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Article in Glasgow Mathematical Journal · March 1999
DOI: 10.1017/S001708959900726

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THE DIMENSION OF A PRIMITIVE INTERIOR $G$-ALGEBRA

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(Received 22 May, 1997)

Abstract. We give the residue class, modulo a certain power of $p$, for the dimension of a primitive interior $G$-algebra in terms of the dimension of the source algebra. To illustrate, we improve a theorem of Brauer on the dimension of a block algebra.

Almost always, the $G$-algebras arising in group representation theory have been interior. Both in applications and in the general theory, it often suffices to consider primitive interior $G$-algebras. One of the themes of the theory is the characterisation of a primitive interior $G$-algebra in terms of its source algebra $S$. Stories revolving around this theme are told in the two books devoted to $G$-algebra theory, namely Külshammer [8], Thévenaz [15] and in the papers listed in their bibliographies. We mention particularly Puig [11], [12]. These stories focus on rich algebraic relationships between $A$ and $S$; for a start, [11, 3.5] tells us that $A$ and $S$ are Morita equivalent. However, many outstanding conjectures, some old and some new, hark back to Brauer’s more arithmetical approach to group representation theory. See, for instance, conjectures in Alperin [1], Dade [4], Feit [6, Section 4.6] and Robinson [13]. In this note, we point out an arithmetical relationship between $A$ and $S$. A simple illustration, we shall discuss a theorem of Knörr on the dimension of a simply defective module, and shall improve a theorem of Brauer on the dimension of a block algebra. See also Ellers [5].

Our notation is as in Thévenaz [15]; we repeat a little of it to set the scene, and extend it slightly. Let $O$ be a complete local noetherian ring with an algebraically closed residue field $k$ of prime characteristic $p$. Let $G$ be a finite group, and let $A$ be an interior $G$-algebra; as usual, we assume that $A$ is finitely generated over $O$, and either free over $O$ or annihilated by $J(O)$. Given a pointed group $H$ on $A$, we choose an element $j \in \beta$, and define $A(j)$ as an interior $H$-algebra. Now let $X$ be an $A$-module; again we assume that $X$ is finitely generated over $O$, and either free over $O$ or annihilated by $J(O)$. We define $X(j) := jX$ as an $A(j)$-module. It is easy to extend the use of embeddings in Puig [12, 2.13.1] to show that $X(j)$ is unique up to a natural isomorphism of $A(j)$-modules.

Henceforth, let us assume that $A$ is primitive. Let $P_{\gamma}$ be a defect pointed group on $A$. The source algebra $A$ associated with $P_{\gamma}$ is an interior $P$-algebra. The multiplicity module $V(\gamma)$ associated with $P_{\gamma}$ is a projective indecomposable $k\tilde{\mathcal{N}}(P_{\gamma})$-module. By the construction of $V(\gamma)$, if $1 = \sum_{t \in T} t$ as a sum of mutually orthogonal primitive idempotents of $A^P$, then $\dim_k V(\gamma) = |\gamma \cap T|$. When $V(\gamma)$ is simple, we say that $A$ is simply defective. This notion has its origins in Knörr [7], and was introduced explicitly in Picaronny-Puig [10]. Necessary and sufficient conditions for $A$ to be simply defective are to be found in [2, 1.3], [10, Proposition 1], and Thévenaz [14, 15, 9.3]. We recall that any block algebra of $G$ over $O$ or over $k$ is simply defective. Also, the linear endomorphism algebras of certain $OG$-modules are simply defective (see below). Whenever $A$ is simply defective, the $p$-part of the dimension of the multiplicity module is
Let \( A \) be a primitive interior \( G \)-algebra, let \( P \gamma \) be a defect pointed group on \( A \), and let \( X \) be an \( A \)-module. Then

\[
\text{rk}_O X \equiv |G : N_G(P \gamma)|. \dim_k V(\gamma). \text{rk}_O X_\gamma \text{ modulo } |G : P| . \text{spr}_G(P).
\]

In particular, if \( A \) is simply defective, then

\[
(\text{rk}_O X)_P \equiv (|G : P|. \text{rk}_O X_\gamma)_P \text{ modulo } |G : P| . \text{spr}_G(P).
\]

**Proof.** If \( P \leq G \), then the points of \( P \) on \( A \) are precisely the \( G \)-conjugates of \( \gamma \). Writing \( 1_A = \sum_{t \in T} t \) as above, we have

\[
\text{rk}_O X = \sum_{g \in N_G(P \gamma) \subseteq G} |T \cap g \gamma|. \text{rk}_O X_{(g \gamma)} = |G : N_G(P \gamma)| . \dim_k V(\gamma). \text{rk}_O X_\gamma.
\]

Now suppose that \( P \not\leq G \). Let \( H := N_G(P) \). By the Green Correspondence Theorem in Thévenaz [15, 20.1], there exists a unique point \( \beta \) of \( H \) on \( A \) such that \( P \gamma \leq H \beta \). Furthermore, \( \beta \) has multiplicity unity; that is to say, if \( 1_A = \sum_{s \in S} s \) as a sum of mutually orthogonal primitive idempotents of \( A^H \), then precisely one element of \( S \) belongs to \( \beta \).

Consider the induced interior \( G \)-algebra \( A' := \text{Ind}_H^G(A_\beta) \). Recall that \( A' = O_G \otimes_{O_H} A_\beta \otimes_{O_G} O_G \) as \( O_G - O_G \)-bimodules, and \( A' \cong \text{Mat}_{|G:H|}(A_\beta) \) as algebras. Let \( X' := O_G \otimes_{O_H} X_\beta \) as an \( A' \)-module. Let \( \gamma' \) and \( \beta' \) be the points of \( P \) and \( H \) on \( A' \) corresponding to \( \gamma \) and \( \beta \), respectively. Since \( P \gamma \) is a defect pointed subgroup of \( H \beta \), the Green Correspondence Theorem implies that there exists a unique point \( \alpha' \) of \( G \) on \( A \) satisfying \( P \gamma \leq G_{\alpha'} \). Furthermore, \( \alpha' \) has multiplicity unity. By Puig [11, 3.6], \( A_{\alpha'} \cong A \) as interior \( G \)-algebras, and via this isomorphism, \( X_{\alpha'} \cong X \) as \( A \)-modules. A routine application of Mackey Decomposition and Rosenberg’s Lemma shows that if \( Q \) is a local pointed group on \( A' \) not \( G \)-conjugate to \( P \gamma \), then \( Q \) is
contained in the intersection of two distinct $G$-conjugates of $P$. Therefore, every point of $G$ on $A'$ distinct from $\alpha'$ has a defect group contained in $P \cap \hat{s}P$ for some $g \in G - H$. By Green’s Indecomposibility Criterion, $|G : P| \cdot \text{spr}_G(P)$ divides $\text{rk}_O X' - \text{rk}_O X$. We also have $\text{rk}_O X' = |G : H| \text{rk}_O X_\beta$ and, by the first paragraph of the argument,

$$\text{rk}_O X_\beta = |H : N_G(P_{\gamma})| \cdot \dim_k V(\gamma) \text{rk}_O X_{\gamma}.$$ 

To illustrate Proposition 1, let us consider an indecomposable $OG$-module $M$ (finitely generated over $O$, and either free over $O$ or annihilated by $J(O)$). Let $P$ be a vertex of $M$, let $U$ be a source $OP$-module of $M$, let $F$ be the inertia group of $U$ in $N_G(P)$, and let $m$ be the multiplicity of $U$ as a direct factor of the restricted $OP$-module of $M$. The linear endomorphism algebra $\text{End}_O(M)$ (interpreted as $\text{End}_k(M)$ when $J(O)$ annihilates $M$) is a primitive interior $G$-algebra with a defect pointed group $P_\gamma$ such that $M_\gamma \cong U$. Also, $N_G(P_\gamma) = F$, and $\dim_k (V(\gamma)) = m$. By [2, 1.4], $\text{End}_O(M)$ is simply defective if and only if $m$ is the multiplicity of $M$ in the induced $OG$-module of $U$. When these equivalent conditions hold, we say that $M$ is simply defective. If $M$ satisfies the hypothesis of Knörr [7, 4.5] (in particular, if $M$ is an irreducible $OG$-module or a simple $kG$-module), then by Picaronny-Puig [10, Proposition 1] $M$ is simply defective. Proposition 1 implies the following result.

**Corollary 2.** Let $M$ be an indecomposable $OG$-module. With the notation above, we have

$$\text{rk}_O M \equiv |G : F| m \text{rk}_O U \mod |G : P| \cdot \text{spr}_G(P).$$

In particular, if $M$ is simply defective, then

$$(\text{rk}_O M)_p \equiv (|G : P| \cdot \text{rk}_O U)_p \mod |G : P| \cdot \text{spr}_G(P).$$

The rider to Corollary 2 relates to [7, 4.5] and [10, Proposition 3], but has slightly weaker hypothesis and conclusion.

**Lemma 3.** Let $G$ and $H$ be finite groups. Let $P_\gamma$ and $Q_\delta$ be defect pointed groups on, respectively, a primitive $G$-algebra $A$ and a primitive $H$-algebra $B$. Then $\gamma \otimes \delta$ is contained in a local point $\varepsilon$ of $P \times Q$ on $A \otimes B$, and $(P \times Q)_\varepsilon$ is a defect pointed group on the primitive $G \times H$-algebra $A \otimes B$.

**Proof.** It is easy to check that $A \otimes B$ is primitive, and that $\gamma \otimes \delta$ is contained in a point $\varepsilon$ of $P \times Q$. By considering the evident isomorphism of Brauer quotients

$$\overline{A}(P) \otimes \overline{B}(Q) \cong \overline{A \otimes B}(P \times Q)$$

we see that $\varepsilon$ is local. On the other hand,

$$1_{A \otimes B} \in \text{Tr}^{G \times H}_{P \times Q}(A^P \otimes B^Q, \varepsilon, A^P \otimes B^Q)$$

so that $(P \times Q)_\varepsilon$ is a defect pointed group. 

\[\square\]
Theorem 4. Given a defect pointed group $P_\gamma$ on a primitive interior $G$-algebra $A$, then
\[
\text{rk}_G A \equiv (|G : N_G(P_\gamma)|, \dim_k V(\gamma))^2 \text{rk}_G A_\gamma \text{ modulo } |G : P|^2 \text{spr}_G(P).
\]
In particular, if $A$ is simply defective, then
\[
(\text{rk}_G A)_p \equiv (|G : P|^2 \cdot \text{rk}_G A_\gamma)_p \text{ modulo } |G : P|^2 \text{spr}_G(P).
\]

Proof. This follows from Proposition 1 and Lemma 3 upon considering $A$ as an $A \otimes A^{op}$-module by left-right translation.

Let us consider a block idempotent $b$ of $OG$ with defect group $P$. Brauer [3, Theorem 1] used character theory to prove that the block algebra $OGb$ satisfies
\[
(\text{rk}_G OGb)_p = (|G| |G : P|)_p.
\]
A module-theoretic demonstration was later given by Michler [9, 2.1], and the result is generalised in Picaronny-Puig [10, Proposition 3]. Since $OGb$ is simply defective, Theorem 4 gives, more precisely, the following result.

Corollary 5. Let $b$ be a block idempotent of $OG$. Let $(P, e)$ be a maximal Brauer pair associated with $b$, let $T$ denote the inertia group of $e$ in $N_G(P)$, and let $W$ be a copy of the isomorphically unique simple $kCG(P)e$-module. Then
\[
\text{rk}_G OGb \equiv (|G| \dim_k W)^2 |Z(P)|/|T||C_G(P)| \text{ modulo } (|G| |G : P|)_p \text{spr}_G(P).
\]

Proof. By an easy adaptation of part of the argument in Michler [9, 2.1], we may and shall assume that $P \leq G$. Thévenaz [15, 40.13] describes a defect pointed group $P_\gamma$ on $OGb$ associated with $(P, e)$, and also informs us that $T = N_G(P_\gamma)$ and $\dim_k W = \dim_k V(\gamma)$. By Puig [12, 6.6, 14.6], we have
\[
\text{rk}_G OGb_\gamma = |N_G(P_\gamma) : PC_G(P)||P| = |T||Z(P)|/|C_G(P)|.
\]

REFERENCES


