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Analysis of Lagrangian Lower Bounds for a Graph Partitioning Problem

Gajendra K. Adil, Jay B. Ghosh,

ANALYSIS OF LAGRANGIAN LOWER BOUNDS FOR A GRAPH PARTITIONING PROBLEM

GAJENDRA K. ADIL

Department of Mechanical and Industrial Engineering, University of Manitoba, Winnipeg, Canada

JAY B. GHOSH

Faculty of Business Administration, Bilkent University, Ankara, Turkey, jghosh@squ.edu.om

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Recently, Ahmadi and Tang (1991) demonstrated how various manufacturing problems can be modeled and solved as graph partitioning problems. They use Lagrangian relaxation of two different mixed integer programming formulations to obtain both heuristic solutions and lower bounds on optimal solution values. In this note, we point to certain inconsistencies in the reported results. Among other things, we show analytically that the first bound proposed is trivial (i.e., it can never have a value greater than zero) while the second is also trivial for certain sparse graphs. We also present limited empirical results on the behavior of this second bound as a function of graph density.

A recent paper by Ahmadi and Tang (1991) demonstrates how various manufacturing problems such as the formation of group technology cells, the loading of tools or operations on a bank of identical flexible machines, and the design of VLSI circuits can be effectively modeled as partitioning problems on an undirected graph. This latter problem essentially involves dividing a graph into a specified number of nonempty components such that the total weight of all edges spanning two distinct components is minimized. For its solution, Ahmadi and Tang propose two different mixed integer programming formulations and their Lagrangian relaxations. The relaxations yield heuristics as well as lower bounds on the optimal solution value. The reported computational results seem to indicate that the heuristics and the lower bounds both perform reasonably well.

In this note, we point to several inconsistencies in the earlier work. We first show analytically that, while the first formulation is correct, the bound that it yields is trivial in that it can never have a positive value. We then observe that the second formulation is not valid but the bound derived from it still is. However, we go on to show that even a stronger version of this bound (which is derived after appropriately correcting the formulation) is also trivial for a class of sparse graphs. Finally, we report the results of a limited empirical study, which trace the strength of the second bound as a function of graph density.

1. PROBLEM DEFINITION

We are given an undirected graph $\mathbf{G} = (\mathbf{N}, \mathbf{A})$, where $\mathbf{N}$ is an index set of the nodes numbered 1 through N and $\mathbf{A}$ is a set of node pairs representing the edges. If there exists an edge between nodes i and j, $i, j \in \mathbf{N}$ and $i < j$, then the pair (i, j) denotes that edge. Associated with each edge (i, j), $(i, j) \in \mathbf{A}$, is a weight w_{ij}, $w_{ij} > 0$.

We let $(\mathbf{N}_1, \ldots, \mathbf{N}_M)$ be an M-way partition of $\mathbf{N}$ for any M, $2 \leq M \leq N$, and consider it feasible if for all k, $1 \leq k \leq M$, the cardinality $|\mathbf{N}_k|$ of $\mathbf{N}_k$ satisfies the inequalities $1 \leq |\mathbf{N}_k| \leq c$ (where c, $c \geq 1$, is a specified size limit). The partition divides $\mathbf{G}$ into M components such that a component $\mathbf{G}_k$, $1 \leq k \leq M$, is given by $\mathbf{G}_k = (\mathbf{N}_k, \mathbf{A}_k)$, where $\mathbf{A}_k = \{(i, j) : (i, j) \in \mathbf{A} \text{ and } i, j \in \mathbf{N}_k\}$. The cutset $\mathbf{C}$ induced by this partition is the set of the edges in $\mathbf{A}$ that span two distinct components and is given by $\mathbf{C} = \mathbf{A} - \cup_{1 \leq k \leq M} \mathbf{A}_k$. The objective in the graph partitioning problem is to find a feasible partition which minimizes $\sum_{(i,j) \in \mathbf{C}} w_{ij}$.

Note that the graph partitioning problem is strongly NP-hard (Garey and Johnson 1979). Also note that it admits a feasible partition if and only if $N \leq Mc$, and further that the Ahmadi-Tang formulations do not guarantee that an optimal partition contains exactly M nonempty components unless $N > (M - 1)c$. For our purposes, we will thus assume that $(M - 1)c < N \leq Mc$.

2. FIRST FORMULATION AND RELAXATION

Define x_{ik} to be 1 if node i, $i \in \mathbf{N}$, belongs to component k, $1 \leq k \leq M$, and 0 otherwise. Similarly, define y_{ij} to be 1 if nodes i and j, $(i, j) \in \mathbf{A}$, belong to two different components. The first Ahmadi-Tang mixed integer programming formulation—call it (IP1) and its solution value V(IP1)—is as follows:

$$\mathrm{V(IP1)} = \min \sum_{(i,j) \in \mathbf{A}} w_{ij} y_{ij}$$

subject to

$$\sum_{1 \leq k \leq M} x_{ik} = 1 \quad \text{for all } i,\ i \in \mathbf{N}, \tag{1}$$

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$$\sum_{i \in N} x_{ik} \leqslant c \quad \text{for all } k,\ 1 \leqslant k \leqslant M, \tag{2}$$

$$y_{ij} \geqslant x_{ik} - x_{jk} \quad \text{for all } i, j,\ (i, j) \in \mathbf{A}, \text{ and all } k,\ 1 \leqslant k \leqslant M, \tag{3}$$

$$x_{ik} \in \{0, 1\} \quad \text{for all } i,\ i \in \mathbf{N}, \text{ and all } k,\ 1 \leqslant k \leqslant M, \tag{4}$$

$$y_{ij} \geqslant 0 \quad \text{for all } i, j,\ (i, j) \in \mathbf{A}. \tag{5}$$

A Lagrangian relaxation of (IP1), call it ($LR1_\lambda$), is obtained by dualizing the constraint set (3) through the use of multipliers λ_{ijk}, $(i, j) \in \mathbf{A}$ and $1 \leqslant k \leqslant M$, such that $\lambda_{ijk} \geqslant 0$. It is not necessary for us to write ($LR1_\lambda$) explicitly; the interested reader may refer to the Ahmadi and Tang (1991) paper. It suffices to observe that ($LR1_\lambda$) separates into two subproblems: a trivial one involving the y_{ij} and another involving the x_{ik}, which is a simple transportation problem. Let V($LR1_\lambda$) be the value of an optimal solution to ($LR1_\lambda$) for the given multiplier vector λ. The Lagrangian dual of (IP1)—call it (LD1) and its solution value V(LD1)—maximizes V($LR1_\lambda$) over all λ. It is well known (see, for example, Fisher 1981 or Nemhauser and Wolsey 1988) that V($LR1_\lambda$) $\leqslant$ V(LD1) $\leqslant$ V(IP1). The Ahmadi and Tang (1991) paper finds a solution to (LD1) using a subgradient optimization algorithm. Let V($\underline{\text{LD1}}$) be the solution value delivered by that algorithm. Clearly, V($\underline{\text{LD1}}$) $\leqslant$ V(LD1) $\leqslant$ V(IP1).

Now consider the continuous relaxation of (IP1), call it (LP1), where the 0-1 constraint set, (4), is replaced by the following:

$$0 \leqslant x_{ik} \leqslant 1 \quad \text{for all } i,\ i \in \mathbf{N}, \text{ and all } k,\ 1 \leqslant k \leqslant M. \tag{6}$$

Let V(LP1) be the optimal solution value of (LP1). We now give the following result.

Result 1. *V($\underline{LD1}$) $\leqslant$ V(LP1) $=$ 0 for any* **G** *as long as* $N \leqslant Mc$.

Proof. We prove the result by giving a feasible solution to (LP1) that has a solution value equal to 0 (i.e., it also happens to be optimal). This solution is:

$$x_{ik} = 1/M \quad \text{for all } i,\ i \in \mathbf{N}, \text{ and all } k,\ 1 \leqslant k \leqslant M, \text{ and}$$

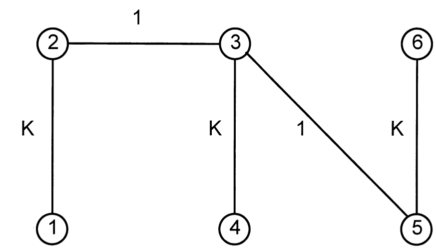
$$y_{ij} = 0 \quad \text{for all } i, j,\ (i, j) \in \mathbf{A}.$$

It is trivially seen that the above solution satisfies the constraint sets (5) and (6). It is similarly easy to see that it also satisfies (1) and (3). To see that it satisfies (2) as well, note that the LHS of (2) equals N/M and that this, under the stated condition (which is needed for the existence of a feasible partition anyway), is less than or equal to c (i.e., the RHS).

Let us now return to ($LR1_\lambda$). It is evident that ($LR1_\lambda$) exhibits the so-called integrality property (see Fisher 1981 or Nemhauser and Wolsey 1988, again), i.e., V($LR1_\lambda$) does not decrease when (4) is replaced with (6). As a consequence of this property, we must have V(LD1) = V(LP1). Hence, V($\underline{\text{LD1}}$) $\leqslant$ V(LD1) $\leqslant$ V(LP) = 0. □

Thus, the lower bound derived from the first Ahmadi-Tang formulation cannot have a positive value and be of any use in an exercise such as the performance evaluation of a heuristic. This finding contradicts the computational results reported in the Ahmadi and Tang (1991) paper, sixth column of their Table 1.

Figure 1. Example graph with edge weights.



3. SECOND FORMULATION AND RELAXATION

This time retain the x_{ik} but remove the y_{ij}. Instead, define z_{ijk} to be 1 if nodes i and j, $i, j \in \mathbf{N}$, belong to the same component k, $1 \leqslant k \leqslant M$, and 0 otherwise. For notational convenience, define the set $\mathbf{D}_i$ of the neighboring nodes of node i, $i \in \mathbf{N}$, as $\mathbf{D}_i = \{j : (i, j) \text{ or } (j, i) \in \mathbf{A}\}$. After minor modifications (which are needed for clarity), the second Ahmadi-Tang mixed integer programming formulation—call it (IP2) and its solution value V(IP2)—can be written as follows:

$$V(IP2) = \min \sum_{(i,j) \in \mathbf{A}} w_{ij} \left[1 - \sum_{1 \leqslant k \leqslant M} z_{ijk} \right],$$

subject to

$$\sum_{j > i, j \in \mathbf{D}_i} z_{ijk} + \sum_{j < i, j \in \mathbf{D}_i} z_{jik} \leqslant (c - 1) x_{ik} \quad \text{for all } i,\ i \in \mathbf{N}, \text{ and all } k,\ 1 \leqslant k \leqslant M, \tag{7}$$

$$z_{ijk} \in \{0, 1\} \quad \text{for all } i, j (i, j), \in \mathbf{A}, \text{ and all } k,\ 1 \leqslant k \leqslant M, \tag{8}$$

and (1) and (4) as above.

The above formulation is, however, incorrect. To see this, refer to Figure 1, which shows a 6-node graph with weights on the edges (the same graph used by Ahmadi and Tang (1991)). Suppose that $M = 2$ and $c = 3$. The true optimal solution value is $K + 1$, given, for instance, by the partition ({1, 2, 6}, {3, 4, 5}). However, the solution given by $x_{11} = x_{21} = x_{32} = x_{42} = x_{52} = x_{62} = 1$ and $z_{121} = z_{342} = z_{352} = z_{562} = 1$ (with all the other variables set equal to 0) satisfies constraints (1), (4), (7), and (8), and is thus feasible with respect to (IP2). But this solution represents the infeasible partition ({1, 2}, {3, 4, 5, 6}) which yields a better-than-optimal solution value of 1.

To correct (IP2), we replace (7) and (8) with the constraint sets (9), (10), and (11) shown below, and call the resulting problem (IP2C) and its solution value V(IP2C):

$$\sum_{j>i, j\in \mathbf{N}} z_{ijk} + \sum_{j<i, j\in \mathbf{N}} z_{jik} \le (c-1)x_{ik}$$
$$\text{for all } i,\ i\in \mathbf{N}, \text{ and all } k,\ 1\le k\le M, \tag{9}$$

$$z_{ijk} \ge x_{ik} + x_{jk} - 1$$
$$\text{for all } i, j,\ i, j\in \mathbf{N} \text{ and } i<j, \text{ and all } k,\ 1\le k\le M, \tag{10}$$

$$z_{ijk} \in \{0, 1\}$$
$$\text{for all } i, j,\ i, j\in \mathbf{N} \text{ and } i<j, \text{ and all } k,\ 1\le k\le M. \tag{11}$$

(IP2C) thus has the same objective function as (IP2) but comprises the constraint sets (1), (4), (9), (10), and (11). Notice that in (9) the sum is taken over all (i, j) such that i, $j \in \mathbf{N}$ and $i < j$, in contrast to (7) where the sum is taken only over $(i, j) \in \mathbf{A}$, to ensure that a partition's size limit is properly enforced. Similarly, (10) is added to ensure that z_{ijk} is 1 if and only if x_{ik} and x_{jk} are both 1. Finally, notice that (11) in (IP2C) can in fact be replaced by its continuous relaxation, which is given below:

$$0 \le z_{ijk} \le 1 \quad \text{for all } i, j,\ i, j\in \mathbf{N} \text{ and } i<j, \text{ and all } k,\ 1\le k\le M. \tag{12}$$

We now discuss the Lagrangian relaxation of (IP2) as obtained in the Ahmadi and Tang (1991) paper, where the constraint set (7) is dualized using multipliers μ_{ik}, $i \in \mathbf{N}$ and $1 \le k \le M$. (Again, we do not write the relaxation explicitly; the interested reader may refer to Ahmadi and Tang 1991.) We simply note that it separates into two subproblems: a trivial one involving the z_{ijk} and another involving the x_{ik} that contains only GUB constraints. By our convention, we call this relaxation (LR2_μ) and its optimal solution value V(LR2_μ) for a given multiplier vector μ. The associated dual problem is called (LD2) and its solution value V(LD2). The solution value returned by the subgradient optimization algorithm is similarly called V($\underline{\text{LD2}}$).

It is easy to show at this point that, although (IP2) is incorrect, V(LR2_μ), V(LD2) and V($\underline{\text{LD2}}$) remain valid lower bounds on the optimal solution value V(LP2C) of the continuous relaxation (LP2C) of (IP2C), which is obtained by replacing (4) with (6) and (11) with (12). Notice also that (LR2_μ), with only GUB constraints, exhibits the integrality property mentioned earlier. This implies that V(LD2) = V(LP2) and thus that V($\underline{\text{LD2}}$) $\le$ V(LD2) = V(LP2) $\le$ V(LP2C) $\le$ V(IP2C).

We now proceed to analyze the quality of the Lagrangian lower bound obtainable from the correct formulation (IP2C) by dualizing in the same manner as in (IP2), i.e., by dualizing (9) and (10) in this case. Notice that the integrality property still holds, because the relaxed problem contains only GUB constraints as before. Thus, if we let (LD2C) with solution value V(LD2C) be the Lagrangian dual of (IP2C), then V(LD2C) = V(LP2C). Because V($\underline{\text{LD2}}$) $\le$ V(LD2) $\le$ V(LP2C), any upper bound we derive on V(LP2C), or equivalently V(LD2C), can be imposed upon the Ahmadi-Tang bound V($\underline{\text{LD2}}$) as well. Let δ_i, $\delta_i = |\mathbf{D}_i|$, be the degree of node i, $i \in \mathbf{N}$. Now define the maximum degree of graph $\mathbf{G}$ to be Δ, where $\Delta = \max_{i\in\mathbf{N}} \{\delta_i\}$. We now give a result for the case when Δ is sufficiently small.

RESULT 2. *V($\underline{LD2}$) $\le$ V(LP2C) $\le$ 0 for any* $\mathbf{G}$ *with* $\Delta \le c-1$.

PROOF. Once again, we prove the result by providing a feasible solution to (LP2C) with value 0 (which may not be optimal). This solution is:

$$x_{ik} = 1/M \text{ for all } i,\ i\in\mathbf{N}, \text{ and all } k,\ 1\le k\le M,$$
$$z_{ijk} = 0 \text{ for all } i, j,\ i, j\in\mathbf{N},$$
$$i<j \text{ and } (i, j)\notin \mathbf{A}, \text{ and all } k,\ 1\le k\le M, \text{ and}$$
$$z_{ijk} = 1/M \text{ for all } i, j,\ (i, j)\in\mathbf{A}, \text{ and all } k,\ 1\le k\le M.$$

This solution clearly satisfies (6) and (12). It is also easy to see that it satisfies (1). The RHS of (10) is $2/M - 1 \le 0$ (since $M \ge 2$), the LHS is 0 or $1/M$, and thus it is satisfied as well. The LHS of (9) is $\delta_i/M \le \Delta/M$, the RHS is $(c-1)/M$, and it too is satisfied under the stated condition. Finally, the solution value can be seen to be 0. □

Notice that the maximum degree Δ of graph $\mathbf{G}$ is at least as much as its average degree (which is proportional to the graph's density). A high Δ does not necessarily imply a dense graph, but a low Δ implies a sparse graph. Thus, Result 2 indicates that the Lagrangian lower bound is going to be trivial for a class of sparse graphs when Δ is smaller than c. But we do not know yet what happens to V(LP2C), or equivalently V(LD2C), for sparse graphs in general. We, therefore, undertake a small computational study. Our study complements the one in the Ahmadi-Tang paper, which is performed over relatively dense graphs.

Let p be the probability that an edge exists between nodes i and j, $i, j \in \mathbf{N}$; notice that p is the expected density of a randomly generated graph. Keeping c fixed at 6, we consider $M = 2$, 4, and 6, and $p = 0.2$, 0.3, and 0.4. This leads to 9 cases; in each case, $N = Mc$ as in Ahmadi and Tang (1991). Ten graphs are generated randomly for each case, with the edge weights drawn from a uniform distribution over (1, 10). (LP2C) is solved for each graph (using a commercial LP solver on a UNIX workstation). We want to find out what percentage of the times in each scenario V(LP2C) is nonpositive. Table 1 records our results and shows what percentage of the times in each scenario the maximum degree Δ of the graph is less than c, i.e., when V(LP2C) is provably nonpositive. Notice that, for a fixed expected density p, if M is increased, we get more and more positive values for V(LP2C). A similar behavior is observed if, for a fixed M (and thus N), the expected density p is increased.

Table 1. Test of the second Lagrangian lower bound with the component size limit (c) fixed at 6.

Percentage of Cases where the Bound is Trivial: Actual [Provable]			
Edge Existence Probability (p)	Number of Graph Components (M)		
	2	4	6
0.2	100	100	80
	[80]	[0]	[0]
0.3	100	80	0
	[40]	[0]	[0]
0.4	100	0	0
	[20]	[0]	[0]

4. CONCLUSION

We show that the first Ahmadi-Tang Lagrangian lower bound is trivial. The second bound, however, holds promise for graphs that are sufficiently dense. It appears that the Ahmadi-Tang computations cover such graphs only. The reported results are encouraging. Also, despite the poor quality of the associated lower bound, the first Lagrangian heuristic continues to perform reasonably well, as has been indicated by Ahmadi and Tang (1991) and as has been independently observed by us in a small computational study. To sum up, in using the Ahmadi-Tang approach, one could choose the first Lagrangian heuristic and the second lower bound. However, in case of the latter, one would have to pay close attention to the density of the graphs under consideration.

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