THREE MANAGEMENT POLICIES FOR A RESOURCE WITH PARTITION CONSTRAINTS

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Abstract

Management of a bufferless resource is considered under non-homogeneous demand consisting of one-unit and two-unit requests. Two-unit requests can be served only by a given partition of the resource. Three simple admission policies are evaluated with regard to revenue generation. One policy involves no admission control and two policies involve trunk reservation. A limiting regime in which demand and capacity increase in proportion is considered. It is shown that each policy is asymptotically optimal for a certain range of parameters. Limiting dynamical behavior is obtained via a theory developed by Hunt and Kurtz. The results also point out the remarkable effect of partition constraints.

Keywords: Resource management; partition constraints; loss networks; multirate networks; admission control; trunk reservation; heavy traffic; time-scale separation

AMS 1991 Subject Classification: Primary 60K30

1. Introduction

This paper investigates effective control policies for a bufferless resource that operates under non-homogeneous dynamic demand. The demand consists of requests of two different types, categorized by the number of resource units required for service. Management of the resource is subject to partition constraints: requests of each type can be serviced only by a block from an associated partition of the resource. We consider in detail the case when one type requires twice as many resource units as the other.

Partition constraints typically arise in time-division-multiplexed multirate communication systems, owing to certain operational limitations. An instance of the problem addressed in this paper arose in the global system for mobile communication (GSM). The system accommodates full-rate users, each of which requires a full-time-slot, as well as half-rate users, each of which requires a half-time-slot. Here a full-time-slot refers to a time-slot in each transmitted frame, and a half-time-slot refers to a time-slot in every other transmitted frame. A pair of half-time-slots can accommodate a full-rate user only if they form a full-time-slot, so the collection of all such pairs constitutes a resource partition for full-rate users.

We consider the following stochastic setting. Let \( \lambda_f \), \( \lambda_h \), and \( C \) be fixed positive numbers, and let \( \gamma \) be a positive scaling factor. There are two types of calls denoted by full-rate calls and half-rate calls. Full-rate calls arrive according to a Poisson process of rate \( \gamma \lambda_f \) and half-rate calls arrive according to a Poisson process of rate \( \gamma \lambda_h \). The two arrival processes are mutually...
The total number of available slots is $\lfloor \gamma C \rfloor$. A slot can be assigned either one full-rate call or at most two half-rate calls. There is no buffering, thus a call is blocked if it is not assigned a slot immediately upon its arrival. Blocked calls cannot be assigned later, and have no effect on the future evolution of the system. A slot is said to be occupied if it is assigned one full-rate call or two half-rate calls, partially occupied if it is assigned one half-rate call, or idle otherwise. A full-rate call is blocked if upon its arrival there are no idle slots, and a half-rate call is blocked if upon its arrival all slots are occupied. Calls can also be blocked in other circumstances depending on the admission policy, which is a decision mechanism to accept or reject an arriving call. For efficient use of capacity, an accepted half-rate call is assigned an idle slot only if there are no partially occupied slots at the time of its arrival.

Each accepted call remains in the system for the duration of its holding time, during which it maintains the same slot assignment. The holding time of a call is an exponentially distributed random variable with unit mean, independent of its type and the history of the system prior to its arrival. If accepted, each full-rate call generates revenue at rate $r_f$ and each half-rate call generates revenue at rate $r_h$ throughout the holding time.

A similar stochastic setting in which calls require either one or six resource units has been a subject of considerable interest in the context of ISDN communication systems. In that setting Ramaswami and Rao (1985) studied approximate call blocking probabilities in the absence of admission control. Reiman and Schmitt (1994) considered trunk reservation type admission policies as well, and studied effective methods to determine the blocking probabilities in the case when call types have vastly different time scales. Ross and Tsang (1989) focused on efficient methods to determine admission policies that maximize resource utilization.

In this paper effectiveness of an admission policy is measured with the revenue generated in the long term. We examine three policies which have desirable features such as simplicity and robustness to traffic parameters. These policies are evaluated in a limiting regime that corresponds to arbitrarily large values of the scaling factor $\gamma$, and it is shown that asymptotically each policy generates revenue at maximum rate for certain values of the parameters $(r_f, r_h)$. In addition to equilibrium properties, explicit descriptions of the transient system behavior are also obtained.

The first policy considered in the paper is trunk reservation for full-rate calls, under which a full-rate call is accepted whenever there is an idle slot, whereas a half-rate call is accepted only if the number of idle slots is larger than a reservation threshold $T(\gamma)$. Note that acceptance of a half-rate call does not depend on the availability of partially occupied slots. The reservation threshold grows unboundedly with $\gamma$ (i.e. $\lim_{\gamma \to \infty} T(\gamma) = \infty$), however slower than $\gamma$ itself (i.e. $\lim_{\gamma \to \infty} T(\gamma)/\gamma = 0$). The second policy, trunk reservation for half-rate calls, prescribes accepting a half-rate call unless all slots are occupied, and accepting a full-rate call only if the number of idle slots is larger than $T(\gamma)$. Finally we consider complete sharing under which no admission control is exercised, so that a call is accepted if there is enough capacity to accommodate it.

Trunk reservation has been studied extensively in stochastic settings that do not involve partition constraints. Miller (1969) showed that under homogeneous traffic a trunk reservation policy maximizes the rate of revenue generation among non-anticipative admission policies. If either the request size or the mean holding time varies with call type, such a conclusion holds in a limiting regime similar to the one considered here, as established by Hunt and Laws (1997). The work of Hunt and Laws (1997) is closely related to the work of Bean et al. (1995, 1997) which studies the limiting behavior of trunk reservation. All three papers are based on the theory developed in Hunt and Kurtz (1994) which provides a detailed description of...
determine the random processes \( X \) of

\[
denotes the equilibrium random vector. Given real numbers \( t \), essential definitions. For each

\[
\lim \sup_{\gamma \to \infty} \frac{J_t}{\gamma},
\]

The main contribution of the paper has two aspects. First, asymptotic optimality of the

\[
\lim \sup_{\gamma \to \infty} \frac{X_t}{\gamma}
\]

Theorem 1.2. Under trunk reservation for half-rate calls (TRH)

\[
\lim \sup_{\gamma \to \infty} \frac{J_t}{\gamma}
\]

\[
\lim \sup_{\gamma \to \infty} \frac{J_t}{\gamma}
\]

\[
\lim \sup_{\gamma \to \infty} \frac{X_t}{\gamma}
\]

Theorem 1.3. Under complete sharing (CS)

\[
\lim \sup_{\gamma \to \infty} \frac{X_t}{\gamma}
\]

In the remainder of this section we state the main results of the paper, starting with some

\[
\lim \sup_{\gamma \to \infty} \frac{J_t}{\gamma}
\]

Remarkable that complete sharing asymptotically achieves full priority for half-rate calls without

\[
\lim \sup_{\gamma \to \infty} \frac{X_t}{\gamma}
\]

Under trunk reservation for full-rate calls (TRF)

\[
\lim \sup_{\gamma \to \infty} \frac{J_t}{\gamma}
\]

Theorem 1.1. dynamical behavior via the theory of Hunt and Kurtz (1994). The need for trunk reservation. Second, a methodical approach is shown to identify the limiting

\[
\lim \sup_{\gamma \to \infty} \frac{X_t}{\gamma}
\]

Policies for a resource with partition constraints

\[
\lim \sup_{\gamma \to \infty} \frac{J_t}{\gamma}
\]
FIGURE 1: Typical trajectories that approximate the transient behavior of the system under (a) trunk reservation for full-rate calls, (b) trunk reservation for half-rate calls, and (c) complete sharing, in the case 
\[ C = \lambda f = \lambda h / 2 = 3. \]

We now briefly comment on the theorems. The vector \( x^* \) (respectively the vector \( x^* \)) characterizes an operating point at which the available capacity is used primarily to accommodate full-rate (half-rate) calls, leaving only the excess capacity for half-rate (full-rate) calls. Moreover half-rate calls are almost perfectly packed so that there is only a marginal number of partially occupied slots. If \( r_f \geq 2r_h \) (\( r_f \leq 2r_h \)) then such an operating point is almost optimal, and by Theorem 1.1 (Theorem 1.2) trunk reservation achieves asymptotic optimality by maintaining the system sufficiently close to it. By Theorem 1.3 the uncontrolled system tends to evolve around the same operating point as the system under the TRH policy, so that complete sharing is also asymptotically optimal if \( r_f \leq 2r_h \).

The partition constraint has a remarkable effect on the natural evolution of the system, as pointed out by Theorem 1.3: in the absence of partition constraints, it follows from Kelly (1986) that complete sharing results in limiting blocking probabilities of 
\[ 1 - q^2 \] and 
\[ 1 - q \] for full-rate and half-rate calls respectively, where \( q \) denotes the positive root of 
\[ \lambda f q^2 + (\lambda h / 2) q - C = 0 \] and \( (\cdot)^+ \) denotes \( \max(\cdot, 0) \). When the partition constraint is imposed, however, full-rate calls may experience a disproportionately large blocking probability, to the extent that they may be totally locked out of the system in the large \( \gamma \) limit.

We finally comment on the transient behavior of the system under the three admission policies. Figure 1 illustrates trajectories that well-approximate the process \( X_\gamma \) for large values of \( \gamma \), in the case 
\[ C = \lambda f = \lambda h / 2 = 3 \] and \( X_\gamma 0 \). An intuitive interpretation of these
Appendix.

Theorems 1.1, 1.2, and 1.3 respectively. Proofs of some auxiliary results are collected in the
advantage is significant enough so that eventually half-rate calls monopolize the entire system.

\[ t \]

\[ X_t \]

\[ \gamma \]

\[ f \]

\[ X_t(\gamma) \]

\[ M_t(\gamma) \]

\[ D_\gamma \]

\[ \lambda \]

\[ \sigma \]

\[ \mathbb{R}^+ \]

\[ G_t \]

\[ K_t \]

\[ F_t \]

\[ T_t \]

\[ \mathbb{N} \]

\[ a \]

\[ c \]

\[ h(\gamma) \]

\[ \int_{0}^{t} \]

\[ G_t + 3 \]

\[ X_t(\gamma) = \mathbb{N} \]

\[ X_t(\gamma) = 0 \]

\[ X_t(\gamma) = 3 \]

\[ X_t(\gamma) = 0 \]

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\[ X_t(\gamma) = 3 \]

\[ X_t(\gamma) = 0 \]
Let the random measure \( V_t \) for each \( t \) and \( \gamma \) be defined by
\[
\gamma \rightarrow \begin{cases} 
0 & \text{if } \nu\gamma (s) \leq 0, \\
\text{tanh}^{-1} \nu\gamma (s) & \text{if } 0 < \nu\gamma (s) < \infty, \\
\infty & \text{if } \nu\gamma (s) = \infty.
\end{cases}
\]
\( \tau \) is tight in the associated product space.

2.1. Convergence

Let \( Z \) be a random variable in the space \( L_2(\Omega, \mathcal{F}, \mathbb{P}) \) such that
\[
\int_{\Omega} |Z|^2 \, d\mathbb{P} < \infty.
\]
Define the sets
\[
\Delta_1 = \left\{ \omega \in \Omega : |Z(\omega)| \leq 1 \right\}, \\
\Delta_2 = \left\{ \omega \in \Omega : |Z(\omega)| > 1 \right\}.
\]
Then
\[
\mathbb{E}[Z^2] = \mathbb{E}[Z|\Delta_1] + \mathbb{E}[Z|\Delta_2],
\]
with the understanding that the integrals are 0 if \( \Delta_1 \) or \( \Delta_2 \) are empty. Therefore, for any \( \epsilon > 0 \), we have
\[
\mathbb{E}[Z^2] > \epsilon \quad \Rightarrow \quad \mathbb{E}[Z|\Delta_2] > \epsilon.
\]

Ethier and Kurtz (1986). Finally, Doob's inequality implies that the sequence
\[
\left( \int_{\Omega} |Z|^2 \, d\mathbb{P} \right)^{1/2}
\]
converges to the point
\[
\lim_{n \to \infty} \int_{\Omega} |Z|^2 \, d\mathbb{P} = \mathbb{E}[Z^2].
\]

In order to better understand the method used here, we start with some essential definitions.
Policies for a resource with partition constraints

(3)

xs(1)
xs(2)
xs(3)
xs(2)
xs(1)
xs(3)
xs(2)

λf
λh

FIGURE 2: Transition diagrams of (a) the process \((Y, X) \in \mathbb{R}^2 \times \mathbb{R} \) and (b) the process \((Y, X) \in \mathbb{R} \times \mathbb{R}^3 \), in the discussion of trunk reservation for full-rate calls.

We now characterize the weak limits of the sequence \((X^x, X^\gamma) : \gamma > 0 \) along convergent subsequences. Let \((x, \nu)\) denote such a limit, and consider first the characterization of the measure \(\nu\). Straightforward adaptation of Theorem 3 of Hunt and Kurtz (1994) yields that \(\nu\) satisfies

\[
\nu(\{0, t\} \times B) = \int_0^t \pi(x) (B) \, ds,
\]

where \(\pi(x)\) is an equilibrium distribution for a Markov process \((Y, X) : t \geq 0\) that takes values in \(E\) and has transition rates given by

\[
\begin{align*}
Y &\rightarrow Y + (0, -1, -1) \text{ at rate } \lambda f \{Y \in A_1\} \\
Y &\rightarrow Y + (0, +1, +1) \text{ at rate } \nu \{Y \in A_2\} \\
Y &\rightarrow Y + (-1, 0, 0) \text{ at rate } \lambda h \{Y \in A_3\} \\
Y &\rightarrow Y + (+1, 0, 0) \text{ at rate } 2 \nu \{Y > 0\}.
\end{align*}
\]

(2.2)

Here and in the rest of the paper it is understood that \(\pm \infty + k = \pm \infty\) for all \(k \in \mathbb{Z}\).

In particular \((Y, X) \in \mathbb{R}^2 \times \mathbb{R}^3\) are effectively two-dimensional Markov processes whose transition diagrams are illustrated by Figures 2(a) and 2(b) respectively.

The process \(Y, X\) is reducible due to the isolated states at infinity; in turn it admits multiple equilibrium distributions. The distribution \(\pi\) is therefore some convex combination of the...
function \( s \) decisions at time changes its value. In particular the instantaneous rates of various admission and allocation two processes separate; the feedback process reaches equilibrium before the system process to the large number of arrivals and departures. In the large

In contrast, within such intervals the feedback process takes on many different values due

Lemma 2.1. The following conditions hold for almost all \( s \)

The lemma is proved in the Appendix.

An analogue of part \( a \) of the same lemma follow by straightforward interpretation of Hunt and Kurtz (1994).

Some of the general ideas used above have analogues in recent work. Hunt and Laws (1997)

We now provide a characterization of the limit trajectory \( x \)

The collection \( \pi_x \) of Ethier and Kurtz (1986)

We adopt the following convention to represent

formally, we adopt the following convention to represent

some probability vector \( p_x \)

\( \lambda \)

An intuitive interpretation of the above description is as follows. For large values of \( X \) the normalized system process

γ

the equilibrium distributions of \( Y_x \) is ergodic, or an arbitrary distribution otherwise. Then

\( \pi_x \) denote the unique equilibrium

\( i \) \( \leq \) \( i \) \( \leq \) \( i \)

\( t \) \( \leq \) \( t \) \( \leq \) \( t \)

\( \leq \) \( \leq \) \( \leq \)

\( + \) \( + \) \( + \)

\( \times \) \( \times \) \( \times \)

\( \Rightarrow \) \( \Rightarrow \) \( \Rightarrow \)

\( - \) \( - \) \( - \)

\( \gamma \) \( \gamma \) \( \gamma \)

\( + \) \( + \) \( + \)

\( \Rightarrow \) \( \Rightarrow \) \( \Rightarrow \)

\( - \) \( - \) \( - \)

\( \gamma \) \( \gamma \) \( \gamma \)

\( + \) \( + \) \( + \)

\( \Rightarrow \) \( \Rightarrow \) \( \Rightarrow \)

\( - \) \( - \) \( - \)

\( \gamma \) \( \gamma \) \( \gamma \)
If $x$ satisfies (2.5)–(2.7) then it is differentiable at almost all $t$ of a function point for the solutions of (2.4)–(2.7). Let

2.2. ODE representation of limit trajectories

Policies for a resource with partition constraints

TABLE 1: Valid expressions for $\lambda_y$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_y(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\in$</td>
<td>+ $\infty$</td>
</tr>
<tr>
<td>$\times{+\infty}$</td>
<td>$\times{+\infty}$</td>
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<tr>
<td>$1$</td>
<td>+ $\times{+\infty}$</td>
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</table>

In particular, if $x$ is ergodic then the long-term rate of up-jumps of $Y_{xt}$ is necessarily equal to the long-term rate of up-jumps of $Y_{xt}$.

$\pi_y$ is ergodic. We consider each row of Table 1 separately.

$\pi_y$ is ergodic then the long-term rate of down-jumps of $Y_{xt}$ is necessarily equal to the long-term rate of down-jumps of $Y_{xt}$.

$x_t$ is ergodic. We consider each row of Table 1 separately.
For any absolutely continuous function \( f \) such that \( f(0) = 0 \) and \( f(t) > 0 \) for every \( t \geq 0 \), the derivative \( \dot{x}_s \) is defined by \( \dot{x}_s(t) = \lim_{h \to 0} \frac{x_s(t+h) - x_s(t)}{h} \). According to Remark \( 2.1 \), \( \dot{x}_s \) should vanish.

Proof. By the arguments stated earlier, one has \( x_t(t) = x_s(t) \) for every \( t \geq 0 \), and \( \dot{x}_s \) should vanish. The lemma now follows by Lemma \( 2.2 \).
Consider the following two cases.

1. \( \dot{x}_t \) is satisfied only if \( \dot{x}_t \) holds:
   - **Lemma 2.5.** For almost all \( t \)
     - This establishes the
     - \( \dot{x}_t \) holds:
     - **Lemma 2.7.** For almost all \( t \)

2. \( \dot{x}_t \) is ergodic only if \( xt_1 \) and \( \dot{x}_t \) holds:
   - **Lemma 2.2.** If \( xt_1 \) and \( \dot{x}_t \)
   - \( xt_2 \) holds:
   - **Lemma 2.3.** If \( xt_1 \) and \( \dot{x}_t \)

Appendix.

Theorem 2: The process \( Y xt_1 \)
- By definition, \( Y xt_1 \) such that \( xt_1 \)
- The condition \( Y xt_1 \)

Hence, by Lemma 2.2 the condition \( Y xt_1 \)
- \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)
- \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)

Proof.
- \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)
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Note that \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)
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Proof.
- \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)
- \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)

\[ \dot{x}_t \in \{ + \infty \} \cap \{ 3 \} \]

By definition, \( \dot{x}_t \) is a condition for \( xt_1 \) such that \( xt_2 \)
- \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)
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- \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)
- \( xt_1 \) is ergodic, and by Lemma 2.2 the condition \( Y xt_1 \)
The above proof indicates that the method employed here may not identify a given initial condition. Here we establish only the weaker claim that each limit trajectory is ergodic, and Lemma 2.2 implies that $\lim_{t \to \infty} f(t) = 0$ is now satisfied only if $h(x) < \lambda$. If $\lambda = 1$, then $h(x) < 1$, and for some probability vector $q \in \mathbb{R}^2$ is now satisfied only if $h(x) < \lambda$. If $\lambda = 1$, then $h(x) < 1$, and for some probability vector $q \in \mathbb{R}^2$.

Lemmas 2.3, 2.4, 2.5, and 2.7 can be shown to identify a unique limit trajectory issued from some probability vector $q \in \mathbb{R}^2$. The process $x_t$ tends to remain in the vicinity of some probability vector $q \in \mathbb{R}^2$. However, the derivative for some probability vector $q \in \mathbb{R}^2$ elsewhere in the paper also (see the proof of Lemma 3.7), however the derivative for some probability vector $q \in \mathbb{R}^2$.
and the arbitrariness of $\epsilon$

For any admission policy (Policies for a resource with partition constraints

807

conditions (2.4) and (2.6) $\lim_{t \to \infty} sup\; \epsilon

Proof of Theorem 1.1.

Proof. $h$

such that $E\in$ and $\epsilon>$ $xt$

and $\epsilon>xt$

respectively. In

$\epsilon>xt$ for almost all such

This completes the proof of the lemma.

$\epsilon>xt$ to choose a

$\epsilon>xt$ and condition (2.6) to choose a

$\epsilon>xt$ for all

$\epsilon>xt$.

$\epsilon>xt$.

$\epsilon>xt$.

$\epsilon>xt$.
If $x$ satisfies (2.5)–(2.7) then it is differentiable at almost all $t$. In effect, (2.8)–(2.10) imply

$$
\frac{d}{dt} x(t) \in \mathbb{E}.
$$

If $x(t)$ is a martingale such that

$$
\int_{0}^{t} \int_{0}^{s} x(s) \, ds \, dt \in \mathbb{E},
$$

then it is differentiable at almost all $t$. In effect, (2.8)–(2.10) imply

$$
\frac{d}{dt} x(t) \in \mathbb{E}.
$$

References


| TABLE 2: Valid expressions for $\pi$ for such a non-observable process with arbitrary $A$ in $(0, \infty)$ and $\lambda$ in $Z$.

<table>
<thead>
<tr>
<th>$\pi$</th>
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<td>$\pi$</td>
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<td>$\pi$</td>
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Adapted from the text, this table indicates the valid expressions for $\pi$ for a non-observable process with arbitrary $A$ in $(0, \infty)$ and $\lambda$ in $Z$. The specific expressions are not detailed in the image but are implied by the context of the text.
Lemma 3.2. By the following lemmas. The proofs of Lemmas 3.2 and 3.3 are identical to the proofs of Lemma 2.1. In fact, they hold for any of the following two conditions

\[ \lambda + f, \pi + h \]

The following two lemmas are used in the proof of Lemma 3.7, and they are proved in the Appendix. The proof of Lemma 3.7 will be given in the next section. The lemma now follows by substituting the expressions for probabilities and in (2.8)–(2.10).
Lemma 3.8 and a direct adaptation of Lemma 7.2 of Alanyali and Hajek (1997). Since

$$X_t > \tau(\epsilon)$$

cases.

Lemma 3.8.

The claim that \( \lim x_t \) exists. Otherwise one can choose

$$\lambda h x_t$$

be such that

$$2a n d \lim h x_t$$

is not ergodic for any

$$x_t 0. Let

$$3$$

such that

$$\delta x_t$$

h

$$\delta x_t$$

\( 1 \to \infty \)

and therefore \( \lim h x_t \).

This completes the proof.
Under complete sharing, \( r \) is tight. The limit \( \lambda \) for all \( \gamma \) is an equilibrium distribution of a Markov process, which takes values in \( (0, \pi \gamma) \). This completes the proof.
Lemma 4.2. For almost all $t_1$ and $t_2$ respectively, except that Lemma 2.2 is replaced by Lemma 4.1.

The lemma now follows by substituting the expressions for probabilities $\pi$ in (2.8)–(2.10).
Lemma 4.1, ergodic for any \( \lambda \).

We next establish a monotonicity property of the probability

\begin{align*}
(\text{a}) & \quad \pi(h, 0) = \pi(h, 1) = 0, \\
(\text{b}) & \quad \pi(h, i) = 0 \quad \text{for each } h \leq 0.
\end{align*}

\textbf{Proof.} If \( C = 0 \), the case when \( h = 0 \) holds:

\[ \pi(h, 0) = \pi(h, 1) = 0, \]

\[ \pi(h, i) = 0 \quad \text{for each } h \leq 0. \]

If \( C \neq 0 \), then \( \pi(h, 0) \) and \( \pi(h, 1) \) are not ergodic for

\[ \lambda = \frac{1}{3}, \quad \lambda = \frac{2}{3}. \]

\textbf{Lemma 4.6.} Let \( h \in [-\infty, \infty] \). Then \( \pi(h, 0) \) and \( \pi(h, 1) \) are not ergodic for

\[ \lambda = \frac{1}{3}, \quad \lambda = \frac{2}{3}. \]

\textbf{Lemma 4.7.} The process \( Y_{xt} \) is ergodic if and only if \( h \in \mathbb{N} \).

\textbf{Remark.} The condition \( h \in \mathbb{N} \) is essential in identifying fixed points of limit trajectories. The proof of the following lemma

\[ \lambda = \frac{1}{3}, \quad \lambda = \frac{2}{3}. \]
Let $U$ be a Markov chain, and the transition probabilities of $Y$ yield that $\pi_0$. Since the convergence of $Y^{xt}$ establishes (b). Proof.

The following remark is useful in several subsequent proofs.

Let $I(\cdot)$ denotes the set of indices of a function $f: \mathcal{X} \to \mathbb{R}$.

In this section we provide the proofs that are deferred in previous sections.

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In particular, $\pi$ follows that $\nu(\cdot)$ converges to $0$ in probability. In particular, $\nu(\cdot)$ follows that $\nu(\cdot)$ converges to $0$ in probability.

The proof of Theorem 1.2 applies by using Lemma 4.8 in place of Lemma 3.8.

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In particular, $\nu(\cdot)$ follows that $\nu(\cdot)$ converges to $0$ in probability. In particular, $\nu(\cdot)$ follows that $\nu(\cdot)$ converges to $0$ in probability.

The proof of Theorem 1.2 applies by using Lemma 4.8 in place of Lemma 3.8.
transition probability matrix of Figure 5, and $Y_{xt}$ by the process $\pi_2$ for all $u$ such that $L_{xt} < \mu_2$. This completes the proof.

Proof of Lemma 4.5.

Let $W_{\sigma (0,0)} = \sigma (0,0) = (0,0)$ and $\sigma (0,1) = (0,1)$. Then $\sigma (0,0) = (0,0)$ and $\sigma (0,1) = (0,1)$. This establishes the lemma.

Proof of Lemma 3.5.

The lemma follows from Lemma 4.5 by taking $f$ is ergodic if and only if $i$ is transient. This completes the proof.

In particular $Y_{xt} = (1995), f = (\lambda_3, \lambda_1), W_{\sigma (0,0)} = \sigma (0,0) = (0,0), \pi_2 = (0,0)$. In the case when $i$ is ergodic let $\delta(\xi) = (0,0)$. Then $\delta(\xi) = (0,0)$.
Policies for a resource with partition constraints

**Lemma 4.5**
Let \( 0 < \sigma \leq \lambda \) such that \( N(\lambda) > 0 \). The proof is completed by constructing a process \( W(\sigma) \) on the same probability space. If \( \theta, \eta \) is ergodic. Since \( W(\sigma) \) is finite. Then the construction is completed.
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References


