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# Alperin's fusion theorem and $G$-posets

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**Abstract.** Some $G$-posets comprising Brauer pairs or local pointed groups belong to a class of $G$-posets which satisfy a version of Alperin's fusion theorem, and as a consequence, have simply connected orbit spaces.

One of the two purposes of this paper is to unify several versions of Alperin's fusion theorem. The other is to appreciate the apparently technical conclusion topologically. Let $G$ be a finite group. A *$G$-poset*, recall, is a partially ordered set upon which $G$ acts as automorphisms. One form of Alperin's fusion theorem derives from Sylow's theorem together with the nilpotency of finite $p$-groups. These two properties of $p$-subgroups are expressed axiomatically in defining a *Sylow $G$-poset* to be a finite set $X$ equipped with $G$-stable relations $\trianglelefteq$ and $\leqslant$ such that

(i) $\leqslant$ is a partial ordering, and is the transitive closure of $\trianglelefteq$,

(ii) $G$ acts transitively on the maximal elements of $X$, and for any $x \in X$, the stabilizer $N_G(x)$ acts transitively on the elements $y \in X$ which are maximal subject to $x \trianglelefteq y$.

Given an upwardly closed $G$-subposet $Y$ of a Sylow $G$-poset $X$ (for all $y \in Y$ and $x \in X$ with $y \leqslant x$, we have $x \in Y$), then $Y$ is a Sylow $G$-poset.

Although our results are expressed in an abstract setting, and the proofs require no specialist knowledge, the following account of a motivation for the notion of a Sylow $G$-poset assumes some familiarity with $p$-local representation theory, particularly as discussed in Knörr–Robinson [2] and Thévenaz [7]. Let $p$ be a prime divisor of $|G|$, and $B$ a positive-defect $p$-block of $G$. Let $\mathscr{B}(B)$ be the $G$-poset of non-trivial Brauer pairs associated with $B$, and $\mathscr{L}(B)$ the $G$-poset of non-trivial local pointed groups associated with $B$. (The *trivial Brauer pair* associated with $B$ is the unique minimal one.) By results of Alperin, Broué, Puig in Thévenaz [7, 40.10, 40.15, 48.1], $\mathscr{B}(B)$ and $\mathscr{L}(B)$ are Sylow. The upwardly closed $G$-subposet $\mathscr{C}(B)$ of $\mathscr{B}(B)$ consisting of

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the self-centralising Brauer pairs associated with $B$ (see Thévenaz [7]) is canonically isomorphic to the upwardly closed $G$-subposet of $\mathscr{L}(B)$ consisting of the self-centralizing local pointed groups associated with $B$.

The idea of reformulating Alperin's weight conjecture using self-centralizing Brauer pairs originated in Knörr–Robinson [2], and was explicit after Robinson–Staszewski [4, 1.1]. (We emphasize this fact because a comment in [1] obscured it.) This idea is developed in Robinson [3, Section 4], where a stronger conjecture is presented, and it is shown that when the stabilizer $N_G(Q,b)$ of a maximal element $(Q,b)$ of $\mathscr{C}(B)$ controls strong fusion in the self-centralizing Brauer subpairs of $(Q,b)$, the stronger conjecture implies that the number of irreducible ordinary characters in $B$ of a given defect $d$ equals the number of irreducible ordinary characters of defect $d$ lying in the block of $N_G(Q,b)$ in Brauer correspondence with $B$. Of course, the weight conjecture itself can be reformulated in the manner of [3, Section 4]. For instance, as has been observed by Puig, techniques in [2] can be used to express the weight conjecture as the assertion that the number of irreducible Brauer characters in $B$ is

$$l(B) = \sum_{\sigma} (-1)^{|\sigma|+1} l(B_\sigma)$$

summed over representatives of the $G$-orbits of chains $\sigma = (Q_1, b_1) < \cdots < (Q_n, b_n)$ in $\mathscr{C}(B)$, with $B_\sigma$ denoting the block of $N_G(\sigma)$ corresponding to $b_n$ as in [2, 3.1].

For the principal $p$-block $B_0$ of $G$, the Sylow $G$-poset $\mathscr{B}(B_0)$ may be identified with the Sylow $G$-poset $\mathscr{S}_p(G)$ consisting of the non-trivial $p$-subgroups of $G$. Symonds [5] proved Webb's conjecture that the orbit space $|\mathscr{S}_p(G)|/G$ is contractible. (Earlier, Thévenaz [6] had verified this conjecture in cases where a control of fusion condition was available, and had suggested to the author that the general case might succumb to Alperin's fusion theorem.) Symonds has asked whether the orbit space $|\mathscr{X}|/G$ is contractible for any upwardly closed $G$-subposet $\mathscr{X}$ of $\mathscr{S}_p(G)$. In Webb [9], reformulations in Knörr–Robinson [2] of the weight conjecture are examined via the Lefschetz invariant $\Lambda_G(\mathscr{S}_p(G))$, and in Webb [10], group cohomology is calculated using $\Lambda_G(\mathscr{S}_p(G))$. Both works employ the fact that, for any $G$-poset $X$, the Lefschetz invariant $\Lambda_G(X)$ depends only on the $G$-homotopy class of the $G$-space associated with $X$ (see below). Thus, questions of fusion in Sylow $G$-posets such as $\mathscr{C}(B)$ are of concern in $p$-local representation theory, and so are questions of $G$-homotopy invariants for these $G$-posets. For instance, we (are are surely not the first to) ask whether $\mathscr{C}(B)$ is $G$-contractible; in the case where $B = B_0$, this question is a special case of Symonds' question. Theorem 1 below reveals nothing new about $\mathscr{C}(B)$ because a stronger result was obtained for $\mathscr{B}(B)$ by J. Alperin and M. Broué, and likewise for $\mathscr{L}(B)$ by L. Puig; see Thévenaz [7, Section 48]. Theorem 3 below tells us, in particular, that if $|\mathscr{C}(B)|/G$ is acyclic, then it is contractible.

For the rest of this paper, the prerequisites are elementary. Given a Sylow $G$-poset $X$, we write $\lhd$ and $<$ for the anti-reflexive relations corresponding in the usual way to $\unlhd$ and $\leqslant$. When $x \unlhd y$ in $X$, we say that $y$ *normalizes* $x$. If $y$ is maximal subject to normalizing $x$, we call $y$ a *maximal normalizer* of $x$. Given $x \leqslant z$ in $X$, we say that $x$ is *fully normalized* in $z$ provided that $z$ contains a maximal normalizer of $x$. See

Thévenaz [7, Section 48] for some historical comments on the following result, in its original form due to Alperin. The presentation in [7] inspired the version here.

**Theorem 1** (Alperin's fusion theorem). *Let $X$ be a Sylow $G$-poset, let $x \leqslant s$ in $X$ with $s$ maximal, and let $g \in G$ such that $x^g \leqslant s$. Then there exist some $n$, and for $1 \leqslant i \leqslant n$, elements $s_i, t_i \in X$ with $s_i$ maximal, and elements $g_i \in N_G(t_i)$ such that*

(a) $s \geqslant t_i \leqslant s_i$, *and $t_i$ is fully normalized in both $s$ and $s_i$, and*

(b) $x^{g_1 \dots g_i} \leqslant t_i$, *and* $g = g_1 \dots g_n$.

*Proof.* We say that the elements $g_1, \dots, g_n$ (when they exist) *accomplish fusion* from $x$ to $x^g$ in $s$. We argue by induction on the depth of $x$ (the maximal length $r$ of a chain $x = x_0 < \cdots < x_r$ starting at $x$). Supposing the depth to be positive, choose $y \in X$ with $x \lhd y \leqslant s$, and let $z$ be a maximal normalizer of $x$ containing $y$. We have $z^h \leqslant s$ for some $h \in G$. Since $y^h \leqslant s$, we may, by induction, write $h$ as a product of elements $h_1, h_2, \dots$ accomplishing fusion from $y$ to $y^h$ in $s$. The elements $h_1, h_2, \dots$ necessarily accomplish fusion from $x$ to $x^h$ in $s$. By replacing $x$, $z$, $g$ with $x^h$, $z^h$, $h^{-1}g$, respectively, we may assume that $z \leqslant s$. A similar argument allows us to assume that $s^{g^{-1}}$ contains a maximal normalizer $z'$ of $x$. We have $z' = z^f$ for some $f \in N_G(x)$. By induction, we can write $gf$ as a product of elements $f_1, f_2, \dots$ accomplishing fusion from $z^{g^{-1}}$ to $z^f$ in $s^{g^{-1}}$. Noting that $x$ is (by our assumptions) fully normalized in $s$ and $s^{g^{-1}}$, we see that $f, f_1^g, f_2^g, \dots$ accomplish fusion from $x$ to $x^g$ in $s$.

Let us review, for a finite $G$-poset $X$, some well known constructions. Write $\mathrm{sd}(X)$ for the $G$-poset consisting of the chains in $X$ (partially ordered by the subchain relation). The $G$-sets $X$ and $\mathrm{sd}(X)$ comprise the vertices and the simplexes, respectively, of a $G$-simplicial complex $\Delta(X)$ whose associated polyhedron $|X|$ is a $G$-space. Since $\Delta(\mathrm{sd}(X))$ is the barycentric subdivision of $\Delta(X)$, there is a $G$-equivariant homeomorphism

$$\phi_X : |\mathrm{sd}(X)| \to |X|$$

linearly extending the function sending each element of $\mathrm{sd}(X)$ to its centroid in $|X|$. We have a canonical projection to the orbit space

$$\theta_X : |X| \to |X|/G$$

and a homeomorphism

$$\bar{\phi}_X : |\mathrm{sd}(X)|/G \to |X|/G$$

such that $\bar{\phi}_X \theta_{\mathrm{sd}(X)} = \theta_X \phi_X$. We also have a homeomorphism

$$\psi_X : |\mathrm{sd}(X)/G| \to |\mathrm{sd}(X)|/G$$

linearly extending the function on the orbit poset $\mathrm{sd}(X)/G$ such that the geometric realization of the $G$-orbit of an element $\zeta \in \mathrm{sd}(X)$ is sent to the $G$-orbit of the geometric realization of $\zeta$ (see Thévenaz [6, Section 1]). Let

$$\rho_X : |\mathrm{sd}(X)| \to |\mathrm{sd}(X)/G|$$

be the projection linearly extending the canonical surjection $\mathrm{sd}(X) \to \mathrm{sd}(X)/G$. It is easy to check that $\psi_X \rho_X = \theta_{\mathrm{sd}(X)}$; hence

$$\theta_X \phi_X = \bar{\phi}_X \psi_X \rho_X.$$

We shall need a lemma explaining how suitable paths in the orbit space $|X|/G$ of a finite $G$-poset $X$ may, up to homotopy, be lifted to $|X|$. First, let us discuss homotopy classes of paths in a finite simplicial complex $K$. It is well known (and easily proved) that for vertices $u, v \in K$, any path from $u$ to $v$ in the polyhedron $|K|$ is homotopic to a path $\alpha$ in the 1-skeleton of $|K|$ such that the preimage under $\alpha$ of the 0-skeleton of $|K|$ is finite. (Homotopies of paths are to preserve end-points.) Given a simplex $\{x, y\}$ in $K$ (allowing the possibility that $x = y$), we choose a path $\sigma(x, y)$ from $x$ to $y$ in $|K|$ whose image is confined to the geometric simplex $|\{x, y\}|$ (which has dimension 0 or 1). The homotopy class of $\sigma(x, y)$ is independent of any choice. We shall sometimes write expressions of the form $\sigma(x_0, x_1)\sigma(x_1, x_2)\dots\sigma(x_{m-1}, x_m)$ where each $\{x_{i-1}, x_i\}$ is a simplex in $K$. Such an expression denotes a concatenation of the paths $\sigma(x_0, x_1)$, $\sigma(x_1, x_2), \dots, \sigma(x_{m-1}, x_m)$, and is well defined up to homotopy. Observe that, given a simplex $\{x, y, z\}$ in $K$ (allowing the possibility of repetitions), we have a homotopy $\sigma(x, y)\sigma(y, z) \simeq \sigma(x, z)$.

**Lemma 2.** *Given a finite $G$-poset $X$, and elements $x, y \in X$, then any path in $|X|/G$ from $\theta_X(x)$ to $\theta_X(y)$ is homotopic to a path of the form*

$$\theta_X(\sigma(z_0, z_1)\sigma(z_1, z_2)\dots\sigma(z_{2m-1}, z_{2m}))$$

*where $x = z_0 \geqslant z_1 \leqslant z_2 \geqslant \cdots \leqslant z_{2m-2} \geqslant z_{2m-1} \leqslant z_{2m}$ and $\theta_X(z_{2m}) = \theta_X(y)$. Furthermore, we may insist that the elements $z_1, z_3, \dots, z_{2m-1}$ are minimal, and that the elements $z_2, z_4, \dots, z_{2m-2}$ are maximal.*

*Proof.* We have $(x) \in \mathrm{sd}(X)$, and $\rho_X((x)) \in \mathrm{sd}(X)/G$. Consider a path in $|\mathrm{sd}(X)/G|$ of the form

$$\bar{\mu} = \sigma(\rho_X(\zeta_0), \rho_X(\zeta_1))\sigma(\rho_X(\zeta_1), \rho_X(\zeta_2))\dots\sigma(\rho_X(\zeta_{2m-1}), \rho_X(\zeta_{2m}))$$

where $\rho_X((x)) = \rho_X(\zeta_0) \geqslant \rho_X(\zeta_1) \leqslant \rho_X(\zeta_2) \geqslant \cdots \leqslant \rho_X(\zeta_{2m}) = \rho_X((y))$, and each $\zeta_i \in \mathrm{sd}(X)$. In view of the identity $\theta_X \phi_X = \bar{\phi}_X \psi_X \rho_X$, together with our above comments about paths, the main assertion will follow when we have shown that $\bar{\mu}$ lifts via $\rho_X$ to a path in $|\mathrm{sd}(X)|$. We can write $(x) = \eta_0 \geqslant \eta_1 \leqslant \eta_2 \geqslant \cdots \leqslant \eta_{2m}$ where each $\eta_i$

is a $G$-conjugate of $\zeta_i$. Putting

$$\mu = \sigma(\eta_0, \eta_1)\sigma(\eta_1, \eta_2) \dots \sigma(\eta_{2m-1}, \eta_{2m})$$

then $\bar{\mu} = \rho_X \mu$, and the main assertion is proved.

For $1 \leqslant j \leqslant m-1$, let $z'_{2j}$ be a maximal element of $X$ containing $z_{2j}$. Then

$$\begin{aligned}\sigma(z_{2j-1}, z_{2j})\sigma(z_{2j}, z_{2j+1}) &\simeq \sigma(z_{2j-1}, z'_{2j})\sigma(z'_{2j}, z_{2j})\sigma(z_{2j}, z'_{2j})\sigma(z'_{2j}, z_{2j+1}) \\ &\simeq \sigma(z_{2j-1}, z'_{2j})\sigma(z'_{2j}, z_{2j+1})\end{aligned}$$

and so we may insist that the elements $z_{2j}$ are maximal. Similarly, we may insist that $z_i$ is minimal for odd $i$.

**Theorem 3.** *Given a Sylow $G$-poset $X$, then the orbit space $|X|/G$ is simply connected.*

*Proof.* The transitivity of $G$ on the maximal elements of $X$ implies that $|X|/G$ is connected. Letting $s$ be a maximal element, we take $\theta_X(s)$ to be the base-point of $|X|/G$. Lemma 2 tells us that any element of the fundamental group $\pi_1(|X|/G)$ is the homotopy class of a path of the form

$$\theta_X(\sigma(x_0, x_1)\sigma(x_1, x_2) \dots \sigma(x_{2m-1}, x_{2m}))$$

where $s = x_0 \geqslant x_1 \leqslant x_2 \geqslant \cdots \leqslant x_{2m}$, each $x_i \in X$, and each $x_{2j}$ is maximal. It suffices to show that $\theta(\sigma(x_{2j-2}, x_{2j-1})\sigma(x_{2j-1}, x_{2j}))$ is null-homotopic. This is equivalent to the assertion that, given $x \leqslant s \geqslant x^g$ with $g \in G$, and writing $\sigma := \sigma(x, s)\sigma(s, x^g)$, then the closed path $\theta_X(\sigma)$ (based at $\theta_X(x)$ ) is null-homotopic.

Let $n$ and the elements $s_i$, $t_i$, $g_i$ be as in Theorem 1. Since $\sigma$ is homotopic to a path passing consecutively through the points $x, s, x^{g_1}, s, \dots, x^{g_1 \cdots g_{n-1}}, s, x^g$, we may assume that $n = 1$. The element $t := t_1$ is fixed by $G$. We have homotopic paths

$$\sigma \simeq \sigma(x, t)\sigma(t, s)\sigma(s, t)\sigma(t, x^g) \simeq \sigma(x, t)(\sigma(t, x))^g$$

whose composites with $\theta_X$ are manifestly null-homotopic.

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