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Multiplicity Computation of Modules over $k[x_1, ..., x_n]$ and an Application to Weyl Algebras

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Abstract Let $A = k[x_1, ..., x_n]$ be the polynomial algebra over a field k of characteristic 0, I an ideal of A, $M = A/I$ and ${}^aHP_I$ the (affine) Hilbert polynomial of M. By further exploring the algorithmic procedure given in [CLO'] for deriving the existence of ${}^aHP_I$, we compute the leading coefficient of ${}^aHP_I$ by looking at the leading monomials of a Gröbner basis of I without computing ${}^aHP_I$. Using this result and the filtered-graded transfer of Gröbner basis obtained in [LW] for (noncommutative) solvable polynomial algebras (in the sense of [K-RW]), we are able to compute the multiplicity of a cyclic module over the Weyl algebra $A_n(k)$ without computing the Hilbert polynomial of that module, and consequently to give a quite easy algorithmic characterization of the "smallest" modules over Weyl algebras. Using the same methods as before, we also prove that the tensor product of two cyclic modules over the Weyl algebras has the multiplicity which is equal to the product of the multiplicities of both modules. The last result enables us to construct examples of "smallest" irreducible modules over Weyl algebras.

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Let k be a field of characteristic 0, and $k[x_1,...,x_n]$ the *commutative* polynomial k-algebra in n variables. If I is an ideal of $k[x_1,...,x_n]$ and $V(I)$ is the affine algebraic set defined by I in the affine n-space $\mathbf{A}_k^n$, then it is well known that the *affine* Hilbert polynomial of the $k[x_1,...,x_n]$-module $M = k[x_1,...,x_n]/I$, denoted

$$^aHP_I = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0, \text{ where } \quad a_i \in \mathbb{Q}\,, \text{ and } a_d > 0,$$

yields two important numbers:

- deg(^aHP_I) $= d =$ dim$V(I)$ (in case k is algebraically closed) where the latter denotes the dimension of $V(I)$, which is also known the Gelfand-Kirillov dimension of the $k[x_1,...,x_n]$-module M, denoted GK.dim(M);
- $d!a_d$, which is usually called the multiplicity of the module M and is denoted by $e(M)$.

Let $G = \{g_1,...,g_s\}$ be a Gröbner basis of I with respect to some *graded monomial ordering* on $k[x_1,...,x_n]$. It follows from ([CLO′] Ch.9 or [BW] ch.9 §3) that deg(^aHP_I) can be easily computed by only looking at the leading monomials of G without computing ^aHP_I. This result has a noncommutative version (see [Li]) for a class of solvable polynomial algebras in the sense of [K-RW] which includes enveloping algebras of Lie algebras, Weyl algebras and certain type of iterated Ore extensions.

It is natural to ask the following question:

- Is there an easy way to compute $e(M)$ (or equivalently, the leading coefficient a_d of ^aHP_I) from G without computing ^aHP_I?

In this note, we give a positive answer to the above question by further exploring the algorithmic procedure given in ([CLO'] ch.9) for deriving the existence of ^aHP_I, and meanwhile we obtain an estimation formula for $e(M)$. This multiplicity computation procedure is described in §1 by only looking at the leading monomials of a Gröbner basis of I without computing ^aHP_I. As an application of the multiplicity computation to noncommutative algebras, in §2 we use the filtered-graded transfer of Gröbner basis obtained in [LW] for (noncommutative) solvable polynomial algebras (in the sense of [K-RW]) to compute the multiplicity of a cyclic module over the Weyl algebra $A_n(k)$i, and consequently to give an quite easy algorithmic characterization (or recognition) of the "smallest" modules over the Weyl algebras $A_n(k)$, namely, the modules of Gelfand-Kirillov dimension n with multiplicity 1 which are known being holonomic and irreducible. Finally, in §3, using the same trick as before we prove that the multiplicity of the tensor product of two cyclic modules over Weyl algebras is the product of the multiplicities of both modules, which enables us to construct examples of the "smallest" irreducible modules over Weyl algebras.

Rings considered in this note are associative rings with 1, and modules are left unitary modules.

§1. Multiplicity Computation

For a general theory of Gröbner bases in commutative polynomial algebras, we refer to [CLO′] and [BW].

Let k be a field of characteristic 0, and $k[x_1, ..., x_n]$ the commutative polynomial k-algebra in n variables. If I is an ideal of $k[x_1, ..., x_n]$ and the affine algebraic set $V(I)$ defined by I in the n-dimensional affine space $\mathbf{A}_k^n$ is d-dimensional, as in the begining of this note we write the (affine) Hilbert polynomial of the $k[x_1, ..., x_n]$-module $M = k[x_1, ..., x_n]/I$ as

$$^aHP_I = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0, \text{ where } \quad a_i \in \mathbb{Q}, \text{ and } a_d > 0,$$

and write the *multiplicity* of M as $e(M) = d!a_d$. In this section we answer the question posed in the beginning of this note and also give an estimation formula for $e(M)$ by looking at a Gröbner basis of I without computing ^aHP_I.

We start with some notation. Let $>$ be a *monomial ordering* on $k[x_1, ..., x_n]$ in the sense of [CLO′]. We write x^α for the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \in k[x_1, ..., x_n]$ with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, the set of n-tuples of integers ≥ 0, and write $|\alpha|$ for the total degree $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ of x^α. If $f \in k[x_1, ..., x_n]$, $f = c_{\alpha(1)}x^{\alpha(1)} + c_{\alpha(2)}x^{\alpha(2)} + \cdots + c_{\alpha(m)}x^{\alpha(m)}$ with $\alpha(1) > \alpha(2) > \cdots > \alpha(m)$, we write $\mathbf{LM}(f)$ for the *leading monomial* $x^{\alpha(1)}$ of f, $\mathbf{LT}(f)$ for the *leading term* $c_{\alpha(1)}x^{\alpha(1)}$ of f. Let $\langle\mathbf{LT}(I)\rangle$ be the ideal generated by the leading terms of I, where $\mathbf{LT}(I) = \{\mathbf{LT}(f) \mid f \in I\}$. Then $\langle\mathbf{LT}(I)\rangle$ is clearly a monomial ideal, i.e., the ideal generated by monomials. Moreover, it is well known that ([CLO′] Ch.9 §3, Proposition 4) under any *graded* monomial ordering $>$

$$(*) \qquad {}^aHP_I = {}^aHP_{\langle\mathbf{LT}(I)\rangle}.$$

Therefore, from now on we focus our attention on a *monomial ideal* I.

For every integer $s \geq 0$, as usual we put

$$k[x_1, ..., x_n]_{\leq s} = \left\{f \in k[x_1, ..., x_n] \;\middle|\; f = \sum c_\alpha x^\alpha,\ |\alpha| \leq s\right\}.$$

Furthermore, we put

$$\begin{aligned} \mathbf{C}(I) &= \left\{\alpha \in \mathbb{Z}_{\geq 0}^n \;\middle|\; x^\alpha \notin I\right\} \\ \mathbf{C}(I)_{\leq s} &= \left\{\alpha \in \mathbf{C}(I) \;\middle|\; |\alpha| \leq s\right\}. \end{aligned}$$

Then the k-subspace $(k[x_1,...,x_n]_{\leq s}+I)/I$ has a finite k-basis $\{\overline{x^\alpha} \mid x^\alpha \in \mathbf{C}(I)_{\leq s}\}$, where $\overline{x^\alpha}$ is the class of x^α in $k[x_1,...,x_n]/I$, and for $s \gg 0$, $\dim_k((k[x_1,...,x_n]_{\leq s}+I)/I) = |\mathbf{C}(I)_{\leq s}| = {}^aHP_I(s)$.

In order to prove the existence of ${}^aHP_I$, the following notation and notions are introduced in [CLO′]. Put

$$\begin{aligned} e_1 &= (1,0,...,0) \\ e_2 &= (0,1,...,0) \\ &\vdots \\ e_n &= (0,0,...,1) \end{aligned}$$

1.1. Definition (i) For $\{e_{i_1},...,e_{i_r}\} \subset \{e_1,...,e_n\}$ with $i_1 < \cdots < i_r$, $r \leq n$, the subset

$$[e_{i_1},...,e_{i_r}] = \left\{ \sum_{j=1}^{r} a_j e_{i_j} \;\middle|\; a_j \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq j \leq r \right\} \subset \mathbb{Z}^n_{\geq 0}$$

is called an r-dimensional *coordinate subspace* of $\mathbb{Z}^n_{\geq 0}$ determined by $e_{i_1},...,e_{i_r}$.
(ii) For $\beta = \sum_{j \notin \{i_1,...,i_r\}} a_j e_j \in \mathbb{Z}^n_{\geq 0}$ with $a_i \in \mathbb{Z}_{\geq 0}$, the subset

$$\beta + [e_{i_1},...,e_{i_r}] = \left\{ \beta + \gamma \;\middle|\; \gamma \in [e_{i_1},...,e_{i_r}] \right\}$$

is called a *translate* of the r-dimensional coordinate subspace $[e_{i_1},...,e_{i_r}]$.

1.2. Lemma ([CLO′] Ch.9, Lemma 5) Let $\beta + [e_{i_1},...,e_{i_r}]$ be a translate of the coordinate subspace $[e_{i_1},...,e_{i_r}] \subset \mathbb{Z}^n_{\geq 0}$ with $\beta = \sum_{j \notin \{i_1,...,i_r\}} a_j e_j$.
(i) The number of points in $\beta + [e_{i_1},...,e_{i_r}]$ of total degree $\leq s$ is equal to

$$\binom{r+s-|\beta|}{s-|\beta|},$$

provided $s > |\beta|$.
(ii) For $s > |\beta|$, the number of points as in (i) above is a polynomial function of s of degree r, and the coefficient of s^r is $\frac{1}{r!}$.

□

Suppose that the affine algebraic set $V(I)$ is of dimension d, denoted $\dim V(I) = d$. It follows from ([CLO′] Ch.9 §2, Proposition 2 and Theorem 3) that $\mathbf{C}(I)$ can be written as the *disjoint* union:

(1) $$\mathbf{C}(I) = C_0 \cup C_1 \cup C_2 \cup \cdots \cup C_d \text{ with } C_d \neq \emptyset,$$

where each C_i is a finite (*not necessarily* disjoint) union of translates T_{i_j} of i-dimensional coordinate subspaces in $\mathbb{Z}^n_{\geq 0}$:

(2) $$C_i = T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_m}.$$

If we put

$$C_i^s = \left\{x^\alpha \in C_i \;\middle|\; |\alpha| \leq s\right\},$$

$$T_{i_j}^s = \left\{x^\alpha \in T_{i_j} \;\middle|\; |\alpha| \leq s\right\},$$

then by Lemma 1.2 and (2) one may directly check (or see the proof of [CLO′] Ch.9 §2, Theorem 6) that

- $|C_i^s|$ is a polynomial of degree i in s, in particular, $|C_d^s|$ is a polynomial of degree d when s is big enough, and the leading term of the Hilbert polynomial ${}^aHP_I$ is given by the leading term of the polynomial $|C_d^s|$ which is of the form $\frac{N}{d!}t^d$, where N is the number of *different* T_{d_j} appearing in the above decomposition (2).

It is then clear that $e(M) = N$. Our *aim* is to see how to compute the number N without computing ${}^aHP_I$. To do this, let us suppose that I is generated by the monomials

$$(3) \qquad \begin{cases} m_1 &= x_1^{\alpha_{11}}x_2^{\alpha_{12}}\cdots x_n^{\alpha_{1n}}, \\ m_2 &= x_1^{\alpha_{21}}x_2^{\alpha_{22}}\cdots x_n^{\alpha_{2n}}, \\ \vdots & \qquad \vdots \\ m_s &= x_1^{\alpha_{s1}}x_2^{\alpha_{s2}}\cdots x_n^{\alpha_{sn}}. \end{cases}$$

Recall from ([CLO′] Ch.9 §1, Proposition 3) that if we put

$$M_k = \left\{i \in \{1, ..., n\} \;\middle|\; \alpha_{ki} \neq 0\right\}, \; k = 1, ..., s,$$

$$\mathcal{M} = \left\{J \subset \{1, ..., n\} \;\middle|\; J \cap M_k \neq \emptyset, \; k = 1, ..., s\right\},$$

then $d = \dim V(I) = n - \min\left\{|J| \;\middle|\; J \in \mathcal{M}\right\}$.

By the definition of $\mathbf{C}(I)$ and the above (1), (2), the first easy but useful observation is recorded as follows.

1.3. Observation With the notation as above, let $\beta + [e_{i_1}, ..., e_{i_r}]$ be a translate contained in C_r with $\beta = \sum_{j \notin \{i_1,...,i_r\}} a_j e_j$. Putting $J = \{1, ..., n\} - \{i_1, ..., i_r\}$, then $J \in \mathcal{M}$.

□

Let $J \in \mathcal{M}$ with $J = \{l_1, ..., l_{n-d}\}$ where $d = \dim V(I)$, and put $\{i_1, ..., i_d\} = \{1, ..., n\} - J$. If we write $\alpha(m_k)$ for $(\alpha_{k1}, \alpha_{k2}, ..., \alpha_{kn})$ in above (3), $k = 1, ..., s$, then we can further rewrite each $\alpha(m_k)$ as

$$\alpha(m_k) = (\alpha_{kl_1}, ..., \alpha_{kl_{n-d}}, \gamma_k)$$

with $\gamma_k = (\alpha_{ki_1}, ..., \alpha_{ki_d}) \in [e_{i_1}, ..., e_{i_d}]$. Put

$$L_p = J - \{l_p\}, \quad p = 1, ..., n-d.$$

Since $\dim V(I) = d$ and $J \cap M_j \neq \emptyset$, $j = 1, ..., s$, there exists some M_k such that $L_p \cap M_k = \emptyset$ and

$$(*) \qquad \alpha(m_k) = (0, ..., 0, \alpha_{kl_p}, 0, ..., 0, \gamma_k) \text{ with } \alpha_{kl_p} \neq 0.$$

Putting

$$(**) \qquad \alpha_{l_p} = \min\left\{\alpha_{kl_p} \;\middle|\; L_p \cap M_k = \emptyset\right\}, \; p = 1, ..., n-d,$$

and reordering (if necessary) the generating set $\{m_1, ..., m_s\}$ of I, we obtain the following array by using the above $(*)$ and $(**)$:

$$(3') \quad \left\{\begin{array}{cclcccccc}
\alpha(m_1) & = & (& \alpha_{l_1}, & 0, & ..., & 0, & \gamma_1 &) \\
\alpha(m_2) & = & (& 0, & \alpha_{l_2}, & ..., & 0, & \gamma_2 &) \\
\vdots & & & & & \vdots & & & \\
\alpha(m_{n-d}) & = & (& 0, & 0, & ..., & \alpha_{l_{n-d}}, & \gamma_{n-d} &) \\
\alpha(m_{n-d+1}) & = & (& \alpha_{n-d+1,l_1}, & \alpha_{n-d+1,l_2}, & ..., & \alpha_{n-d+1,l_{n-d}}, & \gamma_{n-d+1} &) \\
\vdots & & & & & \vdots & & & \\
\alpha(m_s) & = & (& \alpha_{sl_1}, & \alpha_{sl_2}, & ..., & \alpha_{sl_{n-d}}, & \gamma_s &)
\end{array}\right.$$

1.4. Proposition With notation as above, and let us write E for the number of *different* $J \in \mathcal{M}$ with $|J| = n-d$.

(i) If $\beta + [e_{i_1}, ..., e_{i_d}]$ is a translate of some d-dimensional coordinate subspace $[e_{i_1}, ..., e_{i_d}]$ contained in C_d with $\beta = \sum_{p \notin \{i_1, ..., i_d\}} a_p e_p$, then $a_p < \alpha_{l_p}$, where α_{l_p} is as in above $(**)$, $p = 1, ..., n-d$. It follows that

$$e(M) \leq \sum^{E} \prod_{p=1}^{n-d} \alpha_{l_p},$$

where each product $\prod_{p=1}^{n-d} \alpha_{l_p}$ is determined by some $J \in \mathcal{M}$ with $|J| = n-d$.

(ii) $e(M)$ can be computed from the generators of I given in (3) without computing ${}^a HP_I$.

Proof (i) Since $\beta+[e_{i_1},...,e_{i_d}]$ is contained in C_d, the first part follows from the above (3′), and the inequality follows from Observation 1.3. and an easy combinitorial computation.
(ii) For $J \in \mathcal{M}$ with $|J| = n-d$, we assume $J = \{l_1,...,l_{n-d}\} = \{1,...,n\}-\{i_1,...,i_d\}$ and consider $\beta = \sum_{p\notin\{i_1,...,i_d\}} a_p e_p$ with $a_p < \alpha_{l_p}$. Then from (3′) we easily see that $e(M)$ is nothing but the number of all β which cannot be divided by any one of the monomials:

$$\begin{aligned} m'_{n-d+1} &= x_{l_1}^{\alpha_{n-d+1,l_1}} x_{l_2}^{\alpha_{n-d+1,l_2}} \cdots x_{l_{n-d}}^{\alpha_{n-d+1,l_{n-d}}} \\ m'_{n-d+2} &= x_{l_1}^{\alpha_{n-d+2,l_1}} x_{l_2}^{\alpha_{n-d+2,l_2}} \cdots x_{l_{n-d}}^{\alpha_{n-d+2,l_{n-d}}} \\ \vdots \quad &\vdots \quad \vdots \\ m'_s &= x_{l_1}^{\alpha_{sl_1}} x_{l_2}^{\alpha_{sl_2}} \cdots x_{l_{n-d}}^{\alpha_{sl_{n-d}}} \end{aligned}$$

□

Furthermore, for $J \in \mathcal{M}$ with $J = \{l_1,...,l_{n-d}\}$, put

$$(***) \qquad \alpha'_{l_p} = \min\left\{\alpha_{kl_p} \;\middle|\; \alpha_{kl_p} \neq 0 \text{ in above (3)},\ k = 1,...,s\right\},\ p = 1,...,n-d.$$

Again by an easy combinitorial computation and combining Proposition 1.4, we can mention the following multiplicity estimation formula.

1.5. Proposition With notation as above, we have

$$\sum^{E}\prod_{p=1}^{n-d}\alpha_{l_p} \geq e(M) \geq \sum^{E}\prod_{p=1}^{n-d}\alpha'_{l_p},$$

where each product $\prod_{p=1}^{n-d}\alpha_{l_p}$ resp. $\prod_{p=1}^{n-d}\alpha'_{l_p}$ is determined by some $J \in \mathcal{M}$ with $|J| = n-d$, and α_{l_p} resp. α'_{l_p} is as in above $(**)$ resp. as in above $(***)$.
The equalities hold in the above inequalities if $\alpha_{l_p} = \alpha'_{l_p}$, in particular, if I is generated by $n-d$ monomials.

□

The above results have two immediate consequences in certain spacial cases.

1.6. Corollary Let I and M be as before. With notation as above, $e(M) = 1$ if and only if there is only one $J \in \mathcal{M}$ with $|J| = n-d$, say $J = \{l_1,...,l_{n-d}\} \subset \{1,...,n\}$, such that $\alpha_{l_p} = 1$, where α_{l_p} is as in above $(**)$, $p = 1,...,n-d$.

□

1.7. Corollary Let I and M be as before. Suppose $\dim V(I) = n - 1$. Then $e(M) = \sum \alpha_i$, where

$$\alpha_i = \min\left\{\alpha_{ki} \;\middle|\; \alpha_{ki} \text{ is as in above (3)},\ k = 1, ..., s\right\}, \quad i = 1, ..., n.$$

□

Summing up, let L be an *arbitrary* ideal of $k[x_1, ..., x_n]$, $N = k[x_1, ..., x_n]/L$. If $G = \{g_1, ..., g_s\}$ is a Gröbner basis of L with respect to some *graded monomial ordering* on $k[x_1, ..., x_n]$, then since

$$^aHP_L = {}^aHP_{\langle\mathbf{LT}(L)\rangle} = {}^aHP_{\langle\mathbf{LT}(G)\rangle},$$

where $\langle\mathbf{LT}(G)\rangle$ is the monomial ideal generated by the leading monomials $m_1 = \mathbf{LM}(g_1), ..., m_s = \mathbf{LM}(g_s)$, if we put $I = \langle\mathbf{LT}(G)\rangle$, it follows from the foregoing results that the following theorem holds.

1.8. Theorem Let L and N be as above, and suppose $\dim V(L) = d$. With notation as before, we have:

(i)

$$\sum^{E}\prod_{p=1}^{n-d}\alpha_{l_p} \geq e(N) \geq \sum^{E}\prod_{p=1}^{n-d}\alpha'_{l_p},$$

where each product $\prod_{p=1}^{n-d}\alpha_{l_p}$ resp. $\prod_{p=1}^{n-d}\alpha'_{l_p}$ is determined by some $J \in \mathcal{M}$ with $|J| = n - d$, and α_{l_p} resp. α'_{l_p} is as in above $(**)$ resp. as in above $(***)$.
The equalities hold in the above inequalities if $\alpha_{l_p} = \alpha'_{l_p}$, in particular, if I is generated by $n - d$ monomials (i.e., G has exactly $n - d$ members).
(ii) $e(N)$ can be computed by only looking at the leading monomials of G as in the proof of Proposition 1.4(ii).
(iii) $e(N) = 1$ if and only if there is only one $J \in \mathcal{M}$ with $J = \{l_1, ..., l_{n-d}\}$ such that $\alpha_{l_p} = 1$, where α_{l_p} is as in above $(**)$, $p = 1, ..., n - d$.
(iv) If $\dim V(L) = n - 1$, then $e(N) = \sum \alpha_i$, where

$$\alpha_i = \min\left\{\alpha_{ki} \;\middle|\; \alpha_{ki} \text{ is as in above (3)},\ k = 1, ..., s\right\}, i = 1, ..., n.$$

□

§2. An Application to Weyl Algebras

Let k be a field of characteristic 0, and $A_n(k)$ the n-th Weyl algebra over k, namely, the k-algebra generated by $2n$ elements $x_1, ..., x_n, y_1, ..., y_n$ subject to the relations

$$\begin{aligned} x_i x_j - x_j x_i &= y_i y_j - y_j y_i = 0,\ i,j = 1,...,n, \\ y_j x_i &= x_i y_j + \delta_{ij},\ i,j = 1,...,n, \end{aligned}$$

which is also well known as the ring of k-linear differential operators of the polynomial k-algebra $k[t_1,...,t_n]$. For $\alpha = (\alpha_1,...,\alpha_n)$, $\beta = (\beta_1,...,\beta_n) \in \mathbb{Z}$, writing

$$\begin{aligned} x^\alpha y^\beta &= x_1^{\alpha_1}\cdots x_n^{\alpha_n} y_1^{\beta_1}\cdots y_n^{\beta_n}, \\ |\alpha| &= \alpha_1 + \cdots + \alpha_n, \end{aligned}$$

the following are well known (see e.g., [Bj], [MR]):

- The set of monomials

$$\left\{ x^\alpha y^\beta \;\middle|\; \alpha = (\alpha_1,...,\alpha_n), \beta = (\beta_1,...,\beta_n) \in \mathbb{Z}^n_{\geq 0} \right\}$$

  forms a k-basis of $A_n(k)$.
- If $\mathcal{B} = \{\mathcal{B}_p\}_{p\geq 0}$ is the Bernstein filtration on $A_n(k)$ with

$$\mathcal{B}_p = \left\{ \sum x^\alpha y^\beta \;\middle|\; |\alpha| + |\beta| \leq p \right\},\ p \geq 0,$$

  then the associated graded k-algebra of $A_n(k)$, denoted

$$G(A_n(k)) = \oplus_{p\geq 0}(\mathcal{B}_p/\mathcal{B}_{p-1})$$

  , is a commutative polynomial k-algebra in 2n variables $X_1 = \sigma(x_1),...,X_n = \sigma(x_n), Y_1 = \sigma(y_1),...,Y_n = \sigma(y_n)$, where the $\sigma(x_i)$ and $\sigma(y_i)$ are images of x_i and y_i in $G(A_n)_1 = \mathcal{B}_1/\mathcal{B}_0$. (So from now on we always write $k[\mathcal{X},\mathcal{Y}] = k[X_1,...,X_n,Y_1,...,Y_n]$ for $G(A_n(k))$, where $k[\mathcal{X},\mathcal{Y}]$ has the gradation given by the total degree of polynomials.)
- If L is a left ideal of $A_n(k)$ with the filtration FL induced by the Bernstein filtration $\mathcal{B}$: $F_pL = \mathcal{B}_p \cap L$, $p \geq 0$, then $G(L) = \oplus_{p\geq 0}(F_pL/F_{p-1}L)$ is a graded ideal of $G(A_n(k))$, and moreover, the $A_n(k)$-module $M = A_n/L$ and the $G(A_n(k))$-module $G(M) = G(A_n(k))/G(L)$ have the same Gelfand-Kirillov dimension, denoted GK.dim(M) = GK.dim$(G(M))$. Adopting the notation as in the begining of this note, GK.dim(M) is given by the degree of the Hilbert polynomial ${}^aHP_{G(L)}$ and the *multiplicity* of M, which is defined to be $e(M) = e(G(M))$, is determined by the leading term of ${}^aHP_{G(L)}$.
- For any nonzero $A_n(k)$-module M, GK.dim$(M) \geq n$ (Bernstein inequality). A nonzero finitely generated $A_n(k)$-module is said to be *holonomic* if GK.dim$(M) = n$. A holonomic module is always *cyclic*, and hence is of the form $A_n(k)/L$ where L is a left ideal of $A_n(k)$, and is of finite length $\leq e(M)$. If M is a holonomic module with $e(M) = 1$, then M is an irreducible $A_n(k)$-module. So we may say that such modules are the "smallest" modules over $A_n(k)$.

A nice algorithmic property of $A_n(k)$ is that the noncommutative version of Buchberger's algorithm exists in $A_n(k)$ because $A_n(k)$ is a *solvable polynomial algebra* in the sense of [K-RW]. Hence, any nonzero left ideal L of $A_n(k)$ has a (left) Gröbner basis with respect to some monomial ordering $>$ on $A_n(k)$:

- There are $g_1, g_2, ..., g_s \in L$ such that every $f \in L$ has an presentation $\sum_{i=1}^{s} h_i g_i$ with $\mathbf{LM}(f) \geq \mathbf{LM}(h_i g_i)$ whenever $h_i g_i \neq 0$, where $\mathbf{LM}(f)$ denotes the leading monomial of f and similarly for $\mathbf{LM}(h_i g_i)$ (see the definition below).

We refer to [KR-W] for a survey concerning algorithms in the noncommutative solvable plynomial k-algebras. In particular, we refer to the Modula-2 Algebra System (Version 1.00, developed at the University of Passau and released in 1996) for computing Gröbner bases in solvable polynomial algebras.

In this section, by using the results obtained in §1 we give a quite easy algorithmic characterization of the class of $A_n(k)$-modules M with GK.dim$(M) = n$ and $e(M) = 1$.

Let $>_{grlex}$ be the *graded lexicographic* ordering on $A_n(k)$:

$$x^{\alpha}y^{\beta} >_{grlex} x^{\alpha'}y^{\beta'} \Leftrightarrow |\alpha| + |\beta| > |\alpha'| + |\beta'| \text{ or}$$
$$|\alpha| + |\beta| = |\alpha'| + |\beta'| \text{ and } (\alpha, \beta) >_{lex} (\alpha', \beta'),$$

such that $x_1 >_{grlex} x_2 >_{grlex} \cdots >_{grlex} x_n >_{grlex} y_1 >_{grlex} y_2 >_{grlex} \cdots >_{grlex} y_n$, where $>_{lex}$ denotes the *lexicographic ordering* on $\mathbb{Z}^{2n}_{\geq 0}$. If $f \in A_n(k)$, say

$$f = c_1 x^{\alpha(1)}y^{\beta(1)} + c_2 x^{\alpha(2)}y^{\beta(2)} + \cdots + c_m x^{\alpha(m)}y^{\beta(m)} \text{ with}$$
$$x^{\alpha(1)}y^{\beta(1)} >_{grlex} c_2 x^{\alpha(2)}y^{\beta(2)} >_{grlex} \cdots >_{grlex} x^{\alpha(m)}y^{\beta(m)},$$

then we write $\mathbf{LM}(f) = x^{\alpha(1)}y^{\beta(1)}$ for the *leading monomial* of f. Moreover, if $f \in \mathcal{B}_p - \mathcal{B}_{p-1}$, we write $\sigma(f)$ for the image of f in $G(A_n(k))_p = \mathcal{B}_p/\mathcal{B}_{p-1}$, which is usually called the *principal symbol* of f. With these notation in hand, the following lemma is easily verified.

2.1 Lemma (i) $f \in \mathcal{B}_p - \mathcal{B}_{p-1}$ if and only if $|\alpha(1)| + |\beta(1)| = p$.
(ii) Let $f \in \mathcal{B}_p - \mathcal{B}_{p-1}$. Using $>_{grlex}$ on $k[\mathcal{X}, \mathcal{Y}]$ such that $X_1 >_{grlex} X_2 >_{grlex} \cdots >_{grlex} X_n >_{grlex} Y_1 >_{grlex} Y_2 >_{grlex} \cdots >_{grlex} Y_n$, then

$$\sigma(\mathbf{LM}(f)) = \mathbf{LM}(\sigma(f)).$$

□

Let L be a left ideal of $A_n(k)$ generated by $\{f_1, ..., f_m\}$. It is easy to see that generally $G(L)$ cannot be generated by $\{\sigma(f_1), ..., \sigma(f_m)\}$ in $G(A_n(k))$. However, using the

Gröbner basis for left ideals of $A_n(k)$ we do have the following result (see [LW] and [Li]).

2.2. Theorem With notation as above, let L be a nonzero left ideal of $A_n(k)$ with the filtration FL induced by the Bernstein filtration $\mathcal{B}$. If $G = \{g_1, ..., g_s\} \subset L$, then G is a Gröbner basis of L in $A_n(k)$ with respect to $>_{grlex}$ if and only if $\sigma(G) = \{\sigma(g_1), ..., \sigma(g_s)\}$ is a Gröbner basis of $G(L)$ in $G(A_n(k)) = k[\mathcal{X}, \mathcal{Y}]$ with respect to $>_{grlex}$.

□

Now, using Theorem 1.8, Lemma 2.1 and Theorem 2.2 we are able to mention the main result of this section, as follows.

2.3. Theorem Let L be a left ideal of $A_n(k)$ and $G = \{g_1, ..., g_s\}$ a left Gröbner basis of L with respect to $>_{grlex}$ on $A_n(k)$. Put

$$\begin{aligned} M &= A_n(k)/L, \\ I &= G(L), \\ m_1 &= \mathbf{LM}(\sigma(g_1)), m_2 = \mathbf{LM}(\sigma(g_2)), ..., m_s = \mathbf{LM}(\sigma(g_s)). \end{aligned}$$

With notation as in §1, we have:

(i) $e(M)$ can be computed by only looking at $m_1, ..., m_s$ without computing ${}^aHP_{G(L)}$.

(ii) GK.dim$(M) = n$ and $e(M) = 1$ if and only if

(a) $n = \min\{|J| \mid J \in \mathcal{M}\}$; and

(b) there is only one $J \in \mathcal{M}$ with $J = \{l_1, ..., l_n\}$ such that $\alpha_{l_1} = \alpha_{l_2} = \cdots = \alpha_{l_n} = 1$, where α_{l_p} is as in §1 (**), $p = 1, ..., n$.

(iii) If GK.dim$(M) = 2n - 1$, then $e(M) = \sum \alpha_i$, where

$$\alpha_i = \min\left\{\alpha_{ki} \;\middle|\; \alpha_{ki} \text{ is as in §1 (3)},\ k = 1, ..., s\right\},\ i = 1, ..., 2n.$$

□

Finally, we point out that in case $n = 1$, the above (ii) actually gives more about the generating set of a left ideal in $A_1(k)$.

2.4. Proposition Let L be a left ideal of $A_1(k)$ and $G = \{g_1, ..., g_s\}$ a Gröbner basis of L with respect to $>_{grlex}$ on $A_1(k)$. Furthermore let α_1, α_2 be as in Theorem 2.3(ii). If we put $\alpha_1 = \alpha_1$, $\beta_1 = \alpha_2$, and suppose

$$\mathbf{LM}(g_1) = x^{\alpha_1}y^{\beta}, \quad \mathbf{LM}(g_2) = x^{\alpha}y^{\beta_1},$$

then L is generated by $\{g_1, g_2\}$.

Proof Uising the division algorithm in $A_1(k)$, for $f \in L$, if we consider the remainder of f on division by $\{g_1, g_2\}$, it follows from the definition of α_1 and β_1 that L/L' is a finite dimensional k-space, where L' is the left ideal of $A_1(k)$ generated by $\{g_1, g_2\}$. Hence $L = L'$ by Bernstein inequality. □

It is well known (e.g. [Bj]) that every nonzero left ideal of $A_n(k)$ is generated by two elements. Proposition 2.4 may be regarded as an algorithmic realization of this fact in the special $n = 1$ case. We also refer to [Gal] for another algorithmic realization of this special case.

2.5. Remark (i) Let g be any finite dimensional k-Lie algebra and $U(g)$ the enveloping algebra of g. Then $U(g)$ is a solvable polynomial k-algebra with respect to $>_{grlex}$ in the sense of [K-RW]. Since the associated graded algebra of $U(g)$, with respect to the standard filtration on $U(g)$, is a commutative polynomial k-algebra, the similar results as mentioned in Theorem 2.3 hold for $U(g)$.
(ii) If we consider a homogeneous solvable polynomial algebra A (see [Li]) in the sense that $A = k[a_1, ..., a_n]$ is solvable with respect to a monomial ordering and

$$a_j a_i = \lambda_{ij} a_i a_j, \quad 1 \le i < j \le n, \ \lambda_{ij} \in k - \{0\},$$

then it is not hard to see that all results given in §1 may be generalized to A. Thus, if $A = k[a_1, ..., a_n]$ is an affine k-algebra with the standard filtration FA such that the associated graded algebra $G(A)$ is a homogeneous solvable polynomial algebra with respect to $>_{grlex}$ (hence A is a solvable polynomial algebra with respect to $>_{grlex}$ by [LW]), then all results given in §2 may be generalized to A. And furthermore, as pointed out by the referee, these results also hold for weighted degree term orders on A without much difficulty (see [BW] Ch.9, §3 Notes).

§3. Some Related Examples

In this section, we aim to construct some examples of the "smallest" simple modules over Weyl algebras.

Example 1 Consider the first Weyl algebra $A_1(k)$ with generators x, y. In [Dix] it is proved that the module $M = A_1(k)/A_1(k)(xy - \beta)$ with $\beta \in k$ is simple if and only if $\beta \notin \mathbb{Z}$. By Theorem 2.3 of §2 we see immediately that $e(M) = 2$ because $\{xy - \beta\}$ is a Gröbner basis of the left ideal $A_1(k)(xy - \beta)$. However, we claim that for *every* integer $n \ge 1$,

- the module $M = A_1(k)/L$ with L being generated by $\{xy+n, x^n\}$ is a simple $A_1(k)$-module.

Proof By checking the S-polynomial of $xy+n$ and x^n (see [K-RW] for the definition of S-polynomial) it is easy to see that $\{xy+n, x^n\}$ is a Gröbner basis of L. Hence L is a proper left ideal of $A_1(k)$ and GK.dim$(M) = 1$. From Theorem 2.3 of §2 it is also clear that $e(M) = 1$. This shows that M is simple. □

Let us further consider the k-algebra automorphism σ: $A_1(k) \longrightarrow A_1(k)$ with $\sigma(y) = x$, $\sigma(x) = -y$, and the module M in Example 1. It is easy to see that the *twisted module* of M by σ (see the definition in [Cou]), denoted M^σ, is of the form $A_1(k)/L'$ where L' is the left ideal of $A_1(k)$ generated by $\{xy-(n-1), y^n\}$, which is by ([Cou] Ch.5, Proposition 2.1) a simple $A_1(k)$-module, and by directly checking as for Example 1 it is also a simple module of multiplicity 1.

Before giving the next example, we prove a result concerning the dimension and multiplicity of the tensor product of two *cyclic* modules which has its own independent interest. We first look at the commutative case.

Let $k[X] = k[x_1, ..., x_n]$ and $k[Y] = k[y_1, ..., y_m]$ be polynomial k-algebras. Write $k[X,Y]$ for the polynomial k-algebra in the x's and y's. Then both $k[X]$ and $k[Y]$ are subalgebras of $k[X,Y]$. Let I and J be ideals of $k[X]$ and $k[Y]$, respectively. Considering the tensor product $k[X]\otimes_k k[Y]$ of the k-algebras $k[X]$ and $k[Y]$, and the tensor product $(k[X]/I)\otimes_k(k[Y]/J)$ of the $k[X]$-module $k[X]/I$ and the $k[Y]$-module $k[Y]/J$ which is a $k[X]\otimes_k k[Y]$-module under the naturally defined module operation, it is well known that

$$(\mathrm{I})\qquad \begin{cases} k[X]\otimes_k k[Y] & \cong \ k[X,Y] \text{ as algebras,} \\[2ex] \dfrac{k[X]}{I}\otimes_k\dfrac{k[Y]}{J} & \cong \ \dfrac{k[X,Y]}{K} \text{ as } k[X,Y]\text{-modules,} \end{cases}$$

where $K = \langle I, J\rangle$ denotes the ideal of $k[X,Y]$ generated by I and J.

Regarding $k[X]$ and $k[Y]$ as subalgebras of $k[X,Y]$, we may use $>_{grlex}$ on them such that

$$x_1 >_{grlex} x_2 >_{grlex} \cdots >_{grlex} x_n >_{grlex} y_1 >_{grlex} y_2 >_{grlex}, ..., >_{grlex} y_m.$$

3.1. Lemma With notation as above, if $G_1 = \{f_1, ..., fs\}$ is a Gröbner basis of I in $k[X]$ and $G_2 = \{g_1, ..., g_h\}$ is a Gröbner basis of J in $k[Y]$, then $G = \{f_1, ..., f_s, g_1, ..., g_h\}$ is a Gröbner basis of $K = \langle I, J\rangle$ in $k[X,Y]$.

Proof This is straightforward by checking the relative S-polynomials. □

3.2. Theorem With notation as before, we have

$$\begin{aligned} \dim V(K) &= \dim V(I) + \dim V(J), \\ e\left(\frac{k[X]}{I} \otimes_k \frac{k[Y]}{J}\right) &= e\left(\frac{k[X]}{I}\right) e\left(\frac{k[Y]}{J}\right), \end{aligned}$$

where $V(K)$ is the affine algebraic set defined by K in $\mathbf{A}_k^{n+m}$, and $V(I)$ resp. $V(J)$ denotes the affine algebraic set defined by I in $\mathbf{A}_k^n$ resp. the affine algebraic set defined by J in $\mathbf{A}_k^m$.

Proof We are sure that the first equality is known, but here we give an algorithmic proof for completeness.

Note that since the computation of dimension and multiplicity is completely determined by the monomial ideal generated by the leading terms from a Gröbner basis, it follows from Lemma 3.1 that we may assume: *I and J are monomial ideals*, and so K is also a monomial ideal.

Suppose $\dim V(I) = p$, $e(k[X]/I) = a$; $\dim V(J) = q$, $e(k[Y]/J) = b$. Then

$$\begin{aligned} C(I) &= C_0 \cup C_1 \cup \cdots \cup C_p \text{ with } C_p \neq \emptyset, \\ {}^aHP_I &= \frac{a}{p!}t^p + \text{ lower terms in } t; \\ C(J) &= C_0 \cup C_1 \cup \cdots \cup C_q \text{ with } C_q \neq \emptyset, \\ {}^aHP_J &= \frac{b}{q!}t^q + \text{ lower terms in } t, \end{aligned}$$

where $C(I)$ is as defined in §1 (1), similarly for $C(J)$ and the below $C(K)$. We claim that

$$C(K) = C_0 \cup C_1 \cup \cdots \cup C_{p+q} \text{ with } C_{p+q} \neq \emptyset.$$

To see this, let us write $e_1, ..., e_n, e_{n+1}, ..., e_{n+m}$ for the unit vectors in $\mathbb{Z}_{\geq 0}^{n+m}$, as defined in §1, and identify $e_1, ..., e_n$ with the unit vectors in $\mathbb{Z}_{\geq 0}^n$, $e_{n+1}, ..., e_{n+m}$ with the unit vectors in $\mathbb{Z}_{\geq 0}^m$ under the natural embedding of $\mathbb{Z}_{\geq 0}^n$ resp. $\mathbb{Z}_{\geq 0}^m$ into $\mathbb{Z}_{\geq 0}^{n+m}$. Note that if $\beta + [e_{i_1}, ..., e_{i_r}]$ is a translate of the r-dimensional coordinate subspace $[e_{i_1}, ..., e_{i_r}]$ contained in C_r, then $[e_{i_1}, ..., e_{i_r}]$ is also contained in C_r. Hence, if $[e_{i_1}, ..., e_{i_p}]$ is a p-dimensional coordinate subspace contained in C_p and $[e_{j_1}, ..., e_{j_q}]$ is a q-dimensional coordinate subspace contained in C_q, then by the construction of K it is easy to see that $[e_{i_1}, ..., e_{i_p}, e_{j_1}, ..., e_{j_q}]$ is a $p+q$-dimensional coordinate subspace contained in C_{p+q}. This shows that $C_{p+q} \neq \emptyset$. Moreover, if $C(K)$ could contain a $p+q+z$-dimensional coordinate subspace with $z \geq 1$, then again from the construction of K it is easy to see that $C(I)$ would contain a $p+z_1$-dimensional coordinate subspace with $z_1 \geq 1$, or $C(J)$ would contain a $q+z_2$-dimensional coordinate subspace with $z_2 \geq 1$, a contradiction. Therefore, the largest coordinate subspace contained in $C(K)$ is of dimension $p+q$. It follows from ([CLO′] Ch.9 §2, Proposition 2) that $\dim V(K) = p+q = \dim V(I) + \dim V(J)$, as desired.

In order to prove the equality for the multiplicity, let $\alpha + [e_{i_1}, ..., e_{i_p}]$ be a translate in $C_p \subset C(I)$, and $\beta + [e_{j_1}, ..., e_{j_q}]$ is a translate in $C_q \subset C(J)$. Then by the construction of K and the above argumentation we easily see that $(\alpha+\beta) + [e_{i_1}, ..., e_{i_p}, e_{j_1}, ..., e_{j_q}]$ is a translate in $C_{p+q} \subset C(K)$. Conversely, let $\gamma + [e_{k_1}, ..., e_{k_{p+q}}] \in C_{p+q}$ be a translate of the $p+q$-dimensional coordinate subspace $[e_{k_1}, ..., e_{k_{p+q}}]$, where $\gamma = \sum_{j \notin \{k_1,...,k_{p+q}\}} a_j e_j$. Again by the above argumentation $[e_{k_1}, ..., e_{k_{p+q}}]$ must be of the form $[e_{i_1}, ..., e_{i_p}, e_{j_1}, ..., e_{j_q}]$ where $[e_{i_1}, ..., e_{i_p}]$ is a p-dimensional coordinate subspace contained in C_p and $[e_{j_1}, ..., e_{j_q}]$ is a q-dimensional coordinate subspace contained in C_q. If we put

$$\begin{aligned} J_X &= \left\{ j \notin \{k_1, ..., k_{p+q}\} \;\middle|\; e_j \in \{e_1, ..., e_n\} - \{e_{i_1}, ..., e_{i_p}\} \right\} \\ \alpha &= \textstyle\sum_{j \in J_X} a_j e_j, \\ J_Y &= \left\{ j \notin \{k_1, ..., k_{p+q}\} \;\middle|\; e_j \in \{e_{n+1}, ..., e_{n+m}\} - \{e_{j_1}, ..., e_{j_q}\} \right\}, \\ \beta &= \textstyle\sum_{j \in J_Y} a_j e_j, \end{aligned}$$

it is easily seen that $\alpha + [e_{i_1}, ..., e_{i_p}]$ is a translate in C_p, $\beta + [e_{j_i}, ..., e_{j_q}]$ is a translate in C_q. Furthermore, it follows from Observation 1.3 of §1 and ([CLO′] Ch.9 §1, Proposition 3) that $|J_X| = n - p$, $|J_Y| = m - q$. Hence

$$\gamma + [e_{k_1}, ..., e_{k_{p+q}}] = (\alpha+\beta) + [e_{i_1}, ..., e_{i_p}, e_{j_1}, ..., e_{j_q}].$$

Thus we have shown that C_{p+q} contains exactly $ab = e(k[X]/I)e(k[Y]/J)$ different translates of the $p+q$-dimensional coordinate subspaces. Therefore,

$$e\left(\frac{k[X,Y]}{K}\right) = ab = e\left(\frac{k[X]}{I}\right) e\left(\frac{k[Y]}{J}\right),$$

as desired. □

Now we turn to Weyl algebras. Let $A_n(k)$ be the n-th Weyl algebra and $A_m(k)$ the m-th Weyl algebra over k. Then both $A_n(k)$ and $A_m(k)$ are subalgebras of the $n+m$-th Weyl algebra $A_{n+m}(k)$. Let I and J be left ideals of $A_n(k)$ and $A_m(k)$, respectively. Considering the tensor product $A_n(k) \otimes_k A_m(k)$ of the k-algebras $A_n(k)$ and $A_m(k)$, and the tensor product $(A_n(k)/I) \otimes_k (A_m(k)/J)$ of the $A_n(k)$-module $A_n(k)/I$ and the $A_m(k)$-module $A_m(k)/J$ which is a $A_n(k) \otimes_k A_m(k)$-module under the naturally defined module operation, it is well known that:

$$\text{(II)} \qquad \left\{ \begin{aligned} A_n(k) \otimes_k A_m(k) &\cong A_{n+m}(k) \text{ as algebras,} \\ \frac{A_n(k)}{I} \otimes_k \frac{A_m(k)}{J} &\cong \frac{A_{n+m}(k)}{K} \text{ as } A_{n+m}(k)\text{-modules,} \end{aligned} \right.$$

where $K = A_{n+m}I + A_{n+m}J$ denotes the left ideal of $A_{n+m}(k)$ generated by I and J.

Regarding $A_n(k)$ and $A_m(k)$ as subalgebras of A_{n+m}, we may use $>_{grlex}$ on them such that

$$x_1 >_{grlex} x_2 >_{grlex} \cdots >_{grlex} x_{2n} >_{grlex} y_1 >_{grlex} y_2 >_{grlex}, \dots, >_{grlex} y_{2n}.$$

The noncommutative version of Lemma 3.1 is easily verified also by looking at the S-polynomials.

3.3. Lemma With notation as above, if $G_1 = \{f_1, \dots, fs\}$ is a Gröbner basis in $A_n(k)$ and $G_2 = \{g_1, \dots, g_h\}$ is a Gröbner basis of J in $A_m(k)$, then $G = \{f_1, \dots, f_s, g_1, \dots, g_h\}$ is a Gröbner basis of $K = A_{n+m}I + A_{n+m}J$ in $A_{n+m}(k)$. □

In ([Cou] P.128, Theorem 4.1) it is mentioned that if M is a finitely generated $A_n(k)$-module and N is a finitely generated $A_m(k)$-module, then

$$\begin{aligned} \text{GK.dim}(M \otimes_k N) &= \text{GK.dim}(M) + \text{GK.dim}(N), \\ e(M \otimes_k N) &\leq e(M)e(N). \end{aligned}$$

It seems to us that the proof given there does not work for the multiplicity inequality. Nevertheless, from Theorem 2.2 of §2 and the above Theorem 3.2 it follows that we have the following result.

3.4. Theorem Let I and J be left ideals of $A_n(k)$ and $A_m(k)$, respectively. With notation as before, the following equalities hold.

$$\begin{aligned} \text{GK.dim}\left(\frac{A_n(k)}{I} \otimes_k \frac{A_m(k)}{J}\right) &= \text{GK.dim}\left(\frac{A_n(k)}{I}\right) + \text{GK.dim}\left(\frac{A_m(k)}{J}\right), \\ e\left(\frac{A_n(k)}{I} \otimes_k \frac{A_m(k)}{J}\right) &= e\left(\frac{k[X]}{I}\right) e\left(\frac{k[Y]}{J}\right). \end{aligned}$$

□

Finally, using Example 1 and Theorem 3.4 we are able to construct the "smallest" modules over $A_n(k)$, as follows.

Example 2 For $n \geq 1$, regarding the subalgebra $A(j)$ of $A_n(k)$ generated by x_j, y_j as the first Weyl algebra, let $L(j)$ be the left ideal of $A(j)$ generated by $\{x_j y_j + j, x_j^j\}$, $j = 1, \dots, n$, and K the left ideal of $A_n(k)$ generated by $\{x_j y_j + i, x_j^i\}_{i=j=1}^n$. Then

$$\frac{A_n(k)}{K} \cong \frac{A(1)}{L(1)} \otimes_k \frac{A(2)}{L(2)} \otimes_k \cdots \otimes_k \frac{A(n)}{L(n)}$$

is a simple $A_n(k)$-module with Gelfand-Kirillov dimension n and multiplicity 1.

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REFERENCES

[AL] J. Apel and W. Lassner, An extension of Buchberger's algorithm and calculations in enveloping fields of Lie algebras, *J. Symbolic Computation*, 6(1988), 361–370.

[Bj] J-E. Björk, *Rings of Differential Operators*, North-Holland Mathematical Library, Vol. 21, 1979.

[BW] T. Becker and V. Weispfenning, *Gröbner Bases*, Springer-Verlag, 1993.

[CLO′] D. Cox, J. Little and D. O′shea, *Ideals, Varieties, and Algorithms*, Springer-Verlag, 1992.

[Cou] S.C. Coutinho, *A primer of algebraic D-modules*, Cambridge University Press, 1995.

[Dix] J. Dixmier, Sur les algèbres de Weyl II, *Bull. Sci. Math.*, 94(1970), 289–301.

[Gal] A. Galligo, Some algorithmic questions on ideals of differential operators,*Proc. EUROCAL'85* LNCS 204, 1985, 413–421.

[K-RW] A. Kandri-Rody and V. Weispfenning, Non-commutative Gröbner bases in algebras of solvable type, *J. Symbolic Computation*, 9(1990), 1–26.

[Li] Huishi Li, Hilbert Polynomials of modules over homogeneous solvable polynomial algebras, *Comm. Alg.*, 5(27)1999.

[LW] Huishi Li and Wu Yihong, Filtered-graded transfer of Gröbner basis computation in solvable polynomial algebras, *Comm. Alg.*, 1(28)(2000), 15–32.

[MR] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons, 1987.

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