in the above construction, then $h^{(n)}$ is unbounded and Sh is not $2(n+1)$-times differentiable at 0. ∎

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