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Recursion operators of some equations of hydrodynamic type

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We give a general method for constructing recursion operators for some equations of hydrodynamic type, admitting a nonstandard Lax representation. We give several examples for $N=2$ and $N=3$ containing the equations of shallow water waves and its generalizations with their first two general symmetries and their recursion operators. We also discuss a reduction of $N+1$ systems to N systems of some new equations of hydrodynamic type.
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I. INTRODUCTION

Most of the integrable nonlinear partial differential equations admit Lax representations,

$$L_t=[A,L], \tag{1}$$

where L is a pseudo-differential operator of order m and A is a pseudo-differential operator. Recently[1] we established a new method for such integrable equations to construct their recursion operators. This method uses the hierarchy of equations,

$$L_{t_n}=[A_n,L], \tag{2}$$

and the Gel'fand–Dikkii[2] construction of the A_n-operators. Defining an operator R_n in the form

$$A_n=LA_{n-m}+R_n, \tag{3}$$

one then obtains relations among the hierarchies,

$$L_{t_n}=LL_{t_{n-m}}+[R_n;L]. \tag{4}$$

This equation allows to find L_{t_n} in terms of $L_{t_{n-m}}$. It is important to note that one does not need to know the exact form of A_n. For further details of the method see Ref. 1.

In Ref. 1 we introduced a direct method to determine a recursion operator of a system of evolution equations when its Lax representation is known. It has no direct reference to the Hamiltonian operators. Hence one may be able to determine the recursion operators when any one of the Hamiltonian operators are degenerate. In the same paper we gave several applications of the method. In all these examples we have considered the Lax representation is given either in a pseudo-differential operator or in matrix form (taking values in some lower dimensional Lie algebras). We call such Lax representations as standard Lax representation. On the other hand there are some systems of evolution equations, such as the equations of hydrodynamic type, which are obtained by nonstandard Lax represenations used in the present paper. We first show that the method introduced in Ref. 1 is also applicable here in the case of systems of equations of hydrodynamic types and we give several examples for illustration. These equations and their Hamil-

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tonian formulation (sometimes called the dispersion-less KdV system) were studied by Dubrovin and Novikov.[3] See Ref. 4 for more details on this subject (see also Ref. 5). It is known that these equations admit a nonstandard Lax representation,

$$\frac{\partial L}{\partial t}=\{A,L\}_k, \tag{5}$$

where A,L are differentiable functions of t,x,p on a Poisson manifold M with local coordinates (x,p) and $\{,\}_k$ is the Poisson bracket. On M we take this Poisson bracket $\{,\}_k=p^k\{,\}$, where $\{,\}$ is the canonical Poisson bracket and k is an integer. For more information on Poisson manifolds see Refs. 6 and 7. Equations of hydrodynamic type with the above Lax representations were studied in Refs. 8–11. Having such a Lax representation, we can consider a whole hierarchy of equations,

$$\frac{\partial L}{\partial t_n}=\{A_n,L\}_k. \tag{6}$$

We can also represent function A_n in the form given in (3) and apply our method[1] for the construction of a recursion operator for the equation (6). There are some other works[12–14] which also give recursion operators of some equations of hydrodynamic type. The form of these operators are different than the recursion operators presented in this work. Our method[1] produces recursion operators for hydrodynamic type of equations in the form $\mathcal{R}=A+B\,D^{-1}$ where A and B are functions of dynamical variables and their derivatives. All higher symmetries obtained by the repeated application of this recursion operator to translational symmetries also belong to the hydrodynamic type of equations. The recursion operators obtained in Refs. 12–14 are of the form $\mathcal{R}=C\,D+A+B\,D^{-1}\,E$, where A,B,C, and E are functions of dynamical variables and their derivatives.

In the next section we discuss the Lax representation with Poisson brackets for polynomial Lax functions. In Sec. III we give the method of construction of the recursion operators following Ref. 1. In Sec. IV we give several examples for $k=0$ and $k=1$. In Sec. V we consider the Poisson bracket for general k and let

$$L=p+S+Pp^{-1}, \tag{7}$$

and find the Lax equations and the corresponding recursion operator for $N=2$. In Sec. VI we consider the Lax function

$$L=p^{\gamma-1}+u+\frac{v^{\gamma-1}}{(\gamma-1)^2}p^{-\gamma+1}, \tag{8}$$

and take $k=0$. We obtain the equations corresponding to the polytropic gas dynamics and its recursion operators.[6,10] It is interesting to note that the systems of equations and their recursion operators obtained in Secs. V and VI are transformable into each other. In Sec. VII we give a method reduction from an $N+1$ system to an N system and from an $N+1$ system to an $N-1$ system by letting one of the symmetrical variables (defined in the text) either to zero or equating to another variable. The systems obtained by the reduction are equivalent to the systems obtained by the Lax function (in symmetrical variables) having zeros with multiplicities greater than one. Reduced systems are shown to be also integrable, i.e., they admit recursion operators.

II. LAX FORMULATION WITH POISSON BRACKET

We start with the definition of the standard Poisson bracket. Let $f(x,p)$ and $g(x,p)$ be differentiable functions of their arguments. Then the standard Poisson bracket is defined by (see Refs. 6 and 9 for more details)

$$\{f,g\}=\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial g}{\partial p}. \tag{9}$$

We give a slight modification of this bracket as[9]

$$\{f,g\}_k=p^k\{f,g\}, \tag{10}$$

where k is an integer. It is easy to prove that $\{,\}_k$ also defines a Poisson bracket for all $k\in\mathbb{Z}$. Although this bracket is equivalent to $\{,\}$, under $p^k(d/dp)=d/dq$ where q is the new variable, we shall keep using it. The main reason is technical. There is a nice duality between the systems obtained by polynomial Lax representation, $L=p^N+\cdots$, with Poisson bracket $\{,\}_k$ and by Lax represention $L=p^\gamma[p^N+\cdots]$ with Poisson bracket $\{,\}$. For illustration we have examples, equations governing the polytropic gas dynamics, given in Propositions 6 and 7.

For each $k\in\mathbf{Z}$ we can consider hierarchies of equations of hydrodynamic type, defined in terms of the Lax function,

$$L=p^{N-1}+\sum_{i=-1}^{N-2}p^iS_i(x,t), \tag{11}$$

by the Lax equation

$$\frac{\partial L}{\partial t_n}=\{(L^{n/(N-1)})_{\geqslant -k+1};L\}_k, \tag{12}$$

where $n=j+l(N-1)$ and $j=1,2,\ldots,(N-1),l\in\mathbf{N}$. So we have a hierarchy for each k and $j=1,\ldots,(N-1)$. Also, we require $n\geqslant -k+1$ to ensure that $(L^{n/(N-1)})_{\geqslant -k+1}$ is not zero. With the choice of Poisson brackets $\{,\}_k$, we must take a certain part of the series expansion of $L^{n/(N-1)}$ to get the consistent equation (12). This part is $(L^{n/(N-1)})_{\geqslant -k+1}$.

The Lax function (11) can also be written in terms of symmetric variables $u_1,\ldots,u_N$,

$$L=\frac{1}{p}\prod_{j=1}^{N}(p-u_j), \tag{13}$$

that is $u_1,\ldots,u_N$ are roots of the polynomial

$$p^{N-1}+S_{N-2}p^{N-2}+\ldots+S_{-1}p^{-1}.$$

In new variables the equation (12) is invariant under transposition of variables.

III. RECURSION OPERATORS

For each hierarchy of the equations (12), depending on the pair (N,k), we can find a recursion operator.

Lemma 1: For any n,

$$L_n=LL_{n-(N-1)}+\{R_n;L\}_k, \tag{14}$$

where function R_n has a form

$$R_n=\sum_{i=0}^{N-2}p^{i-k}A_i(S_{-1}\ldots S_{N-2},\partial S_{-1}/\partial t_{n-(N-1)}\ldots\partial S_{N-2}/\partial t_{n-(N-1)}). \tag{15}$$

Proof:

$$(L^{n/(N-1)})_{\geqslant -k+1}=[L(L^{n/(N-1)-1})_{\geqslant -k+1}+L(L^{n/(N-1)-1})_{<-k+1}]_{\geqslant -k+1}.$$

So,

$$(L^{n/(N-1)})_{\geqslant -k+1}=L(L^{n/(N-1)-1})_{\geqslant -k+1}+\big(L(L^{n/(N-1)-1})_{<-k+1}\big)_{\geqslant -k+1}$$
$$-\big(L(L^{n/(N-1)-1})_{\geqslant -k+1}\big)_{<-k+1}. \tag{16}$$

If we put

$$R_n=\big(L(L^{n/(N-1)-1})_{<-k+1}\big)_{\geqslant -k+1}-\big(L(L^{n/(N-1)-1})_{\geqslant -k+1}\big)_{<-k+1},$$

then

$$(L^{n/(N-1)})_{\geqslant -k+1}=L(L^{n/(N-1)-1})_{\geqslant -k+1}+R_n.$$

Hence,

$$L_n=\{(L^{n/(N-1)})_{\geqslant -k+1};L\}_k=\{L(L^{n/(N-1)-1})_{\geqslant -k+1}+R_n;L\}_k=LL_{n-(N-1)}+\{R_n;L\}_k, \tag{17}$$

and (14) is satisfied. Evaluating powers of $\big(L(L^{n/(N-1)-1})_{<-k+1}\big)_{\geqslant -k+1}$ and $-\big(L(L^{n/(N-1)-1})_{\geqslant -k+1}\big)_{<-k+1}$ we get that R_n has form (15). □

Lemma 2: A recursion operator for the hierarchy (12) is given by equalities, for $m=N-2,N-3,\dots,-1$,

$$\frac{\partial S_m}{\partial t_n}=\sum_{j=-1}^{m+1} S_j\,\frac{\partial S_{m-j}}{\partial t_{n-(N-1)}}+\sum_{j=-1}^{m+1}(j+1-k)A_{j+1}S_{m-j,x}-\sum_{j=-1}^{m+1}(m-j)A_{j+1,x}S_{m-j}, \tag{18}$$

where to simplify the above formula we have defined that $S_{N-1}=1$ *and* $S_{N-1,x}=0$, $(\partial S_{N-1}/\partial t_n)=0$. *Coefficients* $A_{N-2},A_{N-3},\dots,A_0$ *can be found from the recursion relations, for* $m=N-2,\dots,-1$,

$$(N-1)A_{m,x}=\sum_{j=m}^{N-1} S_j\,\frac{\partial S_{(N-2)+m-j}}{\partial t_{n-(N-1)}}+\sum_{j=m}^{N-2}(j+1-k)A_{j+1}S_{N-2+m-j,x}$$
$$-\sum_{j=m}^{N-2}(N-2+m-j)A_{j+1,x}S_{N-2+m-j}. \tag{19}$$

Proof: Let us write the equality (14), using (15) for R_n,

$$\sum_{i=-1}^{N-2} p^i\,\frac{\partial S_i}{\partial t_n}=\left(p^{N-1}+\sum_{i=-1}^{N-2} p^iS_i\right)\left(\sum_{i=-1}^{N-2} p^i\,\frac{\partial S_{(N-2)+m-j}}{\partial t_{n-(N-1)}}\right)+p^k\left(\sum_{j=0}^{N-1}(j-k)p^{j-k-1}A_j\right)$$
$$\times\left(\sum_{j=-1}^{N-2} p^jS_{j,x}\right)-p^k\left(\sum_{j=0}^{N-1} p^{j-k}A_{j,x}\right)\left((N-1)p^{N-2}+\sum_{j=-1}^{N-2} jp^{j-1}S_j\right).$$

To have the equality, the coefficients of $p^{2N-3},\dots,p^{N-1}$ and p^{-2} must be zero; it gives recursion relations to find $A_{N-2},\dots,A_0$. The coefficients of $p^{N-2},\dots,p^{-1}$ give the expressions for $\partial S_{N-2}/\partial t_{n,\dots,}\partial S_{-1}/\partial t_n$. □

Although the recursion operator $\mathcal{R}$, given by (18), is a pseudo-differential operator, but it gives a hierarchy of local symmetries starting from the equation itself. Indeed, equalities (18), (19) give expressions $\partial S_{N-2}/\partial t_{n,\ldots,}\partial S_{-1}/\partial t_n$ in terms of $S_{N-2},\ldots,S_{-1}$ and $\partial S_{N-2}/\partial t_{n-(N-1),\ldots,}$ $\partial S_{-1}/\partial t_{n-(N-1)}$. Hence, the recursion operator $\mathcal{R}$ is constructed in such a way that

$$\{(L^{n/(N-1)+1})_{\geqslant -k+1};L\}_k=\mathcal{R}(\{(L^{n/(N-1)})_{\geqslant -k+1};L\}_k). \tag{20}$$

IV. SOME INTEGRABLE SYSTEMS

We shall consider first some examples for $k=0$, $k=1$ and the general case in the next section.

A. Multicomponent hierarchy containing also the shallow water wave equations, $k=0$

This hierarchy corresponds to the case $k=0$. Let us give the first equation of hierarchy and a recursion operator for $N=2,3$.

Proposition 1: In the case $N=2$ one has the Lax function,

$$L=p+S+P\,p^{-1},$$

and the Lax equation for $n=2$, given by (47), when $k=0$,

$$\begin{aligned}\tfrac{1}{2}S_t&=SS_x+P_x,\\ \tfrac{1}{2}P_t&=SP_x+PS_x,\end{aligned} \tag{21}$$

and the recursion operator, given by (48),

$$\mathcal{R}=\begin{pmatrix} S+S_xD_x^{-1} & 2\\ 2P+P_xD_x^{-1} & S\end{pmatrix}. \tag{22}$$

These equations are known as the shallow water wave equations or as the equations of polytropic gas dynamics for $\gamma=2$ (See Sec. VI).

The first two symmetries of the system (21) are given by

$$\begin{aligned}S_{t_1}&=(S^3+6SP)_x,\\ P_{t_1}&=(3S^2P+3P^2)_x,\end{aligned} \tag{23}$$

$$\begin{aligned}S_{t_2}&=(S^4+12S^2P+6P^2)_x,\\ P_{t_2}&=(4S^3P+12SP^2)_x.\end{aligned} \tag{24}$$

These are all commuting symmetries.

Remark 1: In symmetric variables the system (21) is written as

$$\begin{aligned}\tfrac{1}{2}u_t&=(u+v)u_x+uv_x,\\ \tfrac{1}{2}v_t&=vu_x+(u+v)v_x,\end{aligned} \tag{25}$$

and the recursion operator (22) takes the form

$$\mathcal{R}=\begin{pmatrix} u+v+u_xD_x^{-1} & 2u+u_xD_x^{-1}\\ 2v+v_xD_x^{-1} & u+v+v_xD_x^{-1}\end{pmatrix}. \tag{26}$$

Proposition 2: In the case $N=3$ one has the Lax function

$$L=p^2+pS+P+p^{-1}Q,$$

and the Lax equation with $n=3$ *is*

$$\begin{aligned}\tfrac{1}{3}S_t&=(\tfrac{1}{2}P-\tfrac{1}{8}S^2)S_x+\tfrac{1}{2}SP_x+Q_x,\\ \tfrac{1}{3}P_t&=\tfrac{1}{2}QS_x+(\tfrac{1}{8}S^2+\tfrac{1}{2}P)P_x+SQ_x,\\ \tfrac{1}{3}Q_t&=\tfrac{1}{4}SQS_x+\tfrac{1}{2}QP_x+(\tfrac{1}{8}S^2+\tfrac{1}{2}P)Q_x.\end{aligned}\tag{27}$$

The recursion operator, corresponding to this equation, is

$$\mathcal{R}=\begin{pmatrix} -\dfrac{S^2}{4}+P+P_xD_x^{-1}-\dfrac{S_x}{4}D_x^{-1}\cdot S & \dfrac{S}{2}+\dfrac{S_x}{2}D_x^{-1} & 3\\ \dfrac{3Q}{2}+\left(Q_x+\dfrac{P_xS}{2}\right)D_x^{-1}-\dfrac{P_x}{4}D_x^{-1}\cdot S & P+\dfrac{P_x}{2}D_x^{-1} & 2S\\ \dfrac{SQ}{4}+\left(\dfrac{SQ_x}{2}+\dfrac{S_xQ}{2}\right)D_x^{-1}-\dfrac{Q_x}{4}D_x^{-1}\cdot S & \dfrac{3Q}{2}+\dfrac{Q_x}{2}D_x^{-1} & P\end{pmatrix}.\tag{28}$$

Proof: Using (19) we find the function R_n and using (18) we find the recursion operator (28). □

Remark 2: In symmetric variables the equation (27) is written as

$$\begin{aligned}\tfrac{1}{3}u_t&=(-\tfrac{1}{8}u^2+\tfrac{1}{2}(uv+uw+vw)+\tfrac{1}{8}(v+w)^2)u_x+(\tfrac{1}{4}u^2+\tfrac{1}{4}uv+\tfrac{3}{4}uw)v_x+(\tfrac{1}{4}u^2+\tfrac{1}{4}uw+\tfrac{3}{4}uv)w_x,\\ \tfrac{1}{3}v_t&=(\tfrac{1}{4}v^2+\tfrac{1}{4}uv+\tfrac{3}{4}vw)u_x+(\tfrac{1}{4}v^2+\tfrac{1}{4}vw+\tfrac{3}{4}uv)w_x+(-\tfrac{1}{8}v^2+\tfrac{1}{2}(uv+uw+vw)+\tfrac{1}{8}(u+w)^2)v_x,\\ \tfrac{1}{3}w_t&=(\tfrac{1}{4}w^2+\tfrac{1}{4}uw+\tfrac{3}{4}wv)u_x+(\tfrac{1}{4}w^2+\tfrac{1}{4}wv+\tfrac{3}{4}uw)v_x+(-\tfrac{1}{8}w^2+\tfrac{1}{2}(uv+uw+vw)+\tfrac{1}{8}(v+u)^2)w_x,\end{aligned}\tag{29}$$

and the recursion operator takes the form (A1) given in the Appendix.

B. Toda hierarchy ($k=1$)

Toda hierarchy corresponds to the case $k=1$.[9] Let us give the first equation of hierarchy and a recursion operator for $N=2$ and $N=3$.

Proposition 3: In the case $N=2$ *and* $n=1$ *one has the Lax function*

$$L=p+S+P\,p^{-1},$$

and the Lax equation for $n=1$, *given by (41),*

$$\begin{aligned}S_t&=P_x,\\ P_t&=PS_x,\end{aligned}\tag{30}$$

and the recursion operator, given by (42),

$$\mathcal{R}=\begin{pmatrix} S & 2+P_xD_x^{-1}\cdot P^{-1}\\ 2P & S+S_xPD_x^{-1}\cdot P^{-1}\end{pmatrix}.\tag{31}$$

The first two symmetries of the equation (30) are given by

$$
\begin{aligned}
S_{t_1} &= (2SP)_x, \\
P_{t_1} &= P(2P+S^2)_x,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
S_{t_2} &= (3S^2P+3P^2)_x, \\
P_{t_2} &= P(6PS+S^3)_x.
\end{aligned}
\tag{33}
$$

Remark 3: In symmetric variables the equation (30) is written as

$$
\begin{aligned}
u_t &= uv_x, \\
v_t &= vu_x,
\end{aligned}
\tag{34}
$$

and the recursion operator (31) takes the form

$$
\mathcal{R} = \begin{pmatrix} u+v+uv_xD_x^{-1}\cdot u^{-1} & 2u+uv_xD_x^{-1}\cdot v^{-1} \\ 2v+vu_xD_x^{-1}\cdot u^{-1} & u+v+vu_xD_x^{-1}\cdot v^{-1} \end{pmatrix}.
\tag{35}
$$

Proposition 4: In the case $N=3$ and $n=1$ one has the Lax function

$$
L = p^2 + pS_1 + P + p^{-1}Q,
$$

and the Lax equation with $n=1$ is

$$
\begin{aligned}
S_t &= P_x - \tfrac{1}{2}SS_x, \\
P_t &= Q_x, \\
Q_t &= \tfrac{1}{2}QS_x.
\end{aligned}
\tag{36}
$$

The recursion operator, corresponding to this equation, is

$$
\mathcal{R} = \begin{pmatrix} P-\frac{1}{4}S^2+(\frac{1}{2}P_x-\frac{1}{4}SS_x)D_x^{-1} & \frac{1}{2}S & 3+2Q_xD_x^{-1}\cdot Q^{-1} \\ \frac{3}{2}Q+\frac{1}{2}Q_xD_x^{-1} & P & 2S+(SQ)_xD_x^{-1}\cdot Q^{-1} \\ \frac{1}{4}SQ+\frac{1}{4}S_xQD_x^{-1} & \frac{3}{2}Q & P+P_xQD_x^{-1}\cdot Q^{-1} \end{pmatrix}.
\tag{37}
$$

Proof: Using equalities (19) we find the function R_n and using (18) we find the recursion operator (37). □

Remark 4: In symmetric variables the equation (36) is written as

$$
\begin{aligned}
u_t &= \tfrac{1}{2}u(-u_x+v_x+w_x), \\
v_t &= \tfrac{1}{2}v(+u_x-v_x+w_x), \\
w_t &= \tfrac{1}{2}w(+u_x+v_x-w_x),
\end{aligned}
\tag{38}
$$

and the recursion operator takes the form (A2) given in the Appendix.

V. LAX EQUATION FOR GENERAL *k*

We shall only consider the case where $N=2$. We have the Lax function

$$
L = p + S + Pp^{-1},
\tag{39}
$$

and the Lax equation

$$\frac{\partial L}{\partial t_n}=\{(L^n)_{\geqslant -k+1};L\}_k. \tag{40}$$

We consider two cases $k\geqslant 1$ and $k\leqslant 0$.

A. The first case $k\geqslant 1$

Proposition 5: In the case $N=2$ and $k\geqslant 1$ one has the Lax equation

$$\begin{aligned} S_t&=kP^{k-1}P_x, \\ P_t&=kP^kS_x, \end{aligned} \tag{41}$$

and the recursion operator for this equation is

$$\mathcal{R}=\begin{pmatrix} S+(1-k)S_xD_x^{-1} & 2+kP^{k-1}P_xD_x^{-1}\cdot P^{-k} \\ 2P+(1-k)P_xD_x^{-1} & S+kS_xP^kD_x^{-1}\cdot P^{-k} \end{pmatrix}. \tag{42}$$

Proof: The smallest power of p in L^n is $-n$. To have powers less than $-k+1$ we must put $n=k$. If there are no such powers then Poisson brackets are $\{(L^n);L\}_k=0$.

Let us calculate the Lax equation,

$$L_t=\{(L^k)_{\geqslant -k+1};L\}_k=-\{(L^k)_{\leqslant -k};L\}_k.$$

We have $(L^k)_{\leqslant -k}=[(p+S+Pp^{-1})^k]_{\leqslant -k}=P^kp^{-k}$, thus

$$L_t=-\{P^kp^{-k};p+S+Pp^{-1}\}_k.$$

And we get the equation (41). Using (18), (19) we find the recursion operator (42). □

First two symmetries are given as follows:

$$\begin{aligned} S_{t_1}&=(k+1)(P^kS)_x, \\ P_{t_1}&=(k+1)P^k\left(P+\frac{k}{2}S^2\right)_x. \end{aligned} \tag{43}$$

$$\begin{aligned} S_{t_2}&=(k+1)(k+2)\left(\frac{1}{2}P^kS^2+\frac{1}{k+1}P^{k+1}\right)_x, \\ P_{t_2}&=(k+1)(k+2)P^k\left(PS+\frac{k}{6}S^3\right)_x. \end{aligned} \tag{44}$$

Remark 5: In symmetric variables the equation (41) is written as

$$\begin{aligned} u_t&=ku^kv^{k-1}v_x, \\ v_t&=ku^{k-1}v^ku_x, \end{aligned} \tag{45}$$

and the recursion operator (42) takes the form

$$\mathcal{R}=\begin{pmatrix} u+v+(1-k)u_xD_x^{-1}+ & 2u+(1-k)u_xD_x^{-1}+ \\ ku^kv^{k-1}v_xD_x^{-1}\cdot u^{-k}v^{-k+1} & ku^kv^{k-1}v_xD_x^{-1}\cdot u^{-k+1}v^{-k} \\ 2v+(1-k)v_xD_x^{-1}+ & u+v+(1-k)v_xD_x^{-1}+ \\ ku^{k-1}v^ku_xD_x^{-1}\cdot u^{-k}v^{-k+1} & ku^{k-1}v^ku_xD_x^{-1}\cdot u^{-k+1}v^{-k} \end{pmatrix}. \tag{46}$$

B. The second case $k\leqslant 0$

Proposition 6: In the case $N=2$ and $k\leqslant 0$ one has the Lax equation

$$\begin{aligned} S_t&=(-k+2)(-k+1)SS_x+(-k+2)P_x, \\ P_t&=(-k+2)(-k+1)SP_x+(-k+2)S_xP, \end{aligned} \tag{47}$$

and the recursion operator for this equation is

$$\mathcal{R}=\begin{pmatrix} S+(1-k)S_xD_x^{-1} & 2+kP^{k-1}P_xD_x^{-1}\cdot P^{-k} \\ 2P+(1-k)P_xD_x^{-1} & S+kS_xP^kD_x^{-1}\cdot P^{-k} \end{pmatrix}. \tag{48}$$

Proof: The largest power of p in L^n is p^n. To have powers larger than $-k+1$ we must put $n=-k+1$. Then we have

$$(L^{-k+1})_{\geqslant -k+1}=[(p+S+Pp^{-1})^{-k+1}]_{\geqslant -k+1}=p^{-k+1};$$

thus

$$L_t=\{p^{-k+1};p+S+Pp^{-1}\}_k.$$

Then the Lax equation becomes

$$S_t=S_x,$$

$$P_t=P_x.$$

This is a trivial equation; let us calculate the second symmetry. We have $(L^{-k+2})_{\geqslant -k+1}=[(p+S+Pp^{-1})^{-k+1}]_{\geqslant -k+1}=p^{-k+2}+(-k+2)Sp^{-k+1}$, thus

$$L_t=\{p^{-k+2}+(-k+2)Sp^{-k+1};p+S+Pp^{-1}\}_k.$$

We get the equation (47). Using (18), (19) we find the recursion operator (48). □

First two symmetries are given as follows:

$$\begin{aligned} S_{t_1}&=(k-2)(k-3)(P\,S+\tfrac{1}{6}(1-k)S^3)_x, \\ P_{t_1}&=(k-2)(k-3)(SS_xP+\tfrac{1}{2}(1-k)S^2\,P_x+PP_x), \end{aligned} \tag{49}$$

$$\begin{aligned} S_{t_2}&=(2-k)(3-k)(4-k)\left(\frac{1}{2}S^2P+\frac{1}{6}S^4+\frac{1}{2(2-k)}P^2\right)_x, \\ P_{t_2}&=(2-k)(3-k)(4-k)\left(\frac{1}{2}S^2S_xP+\frac{1}{6}(1-k)S^3\,P_x+SPP_x+\frac{1}{(2-k)}P^2S_x\right). \end{aligned} \tag{50}$$

Remark 6: In symmetric variables the equation (47) is written as

$$u_t=(-k+2)(1-k)(u+v)u_x+(-k+2)uv_x,$$

$$v_t=(-k+2)vu_x+(-k+2)(1-k)(u+v)v_x, \tag{51}$$

and the recursion operator (48) takes the form

$$\mathcal{R}=\begin{pmatrix} u+v+(1-k)u_xD_x^{-1}+ & 2u+(1-k)u_xD_x^{-1}+ \\ ku^kv^{k-1}v_xD_x^{-1}\cdot u^{-k}v^{-k+1} & ku^kv^{k-1}v_xD_x^{-1}\cdot u^{-k+1}v^{-k} \\ 2v+(1-k)v_xD_x^{-1}+ & u+v+(1-k)v_xD_x^{-1}+ \\ ku^{k-1}v^ku_xD_x^{-1}\cdot u^{-k}v^{-k+1} & ku^{k-1}v^ku_xD_x^{-1}\cdot u^{-k+1}v^{-k} \end{pmatrix}. \tag{52}$$

In this section, to obtain the recursion operators we have considered two different cases $k\leqslant 0$ and $k\geqslant 1$ to simplify some technical problems in the method. At the end we obtained recursion operators having the same forms (42) and (48). Hence any one of these represent the recursion operator for $k\in\mathbb{Z}$. It seems, comparing the results, that the systems of equations in one case are symmetries of the other case. For instance, the system (47) is a symmetry of system (41). Hence we may consider only one case with recursion operator (42) for all integer values of k.

VI. LAX FUNCTION FOR POLYTROPIC GAS DYNAMICS

In this section we consider another Lax function, introduced in Ref. 10,

$$L=p^{\gamma-1}+u+\frac{v^{\gamma-1}}{(\gamma-1)^2}p^{-\gamma+1}, \tag{53}$$

and the Lax equation

$$\frac{\partial L}{\partial t}=\frac{\gamma-1}{\gamma}\{(L^{\gamma/(\gamma-1)})_{\geqslant 1},L\}_0, \tag{54}$$

gives the equations of the polytropic gas dynamics.

Proposition 7: The Lax equation corresponding to (54) is

$$u_t+uu_x+v^{\gamma-2}v_x=0,$$
$$v_t+(uv)_x=0. \tag{55}$$

Proof: Expanding the function (53) around the point $p=\infty$, we have

$$\left(p^{\gamma-1}+u+\frac{v^{\gamma-1}}{(\gamma-1)^2}p^{-\gamma+1}\right)^{\gamma/(\gamma-1)}=p^{\gamma}+\frac{\gamma}{\gamma-1}pu+\dots;$$

all other terms have negative powers of p. Therefore

$$(L^{\gamma/(\gamma-1)})_{\geqslant 1}=p^{\gamma}+\frac{\gamma}{\gamma-1}pu,$$

and the Lax equation (54) corresponds to (55). □

Proposition 8: The recursion operator for the equation (55) is

$$\mathcal{R}=\begin{pmatrix} u+\dfrac{u_x}{\gamma-1}D_x^{-1} & \dfrac{2v^{\gamma-2}}{\gamma-1}+\dfrac{(v^{\gamma-2})_x}{\gamma-1}D_x^{-1} \\ \dfrac{2v}{\gamma-1}+\dfrac{v_x}{\gamma-1}D_x^{-1} & u+\dfrac{\gamma-2}{\gamma-1}u_xD_x^{-1} \end{pmatrix}. \tag{56}$$

Proof: Using the equation

$$\frac{\partial L}{\partial t_{n+1}}=L\,\frac{\partial L}{\partial t_n}+\{R_n,L\},$$

in the same way as for the polynomial Lax function one can find the recursion operator (56). □

It is interesting to note that the equation (47) and equations of polytropic gas dynamics (55) are related by the following change of variables:

$$S=\frac{u}{(-k+2)(-k+1)},\qquad P=\frac{v^{1/(-k+1)}}{(-k+2)^2}, \tag{57}$$

where $\gamma=(-k+2)/(-k+1)$. We note that under this change of variables recursion operator (48) is mapped to the recursion operator (56).

VII. REDUCTION

In this section we consider reductions of the equation (12), written in symmetric variables, by setting $u_1=0$, or $u_1=u_N,\dots,$ or $u_1=u_2=\cdots,=u_N$. These reductions correspond to the Lax equations with different Lax functions. For reduction $u_1=0$ we have a polynomial Lax function with simple roots $L=(p-u_N)\cdots(p-u_2)$ and for reduction $u_N=u_1$ we have a polynomial Lax function with a root of multiplicity two $L=(1/p)\,(p-u_N)^2(p-u_{N-1})\dots(p-u_2)$, etc. We note that instead of working on the Lax functions with higher multiplicities like the last example one can take a polynomial Lax function without any multiplicities and perform the reductions we propose in this section.

A. Reduction $u_1=0$

Let us write the equation (12) as

$$\Delta(u_N,\dots,u_1)=0, \tag{58}$$

where Δ is a differential operator. Then

$$\Delta(u_N,\dots,u_1)|_{u_1=0}=\left(\frac{\tilde{\Delta}(u_N,\dots,u_2),}{0}\right), \tag{59}$$

where $\tilde{\Delta}$ is another differential operator. Indeed, following Ref. 8 for the Lax function $L=(1/p)\,\Pi_{j=1}^{N}(p-u_j)$ we have

$$\frac{\partial L}{\partial t}=L\sum_{j=1}^{N}\frac{u_{j,t}}{p+u_j},$$

$$\frac{\partial L}{\partial x}=L\sum_{j=1}^{N}\frac{u_{j,x}}{p+u_j},$$

and

$$\frac{\partial L}{\partial p}=L\left(-\frac{1}{p}+\sum_{j=1}^{N}\frac{1}{p+u_j}\right).$$

Thus $u_{j,t}=\mathrm{Res}_{p=-u_j}\{M,L\}_k$, where $M=(L^{n/(N-1)})_{\geqslant -k+1}$. The Lax equation (12) can be written as

$$\sum_{j=1}^{N}\frac{u_{j,t}}{p+u_j}=p^kM_p\sum_{j=1}^{N}\frac{u_{j,x}}{p+u_j}-p^kM_x\left(-\frac{1}{p}+\sum_{j=1}^{N}\frac{1}{p+u_j}\right). \tag{60}$$

Note that p^kM_x and p^kM_p are polynomials. So, if we put $u_1=0$ and calculate the residue of the right hand side at $p=0$ we get (59). A new equation,

$$\tilde{\Delta}(u_N,\ldots,u_2)=0, \tag{61}$$

is also integrable and a recursion operator of this equation can be obtained as a reduction of the recursion operator of the equation (58). Let $\mathcal{R}$ be the recursion operator of (58) given by Lemma 2, then

$$\mathcal{R}|_{u_1=0}=\left(\begin{array}{c|c}\tilde{R} & * \\ \hline 0\cdots0 & 0\end{array}\right). \tag{62}$$

Indeed, we found the recursion operator using formula (14). This formula can be written as

$$\sum_{j=1}^{N}\frac{u_{j,t_n}}{p+u_j}=LL_{n-(N-1)}+p^kR_{n,p}\sum_{j=1}^{N}\frac{u_{j,x}}{p+u_j}-p^kR_{n,x}\left(-\frac{1}{p}+\sum_{j=1}^{N}\frac{1}{p+u_j}\right) \tag{63}$$

and in the same way as for the reduction of (58) we have (62); note, that $p^kR_{n,x}$ and $p^kR_{n,p}$ are also polynomials.

Lemma 3: The operator $\tilde{R}$ is a recursion operator of the equation (61).

Proof: Equation (61) is an evolution equation, so, to prove that $\tilde{R}$ is a recursion operator we must prove that for any solution $(u_N,\ldots,u_2)$ of (61) the following equality holds (see Ref. 6):

$$D_{\tilde{\Delta}}\tilde{R}=\tilde{R}D_{\tilde{\Delta}},$$

where $D_{\tilde{\Delta}}$ is a Frechet derivative of $\tilde{\Delta}$.

If $(u_N,\ldots,u_2)$ is a solution of (61) then $(u_N,\ldots,u_2,u_1=0)$ is a solution of (58) and for the solution $(u_N,\ldots,u_2,u_1=0)$ we have

$$D_{\Delta}\mathcal{R}=\mathcal{R}D_{\Delta}. \tag{64}$$

Next

$$D_{\Delta}|_{u_1=0}=\left(\begin{array}{c|c}\tilde{D} & * \\ \hline 0\cdots0 & *\end{array}\right)$$

and

$$\mathcal{R}|_{u_1=0}=\left(\begin{array}{c|c}\tilde{R} & * \\ \hline 0\cdots0 & 0\end{array}\right).$$

Hence by (64) we have $\tilde{D}\tilde{R}=\tilde{R}\tilde{D}$. Calculating the Frechet derivative, we take derivatives with respect to one variable, considering other variables as constants. Thus, to calculate $\tilde{D}$ we can put $u_1=0$ and differentiate with respect to other variables or we can first differentiate and then put $u_1=0$. It means that $\tilde{D}=D_{\tilde{\Delta}}$ and

$$D_{\tilde{\Delta}}\tilde{R}=\tilde{R}D_{\tilde{\Delta}}.$$

□

Let us consider the reduction of systems, given by Remark 2 and Remark 4 and their recursion operators.

Proposition 9: Putting $w=0$ in (38) and (A2) we obtain a new system,

$$\begin{aligned} u_t &= \tfrac{1}{2}u(-u_x+v_x), \\ v_t &= \tfrac{1}{2}v(+u_x-v_x), \end{aligned} \tag{65}$$

and its recursion operator,

$$\mathcal{R}=\begin{pmatrix} -uv+\dfrac{u}{4}(u+v) & -\dfrac{u}{4}(u+v) \\ +\dfrac{u}{4}(u_x-v_x)D_x^{-1} & +\dfrac{u}{4}(u_x-v_x)D_x^{-1} \\ -\dfrac{v}{4}(u+v) & -uv+\dfrac{v}{4}(u+v) \\ +\dfrac{v}{4}(-u_x+v_x)D_x^{-1} & +\dfrac{v}{4}(-u_x+v_x)D_x^{-1} \end{pmatrix}, \tag{66}$$

respectively. □

Proposition 10: Putting $w=0$ in (29) and (A1) we obtain a new system,

$$\begin{aligned} \tfrac{1}{3}u_t &= (-\tfrac{1}{8}u^2+\tfrac{1}{2}uv+\tfrac{1}{8}v^2)u_x+(\tfrac{1}{4}u^2+\tfrac{1}{4}uv)v_x, \\ \tfrac{1}{3}v_t &= (\tfrac{1}{4}v^2+\tfrac{1}{4}uv)u_x+(-\tfrac{1}{8}v^2+\tfrac{1}{2}uv+\tfrac{1}{8}u^2)v_x, \end{aligned} \tag{67}$$

and its recursion operator,

$$\mathcal{R}=\begin{pmatrix} -\dfrac{u^2}{4}+\dfrac{3uv}{4}+\left(\dfrac{u_xv}{2}+\dfrac{uv_x}{2}\right)D_x^{-1} & \dfrac{u}{4}(u+v)+\left(\dfrac{u_xv}{2}+\dfrac{uv_x}{2}\right)D_x^{-1} \\ -\dfrac{u_x}{4}D_x^{-1}\cdot u+\dfrac{u_x}{4}D_x^{-1}\cdot v & +\dfrac{u_x}{4}D_x^{-1}\cdot u-\dfrac{u_x}{4}D_x^{-1}\cdot v \\ \dfrac{v}{4}(u+v)+\left(\dfrac{uv_x}{2}+\dfrac{u_xv}{2}\right)D_x^{-1} & -\dfrac{v^2}{4}+\dfrac{3uv}{4}+\left(\dfrac{uv_x}{2}+\dfrac{u_xv}{2}\right)D_x^{-1} \\ -\dfrac{v_x}{4}D_x^{-1}\cdot u+\dfrac{v_x}{4}D_x^{-1}\cdot v & +\dfrac{v_x}{4}D_x^{-1}\cdot u-\dfrac{v_x}{4}D_x^{-1}\cdot v \end{pmatrix}, \tag{68}$$

respectively. □

It is worth mentioning that by reduction we obtain a new equation. For example, consider the case $k=0$. The equation (25), corresponding to $N=2$, and reduction of the equation (29), corresponding to $N=3$, are not related by a linear transformation of variables. Indeed, in the equation (25) coefficients of u_x, v_x are linear in u,v but in the equation (67) coefficients of u_x, v_x contain quadratic terms. Hence they cannot be related by a linear transformation.

B. Reduction $u_N=u_1$

It follows from (60) that the Lax equation (12) can be written as

$$u_{i,t}=\sum_{j=1}^{N} h_i^j(u_N,\ldots,u_1)u_{j,x}, \tag{69}$$

where $i,j=1,\ldots,N$ and $h_i^j=h_1(u_i,u_N,\ldots,\hat{u}_i,\ldots,u_1)$ when $i\neq j$ and $h_i^i=h_2(u_i,u_N,\ldots,\hat{u}_i,\ldots,u_1)$, the overcaret denotes the absence of the corresponding variable. It also follows from (60) that the functions $h_1(x_N,\ldots,x_1)$ and $h_2(x_N,\ldots,x_1)$ are symmetric under permutations of variables $x_{N-1},\ldots,x_1$.

Reduction $u_N=u_1$ gives us a new integrable equation,

$$
\begin{aligned}
u_{N,t}=&(h_N^N(u_N,u_{N-1},\dots,u_2,u_N)+h_N^1(u_N,u_{N-1},\dots,u_2,u_N))u_{N,x}\\
&+\sum_{j=2}^{N-1} h_N^j(u_N,u_{N-1},\dots,u_2,u_N)u_{j,x},\\
u_{i,t}=&2h_i^N(u_N,u_{N-1},\dots,u_2,u_N)u_{N,x}+\sum_{j=2}^{N-1} h_i^j(u_N,u_{N-1},\dots,u_2,u_N)u_{j,x},
\end{aligned}
\tag{70}
$$

where $i=(N-1),\dots,2$.

The Frechet derivative of (69), under condition $u_N=u_1$, has the form

$$
D_\Delta|_{u_N=u_1}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(N-1)} & a_{1N}\\ a_{21} & a_{22} & \cdots & a_{2(N-1)} & a_{21}\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ a_{(N-1)1} & a_{(N-1)2} & \cdots & a_{(N-1)(N-1)} & a_{(N-1)1}\\ a_{1N} & a_{12} & \cdots & a_{1(N-1)} & a_{11}\end{pmatrix}, \tag{71}
$$

where a_{ij}, $i,j=1,\dots,N$ are differential operatos. So, the Frechet derivative of (70) can be writen as

$$
D_{\bar{\Delta}}=\begin{pmatrix} a_{11}+a_{1N} & a_{12} & \cdots & a_{1(N-1)}\\ 2a_{21} & a_{22} & \cdots & a_{2(N-1)}\\ \vdots & \vdots & \cdots & \vdots\\ 2a_{(N-1)1} & a_{(N-1)2} & \cdots & a_{(N-1)(N-1)}\end{pmatrix}. \tag{72}
$$

Now let us write the recurcion operator of (69), given by Lemma 2. From (63) it follows that, under condition $u_N=u_1$, it has the form

$$
\mathcal{R}|_{u_N=u_1}=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1(N-1)} & b_{1N}\\ b_{21} & b_{22} & \cdots & b_{2(N-1)} & b_{21}\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ b_{(N-1)1} & b_{(N-1)2} & \cdots & b_{(N-1)(N-1)} & b_{(N-1)1}\\ b_{1N} & b_{12} & \cdots & b_{1(N-1)} & b_{11}\end{pmatrix}, \tag{73}
$$

where b_{ij}, $i,j=N,\dots,1$ are differential operators.

Now we can write a recursion operator for Eq. (70),

$$
\bar{\mathcal{R}}=\begin{pmatrix} b_{11}+b_{1N} & b_{12} & \cdots & b_{1(N-1)}\\ 2b_{21} & b_{22} & \cdots & b_{2(N-1)}\\ \vdots & \vdots & \cdots & \vdots\\ 2b_{(N-1)1} & b_{(N-1)2} & \cdots & b_{(N-1)(N-1)}\end{pmatrix}. \tag{74}
$$

The form of (74) can be deduced by applaing operator $\mathcal{R}|_{u_N=u_1}$ to a symmetry $(\partial u_N/\partial t_n, \partial u_{N-1}/\partial t_{n,\dots,}\, \partial u_2/\partial t_n, \partial u_n/\partial t_n)$.

Lemma 4: The operator $\bar{R}$ in (74) is a recursion operator of the equation (70).

Proof: Equation (70) is an evolution equation, so, to prove that $\bar{R}$ is a recursion operator we must prove that for any solution $(u_N,\dots,u_2)$ of (70) the following equality holds (see Ref. 6):

$$
D_{\bar{\Delta}}\bar{R}=\bar{R}D_{\bar{\Delta}}.
$$

If $(u_N,\ldots,u_2)$ is a solution of (70) then $(u_N,\ldots,u_2,u_1=u_N)$ is a solution of (69) and for the solution $(u_N,\ldots,u_2,u_1=u_N)$ we have

$$D_\Delta\,\mathcal{R}=\mathcal{R}\,D_\Delta\,. \tag{75}$$

One can show that from commutation of (71) and (73) follows the commutation of (72) and (74) that is equality (75). □

Let us consider reduction of systems, given by Remark 2 and Remark 4 and their recursion operators.

Proposition 11: Putting $w=u$ in (38) and (A2) we obtain a new system,

$$\begin{aligned} u_t&=\tfrac{1}{2}uv_x\,,\\ v_t&=\tfrac{1}{2}v(2u_x-v_x), \end{aligned} \tag{76}$$

and its recursion operator

$$\mathcal{R}=\begin{pmatrix} -(2uv+u^2)-\frac{3}{2}uv & -\frac{1}{4}u(2u+v)-\frac{3}{2}u^2 \\ +\frac{1}{2}uv_xD_x^{-1} & +\frac{1}{4}u(2u_x-v_x)D_x^{-1} \\ -2u(uv)_xD_x^{-1}\cdot u^{-1} & -u(uv)_xD_x^{-1}\cdot v^{-1} \\ -\frac{1}{2}v(2u+v)-3uv & -(2uv+u^2)+\frac{1}{4}v(2u+v) \\ +\frac{1}{2}v(-2u_x+v_x)D_x^{-1} & +\frac{1}{4}v(2-u_x+v_x)D_x^{-1} \\ -2v(uv)_xD_x^{-1}\cdot u^{-1} & -v(u^2)_xD_x^{-1}\cdot v^{-1} \end{pmatrix}. \tag{77}$$

□

Proposition 12: Putting $w=u$ in (29) and (A1) we obtain a new system,

$$\begin{aligned} \tfrac{1}{3}u_t&=(u^2+2uv+\tfrac{1}{8}v^2)u_x+(u^2+\tfrac{1}{4}uv)v_x\,,\\ \tfrac{1}{3}v_t&=(\tfrac{1}{2}v^2+2uv)u_x+(-\tfrac{1}{8}v^2+uv+u^2)v_x\,, \end{aligned} \tag{78}$$

and its recursion operator,

$$\mathcal{R}=\begin{pmatrix} u^2+\frac{7}{2}uv+(u^2+uv)_xD_x^{-1} & 2u^2+\frac{1}{4}uv+\frac{1}{2}(u^2+uv)D_x^{-1} \\ +\frac{1}{2}u_xD_x^{-1}\cdot v & +\frac{1}{2}u_xD_x^{-1}\cdot u-\frac{1}{4}u_xD_x^{-1}\cdot v \\ 4uv+\frac{1}{2}v^2+2(uv)_xD_x^{-1} & -\frac{1}{4}v^2+\frac{3}{2}uv+u^2+(uv)_xD_x^{-1} \\ +\frac{1}{2}v_xD_x^{-1}\cdot v & +\frac{1}{2}v_xD_x^{-1}\cdot u-\frac{1}{4}v_xD_x^{-1}\cdot v \end{pmatrix}. \tag{79}$$

□

We may go on introducing new reductions. For instance a reduction of the type $u_1=u_2=u_N$, $(N>3)$, reduces an N-system to an $(N-2)$-system. One may obtain this $(N-2)$-system also from the polynomial Lax function having the form $L=p^{-1}\,(p-u_1)^3(p-u_3)\cdots(p-u_{N-1})$ (a zero of L with multiplicity three). In this way one obtains an infinite number of different classes of $N=2$, $N=3$ systems.

VIII. CONCLUSION

We have constructed the recursion operators of some equations of hydrodynamic type. The form of the these operators fall into the class of pseudo-differential operators $A+B\,D^{-1}$ where A and B are functions of dynamical variables and their derivatives. The generalized symmetries of these equations are local and all belong to the same class (i.e., they are also equations of hydrodynamic type). We have introduced a method of reduction which leads also to integrable classes. Depending upon the type of reductions we may obtain infinitely many different classes of $N=k$ systems. These properties, the bi-Hamiltonian structure of the equations we obtained and equations with rational Lax functions, will be communicated elsewhere.

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APPENDIX: RECURSION OPERATORS FOR $N=3$ SYSTEMS (29) AND (38)

Recursion operators of the systems (29) and (38) are, respectively, given by

$$
\mathcal{R}=\begin{pmatrix}
\begin{array}{c} -\frac{u^2}{4}+\frac{3}{4}(uv+uw)+wv \\ +\frac{u_x}{2}(v+w)D_x^{-1} \\ +\frac{u}{2}(v_x+w_x)D_x^{-1} \\ -\frac{u_x}{4}D_x^{-1}\cdot u+\frac{u_x}{4}D_x^{-1}\cdot v \\ +\frac{u_x}{4}D_x^{-1}\cdot w \end{array} &
\begin{array}{c} \frac{u}{4}(u+v+w)+\frac{3uw}{2} \\ +\frac{u_x}{2}(v+w)D_x^{-1} \\ +\frac{u}{2}(v_x+w_x)D_x^{-1} \\ +\frac{u_x}{4}D_x^{-1}\cdot u-\frac{u_x}{4}D_x^{-1}\cdot v \\ +\frac{u_x}{4}D_x^{-1}\cdot w \end{array} &
\begin{array}{c} \frac{u}{4}(u+v+w)+\frac{3uv}{2} \\ +\frac{u_x}{2}(v+w)D_x^{-1} \\ +\frac{u}{2}(v_x+w_x)D_x^{-1} \\ +\frac{u_x}{4}D_x^{-1}\cdot u+\frac{u_x}{4}D_x^{-1}\cdot v \\ -\frac{u_x}{4}D_x^{-1}\cdot w \end{array} \\
\begin{array}{c} \frac{v}{4}(u+v+w)+\frac{3vw}{2} \\ +\frac{v_x}{2}(u+w)D_x^{-1} \\ +\frac{v}{2}(u_x+w_x)D_x^{-1} \\ -\frac{v_x}{4}D_x^{-1}\cdot u+\frac{v_x}{4}D_x^{-1}\cdot v \\ +\frac{v_x}{4}D_x^{-1}\cdot w \end{array} &
\begin{array}{c} -\frac{v^2}{4}+\frac{3}{4}(uv+vw)+uw \\ +\frac{v_x}{2}(u+w)D_x^{-1} \\ +\frac{v}{2}(u_x+w_x)D_x^{-1} \\ +\frac{v_x}{4}D_x^{-1}\cdot u-\frac{v_x}{4}D_x^{-1}\cdot v \\ +\frac{v_x}{4}D_x^{-1}\cdot w \end{array} &
\begin{array}{c} \frac{v}{4}(u+v+w)+\frac{3uv}{2} \\ +\frac{v_x}{2}(u+w)D_x^{-1} \\ +\frac{v}{2}(u_x+w_x)D_x^{-1} \\ +\frac{v_x}{4}D_x^{-1}\cdot u+\frac{v_x}{4}D_x^{-1}\cdot v \\ -\frac{v_x}{4}D_x^{-1}\cdot w \end{array} \\
\begin{array}{c} \frac{w}{4}(u+v+w)+\frac{3vw}{2} \\ +\frac{w_x}{2}(u+v)D_x^{-1} \\ +\frac{w}{2}(u_x+v_x)D_x^{-1} \\ -\frac{w_x}{4}D_x^{-1}\cdot u+\frac{w_x}{4}D_x^{-1}\cdot v \\ +\frac{w_x}{4}D_x^{-1}\cdot w \end{array} &
\begin{array}{c} \frac{w}{4}(u+v+w)+\frac{3uw}{2} \\ +\frac{w_x}{2}(u+v)D_x^{-1} \\ +\frac{w}{2}(u_x+v_x)D_x^{-1} \\ +\frac{w_x}{4}D_x^{-1}\cdot u-\frac{w_x}{4}D_x^{-1}\cdot v \\ +\frac{w_x}{4}D_x^{-1}\cdot w \end{array} &
\begin{array}{c} -\frac{w^2}{4}+\frac{3}{4}(uw+vw)+uv \\ +\frac{w_x}{2}(u+v)D_x^{-1} \\ +\frac{w}{2}(u_x+v_x)D_x^{-1} \\ +\frac{w_x}{4}D_x^{-1}\cdot u+\frac{w_x}{4}D_x^{-1}\cdot v \\ -\frac{w_x}{4}D_x^{-1}\cdot w \end{array}
\end{pmatrix}, \tag{A1}
$$

$$\mathcal{R}=\begin{pmatrix}
\begin{array}{c}-(uv+uw+vw)\\ +\frac{u}{4}(u+v+w)\\ +\frac{u}{4}(u_x-v_x-w_x)D_x^{-1}\\ -u(wv_x+vw_x)D_x^{-1}\cdot u^{-1}\end{array} & \begin{array}{c}-\frac{u}{4}(u+v+w)\\ -\frac{3uw}{2}\\ +\frac{u}{4}(u_x-v_x-w_x)D_x^{-1}\\ -u(wv_x+vw_x)D_x^{-1}\cdot v^{-1}\end{array} & \begin{array}{c}-\frac{u}{4}(u+v+w)\\ -\frac{3uv}{2}\\ +\frac{u}{4}(u_x-v_x-w_x)D_x^{-1}\\ -u(wv_x+vw_x)D_x^{-1}\cdot w^{-1}\end{array}\\
\begin{array}{c}-\frac{v}{4}(u+v+w)\\ -\frac{3vw}{2}\\ +\frac{v}{4}(-u_x+v_x-w_x)D_x^{-1}\\ -v(wu_x+uw_x)D_x^{-1}\cdot u^{-1}\end{array} & \begin{array}{c}-(uv+uw+vw)\\ +\frac{v}{4}(u+v+w)\\ +\frac{v}{4}(-u_x+v_x-w_x)D_x^{-1}\\ -v(wu_x+uw_x)D_x^{-1}\cdot v^{-1}\end{array} & \begin{array}{c}-\frac{v}{4}(u+v+w)\\ -\frac{3uv}{2}\\ +\frac{v}{4}(-u_x+v_x-w_x)D_x^{-1}\\ -v(wu_x+uw_x)D_x^{-1}\cdot w^{-1}\end{array}\\
\begin{array}{c}-\frac{w}{4}(u+v+w)\\ -\frac{3uw}{2}\\ +\frac{w}{4}(-u_x-v_x+w_x)D_x^{-1}\\ -w(uv_x+vu_x)D_x^{-1}\cdot u^{-1}\end{array} & \begin{array}{c}-\frac{w}{4}(u+v+w)\\ -\frac{3vw}{2}\\ +\frac{w}{4}(-u_x-v_x+w_x)D_x^{-1}\\ -w(uv_x+vu_x)D_x^{-1}\cdot v^{-1}\end{array} & \begin{array}{c}-(uv+uw+vw)\\ +\frac{w}{4}(u+v+w)\\ +\frac{w}{4}(-u_x-v_x+w_x)D_x^{-1}\\ -w(uv_x+vu_x)D_x^{-1}\cdot w^{-1}\end{array}
\end{pmatrix}. \tag{A2}$$

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