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On the Integrability of a Class of Monge-Ampère Equations

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Abstract

We give the Lax representations for for the elliptic, hyperbolic and homogeneous second order Monge-Ampère equations. The connection between these equations and the equations of hydrodynamical type give us a scalar dispersionless Lax representation. A matrix dispersive Lax representation follows from the correspondence between sigma models, a two parameter equation for minimal surfaces and Monge-Ampère equations. Local as well nonlocal conserved densities are obtained.

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1. Introduction

The nonlinear partial differential equation in $1+1$ dimensions

$$U_{tt}U_{xx} - U_{tx}^2 = -k \tag{1}$$

is the second order Monge-Ampère equation. Here we will be interested in the case where k is a constant. For $k = 1$ we have the hyperbolic Monge-Ampère equation which is equivalent [1] to the Born-Infeld equation [2]. The choice $k = -1$ yields the elliptic Monge-Ampère equation that is related [3,4] to the equation for minimal surfaces [5]. Finally, $k = 0$ corresponds to the homogeneous Monge-Ampère equation that can be shown to be related to the Bateman equation [6]. The Born-Infeld, minimal surfaces and Bateman equations can be treated simultaneously as

$$(k^2 + \phi_x^2)\phi_{tt} - 2\phi_x\phi_t\phi_{xt} + (k^2\alpha + \phi_t^2)\phi_{xx} = 0 \tag{2}$$

where

$$\alpha \equiv k^2 - k - 1 \tag{3}$$

and we should keep in mind the trivial identities $\alpha k^2 = -k$ and $\alpha k = -k^2$.

The Born-Infeld equation was introduced in 1934 as a nonlinear generalization of Maxwell's electrodynamics. It is the simplest wave equation in 1+1 dimensions, that preserves Lorentz invariance and is nonlinear. This equation is integrable [7,8] and has a multi-Hamiltonian structure [9]. The Bateman equation was introduced in 1929 and is related with hydrodynamics. This equation has a very interesting behavior [10]. If $\phi(x,t)$ is a solution of (2), for $k = 0$, so is any function of it (covariance of (2)). Also, (2) can be derived from an infinite class of inequivalent Lagrangian densities and is form invariant under arbitrary linear transformations of the (x,t) coordinates. The equation for minimal surfaces gives the surface $z = \phi(x,t)$ in the three-dimensional space that spans a given contour and has the minimum area. This is the Plateau's problem and has interest both in physics and mathematics.

In this paper we will obtain Lax representations for (1) and (2) since both systems are related. A scalar dispersionless Lax representation as well a matrix dispersive Lax representation will be given. As far as the authors can say this is the first example of a system where both Lax pairs are present. In fact our results suggest that many other systems, which have both an infinite number of local and nonlocal charges, are likely to have such characteristic.

This paper is organized as follows. In Section 2 we review the Bianchi transformation which relates (1) and (2). This is the Proposition 2.1 that unifies the results obtained in [1,3,4]. With this transformation we can easily translate results from the system (1) to system (2) and vice-versa. The existence of this Bianchi transformation is due to the fact that both (1) and (2) can be rewritten in a hydrodynamic type equation (polytropic gas). In Section 3, using results from [8,11,12], we obtain the dispersionless Lax representation of (1) (Proposition 3.1) and write the two sets of local conserved charges densities for the Monge-Ampère equation. In Section 4 we generalize the results of [5,13,14] concerning the matrix Lax representation for minimal surfaces through its correspondence with the sigma model. We obtain a matrix Lax representation for a two parameter equation for minimal surfaces which includes (2) for particular choices of the parameter (Proposition 4.4). From this Lax representation we give the nonlocal conserved charges densities of the system. In Section 5 we write explicitly the Lax representations, obtained in the previous sections, for the Monge-Ampère system (1) using the Bianchi transformation (Proposition 5.1). Finally we present our conclusions in Section 6.

2. Bianchi Transformation

In order to see the connection between (1) and (2) (see Equation (16))we have to express these equations in the form of equations of hydrodynamic type [15]. Following [9,16] we first introduce the potentials a and b, defined as

$$\begin{aligned} a =& U_x \\ b =& U_t \end{aligned} \tag{4}$$

Then, Equation (1) can be expressed as a first order system

$$\begin{aligned} k(a_t - b_x) &= 0 \\ b_t &= \frac{1}{a_x}(b_x^2 - k) \end{aligned} \tag{5}$$

which is the natural starting point for a Hamiltonian treatment of Monge-Ampère equations (1).

Now, introducing

$$\begin{aligned} u =& - \frac{b_x}{a_x} \\ v =& a_x \end{aligned} \tag{6}$$

the Monge-Ampère equation can be written in the following hydrodynamic type equation form

$$\begin{aligned} u_t + uu_x + kv^{-3}v_x =& 0 \\ k(v_t + (uv)_x) =& 0 \end{aligned} \tag{7}$$

Equation (2) follows from the Lagrangian

$$\mathcal{L} = \sqrt{k^2 + \phi_x^2 + \alpha\phi_t^2} \tag{8}$$

We stress that the Bateman equation can be obtained from a large class of inequivalent Lagrangian. However, we will use this one and the limit $k \to 0$ will give us results for the Bateman equation.

Since (8) has no ϕ dependence (2) can be written as a conservation law given by

$$\partial_x \left(\frac{\partial \mathcal{L}}{\partial \phi_x} \right) + \partial_t \left(\frac{\partial \mathcal{L}}{\partial \phi_t} \right) = 0 \tag{9}$$

This result allows us to rewrite (2) as a set of coupled first order nonlinear equations. Following [9,16] let us express (2) as the integrability condition of a first-order system given by

$$\begin{aligned} \psi_x =& - \frac{\partial \mathcal{L}}{\partial \phi_t} = - \frac{\alpha\phi_t}{\sqrt{k^2 + \phi_x^2 + \alpha\phi_t^2}} \\ \psi_t =& \frac{\partial \mathcal{L}}{\partial \phi_x} = \frac{\phi_x}{\sqrt{k^2 + \phi_x^2 + \alpha\phi_t^2}} \end{aligned} \tag{10}$$

Introducing the variables

$$\begin{aligned} r =& \phi_x \\ s =& \psi_x \end{aligned} \tag{11}$$

we get from (10)

$$\begin{aligned} \phi_t =& - \alpha s \sqrt{\frac{k^2 + r^2}{1 - \alpha s^2}} \\ \psi_t =& r \sqrt{\frac{1 - \alpha s^2}{k^2 + r^2}} \end{aligned} \tag{12}$$

So, the one-forms

$$
\begin{aligned}
d\phi =& r\,dx - \alpha s\sqrt{\frac{k^2+r^2}{1-\alpha s^2}}\,dt \\
d\psi =& s\,dx + r\sqrt{\frac{1-\alpha s^2}{k^2+r^2}}\,dt
\end{aligned}
\tag{13}
$$

are exact and its closure give us the equations

$$
\begin{aligned}
r_t =& -\frac{\alpha r s}{\sqrt{(k^2+r^2)(1-\alpha s^2)}}\,r_x - \alpha\sqrt{\frac{k^2+r^2}{(1-\alpha s^2)^3}}\,s_x \\
s_t =& k^2\sqrt{\frac{1-\alpha s^2}{(k^2+r^2)^3}}\,r_x - \frac{\alpha r s}{\sqrt{(k^2+r^2)(1-\alpha s^2)}}\,s_x
\end{aligned}
\tag{14}
$$

Now the amazing fact is that Equation (14) is also related with Equation (7) by a special transformation. For the case $k = 1$ this transformation is known as the Verosky transformation [9]. We can easily check that the following k generalized Verosky transformation

$$
\begin{aligned}
u =& \frac{\alpha r s}{\sqrt{(k^2+r^2)(1-\alpha s^2)}} \\
kv =& -k\sqrt{(k^2+r^2)(1-\alpha s^2)}
\end{aligned}
\tag{15}
$$

links (14) with (7). From the diagram

$$
\left.\begin{array}{c}
\text{Eq.(1)} \;\Rightarrow\; U \xrightarrow{\text{Eq.(4)}} a,b \xrightarrow{\text{Eq.(6)}} u,v \;\Rightarrow\; U = U(u,v) \\
\Downarrow \\
\text{Eq.(7)} \\
\Uparrow \\
\text{eq.(2)} \;\Rightarrow\; \phi \xrightarrow{\text{Eq.(11)}} r,s \xrightarrow{\text{Eq.(15)}} u,v \;\Rightarrow \begin{cases} u = u(\phi) \\ v = v(\phi) \end{cases}
\end{array}\right\} \Rightarrow U = U(\phi)
$$

we are led to the proposition [1,3,4]:

Proposition 2.1 *The Monge-Ampère Equation* (1) *and Equation* (2) *are related by the following Bianchi transformation*

$$
\begin{aligned}
U_{tt} =& \frac{k-\phi_t^2}{\sqrt{k^2+\phi_x^2+\alpha\phi_t^2}} \\
U_{tx} =& \frac{-\phi_x\phi_t}{\sqrt{k^2+\phi_x^2+\alpha\phi_t^2}} \\
U_{xx} =& \frac{-(k^2+\phi_x^2)}{\sqrt{k^2+\phi_x^2+\alpha\phi_t^2}}
\end{aligned}
\tag{16}
$$

3. Dispersionless Lax Representation: Local Conserved Charges

Equation (7) for $k = 1$ corresponds to the equations of isentropic, polytropic gas dynamics with the adiabatic index $\gamma = -1$ [9]. This system is known as a Chaplygin gas [17]. For $k = 0$ (7) is the Riemann equation [11] and in this case the transformation (15) give us $u = -\frac{\phi_t}{\phi_x}$.

In [12] the polytropic gas dynamics [18] equations

$$\begin{aligned} u_t + uu_x + v^{\gamma-2}v_x =& 0 \ , \quad \gamma \geq 2 \\ v_t + (uv)_x =& 0 \end{aligned} \tag{17}$$

were derived from the following dispersionless nonstandard Lax representation

$$\begin{aligned} L =& p^{\gamma-1} + u + \frac{v^{\gamma-1}}{(\gamma-1)^2}p^{-(\gamma-1)} \ , \quad \gamma \geq 2 \\ \frac{\partial L}{\partial t} =& \frac{(\gamma-1)}{\gamma}\left\{\left(L^{\frac{\gamma}{\gamma-1}}\right)_{\geq 1}, L\right\} \end{aligned} \tag{18}$$

Here $\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial p}$ and $\left(L^{\frac{\gamma}{\gamma-1}}\right)_{\geq 1}$ stands for the purely nonnegative (without p^0 terms) part of the polynomial in p. In (18) $L^{\frac{1}{\gamma-1}}$ was expanded around $p = \infty$. A Lax description for the Chaplygin gas like equations

$$\begin{aligned} u_t + uu_x + \frac{v_x}{v^{\beta+2}} =& 0 \ , \quad \beta \geq 1 \\ v_t + (uv)_x =& 0 \end{aligned} \tag{19}$$

was obtained in [8] in connection with the Born-Infeld equation and it is given by

$$L = p^{-(\beta+1)} + u + \frac{v^{-(\beta+1)}}{(\beta+1)^2}p^{\beta+1} \ , \quad \beta \geq 1 \tag{20}$$

with

$$\frac{\partial L}{\partial t} = \frac{(\beta+1)}{\beta}\left\{\left(L^{\frac{\beta}{\beta+1}}\right)_{\leq 1}, L\right\} \tag{21}$$

where $L^{\frac{1}{\beta+1}}$ is expanded around $p = 0$.

In view of these results we have the proposition:

Proposition 3.1 *For $\beta = 1$, the Lax operator*

$$L = p^{-2} + u + \frac{k}{4v^2}p^2 \tag{22}$$

where

$$\left(L^{1/2}\right)_{\leq 1} = p^{-1} + \frac{1}{2}up$$

reproduces (7)*. In terms of the variables a and b the Lax representation* (22) *assumes the form*

$$\begin{aligned} L =& p^{-2} - \frac{b_x}{a_x} + \frac{k}{4a_x^2}p^2 \\ \frac{\partial L}{\partial t} =& 2\left\{\left(L^{1/2}\right)_{\geq 1}, L\right\} \end{aligned} \tag{23}$$

and yields the Monge-Ampère equation as expressed in (5)*.*

This proposition is the first main result of our paper. This is a dispersionless Lax representation, a dispersive one will be obtained in Section 5 (see Proposition 5.1).

Conserved charges for the Chaplygin gas like equations (19) can be easily obtained from (20) through [8,12]

$$\mathcal{H}_n = \mathrm{Tr}\, L^{n+\frac{\beta+2}{\beta+1}},\ n = 0, 1, 2, 3, \ldots \tag{24}$$

This conserved charges were obtained by expanding $L^{\frac{1}{\beta+1}}$ around $p = 0$. An alternate expansion around $p = \infty$ is possible and it gives us a second set of conserved charges through

$$\widetilde{\mathcal{H}}_n = \mathrm{Tr}\, L^{n-\frac{1}{\beta+1}},\ n = 0, 1, 2, 3, \ldots \tag{25}$$

Both set of densities for (24) and (25) can be expressed in closed form [8]. They are

$$\begin{aligned} H_n =& (n+1)!\,\mathrm{C}_{n+1}^{\frac{(n+1)(\beta+1)+1}{(\beta+1)}} \sum_{m=0}^{\left[\frac{n+1}{2}\right]} \left(-\prod_{\ell=0}^{m} \frac{-1}{\ell(\beta+1)+1}\right) \frac{u^{n-2m+1}}{m!(n-2m+1)!} \frac{v^{-m(\beta+1)}}{(-\beta-1)^m} \\ \widetilde{H}_n =& n!(-\beta-1)^{\frac{2}{\beta+1}}\,\mathrm{C}_n^{\frac{n(\beta+1)-1}{(\beta+1)}} \sum_{m=0}^{\left[\frac{n}{2}\right]} \left(\prod_{\ell=0}^{m} \frac{-1}{\ell(\beta+1)-1}\right) \frac{u^{n-2m}}{m!(n-2m)!} \frac{v^{-m(\beta+1)+1}}{(-\beta-1)^m} \end{aligned} \tag{26}$$

The first densities H_n for the Monge-Ampère are

$$\begin{aligned} H_0 =& -\frac{3}{2}\frac{b_x}{a_x} \\ H_1 =& \frac{5}{8}\frac{1}{a_x^2}\left(3b_x^2 + k\right) \\ H_2 =& -\frac{35}{16}\frac{b_x}{a_x^3}\left(b_x^2 + k\right) \\ H_3 =& \frac{63}{128}\frac{1}{a_x^4}\left(5b_x^4 + 10b_x^2 k + k^2\right) \\ & \vdots \end{aligned} \tag{27}$$

and the first densities $\widetilde{H}_n$ are

$$\begin{aligned} \widetilde{H}_0 =& -2k^2 a_x \\ \widetilde{H}_1 =& -k^2 b_x \\ \widetilde{H}_2 =& -\frac{3}{4}k^2\frac{1}{a_x}\left(b_x^2 + k\right) \\ \widetilde{H}_3 =& \frac{5}{8}k^2\frac{b_x}{a_x^2}\left(b_x^3 + 3k\right) \\ & \vdots \end{aligned} \tag{28}$$

4. Minimal Surfaces and Sigma Models

In this section we will generalize some results of [5,13,14] where a matrix Lax representation for the minimal surface equation (Eq. (2) with $k = -1$) was obtained.

Let g be a 2×2 matrix function with components

$$g_{11} = \frac{k_1 + a^2}{\omega}\,, \quad g_{12} = g_{21} = \frac{ab}{\omega} \quad \text{and} \quad g_{22} = \frac{k_2 + b^2}{\omega}\,,$$

where k_1 and k_2 are arbitrary constants, not vanishing simultaneously and

$$\varepsilon\omega^2 = k_1 k_2 + k_1 b^2 + k_2 a^2\,, \quad \text{where} \quad \varepsilon = \pm 1.$$

Thus, $\det g = \varepsilon$. Note that ε is not fully independent of k_1 and k_2. $\omega^2 > 0$ when we are dealing with real fields. In the case of complex fields ε is independent of k_1 and k_2 .

The sigma model equation can be written as

$$\partial_\alpha(g^{\alpha\beta}g^{-1}\partial_\beta g) = 0\,, \tag{29}$$

where $g^{\alpha\beta}$ are the components of g^{-1}. As shown in [5] and [13] the Lax representation of (29) is

$$\varepsilon^{\alpha\beta}\partial_\beta\psi = \frac{1}{\lambda^2+\varepsilon}\left[\lambda g^{\alpha\beta} - \varepsilon\varepsilon^{\alpha\beta}\right](g^{-1}\partial_\beta g)\psi, \tag{30}$$

where $\varepsilon^{\alpha\beta}$ is Levi-Civita tensor with $\varepsilon^{12}=1$, λ is the spectral parameter, $\det g = \varepsilon$ and $\varepsilon = \pm 1$.

Now, let us see how a Lax representation for (2) can be obtained from (30). First, let M_3 be a 3-dimensional manifold with metric

$$ds^2|_{M_3} = k_1 dt^2 + k_2 dx^2 + dz^2 \quad (k_1 \neq 0,\ k_2 \neq 0)$$

and $z = \phi(t,x)$ define a graph of a regular surface S in M_3. The induced metric on S is given by

$$ds^2|_S = (k_1 + \phi_t^2)dt^2 + (k_2 + \phi_x^2)dx^2 + 2\phi_x\phi_t dx dt.$$

If $a = \phi_t$ and $b = \phi_x$, then g is a metric tensor on S. Surface S is called minimal if its mean curvature H vanishes. Minimality condition leads to the equation

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = 0,$$

or

$$(k_1 + \phi_t^2)\phi_{xx} - 2\phi_x\phi_t\phi_{xt} + (k_2 + \phi_x^2)\phi_{tt} = 0\,. \tag{31}$$

There is a parametrization of the minimal surfaces where the minimality condition reduces to the Laplace equation in 2-dimensions. Let $X : S \to M_3$ define a parametrization of S in M_3. This parametrization is called isothermal [19,20], if

$$\langle X_u\, X_u\rangle = \varepsilon\langle X_v\, X_v\rangle \tag{32}$$

$$\langle X_u\, X_v\rangle = 0 \qquad (\varepsilon = \pm 1) \tag{33}$$

Proposition 4.1. *S is a minimal surface if and only if $X_{uu} + \varepsilon X_{vv} = 0$, where X is an isothermal parametrization.*

A connection between the above two different parametrizations may be obtained from the following two propositions:

Proposition 4.2. *Let $z = \phi(t, x)$ define a regular surface S. Parametrization $X : S \to M_3$ is isothermal if and only if the following equations are satisfied*

$$\begin{aligned} (k_1 + \phi_t^2)t_u &= -\omega x_v - \phi_t\phi_x x_u \,, \\ (k_2 + \phi_t^2)t_v &= -\omega x_u - \phi_t\phi_x x_v \,. \end{aligned} \tag{34}$$

The proof of the Proposition 4.2 can be done in the following way. The Equation (33) can be written as

$$x_u(k_2 x_v + \phi_x^2 x_v + \phi_t\phi_x t_v) + t_u(k_1 t_v + \phi_t\phi_x x_v + \phi_t^2 t_v) = 0$$

and it is equivalent to the system

$$\begin{cases} t_u = \lambda^{-1}[(k_2 + \phi_x^2)x_v + \phi_t\phi_x t_v] \\ x_u = \lambda^{-1}[(k_1 + \phi_t^2)t_v + \phi_t\phi_x x_v] \,. \end{cases}$$

Inserting expressions for t_u and x_u into (32), it can be found that

$$\varepsilon\lambda^2 = (k_1 k_2 + k_1\phi_x^2 + k_2\phi_t^2).$$

Hence, $\lambda = \omega$ and

$$\begin{cases} t_u = \omega^{-1}[(k_2 + \phi_x^2)x_v + \phi_t\phi_x t_v] \\ x_u = \omega^{-1}[(k_1 + \phi_t^2)t_v + \phi_t\phi_x x_v]. \end{cases}$$

That is equivalent to (34).

Proposition 4.1 and 4.2 imply next proposition:

Proposition 4.3. *Let x and t be harmonic functions of u and v. Let a differentiable function $\phi(t, x)$ be defined by* (34)*. Then the function $\phi(t, x)$ is a harmonic function of u and v if and only if it satisfies the minimality condition* (31)*.*

Let us consider Equation (31), where k_1 and k_2 are arbitrary constants. We have four distinct cases:

(i) $k_1k_2 > 0$.

(1) $k_1 > 0, k_2 > 0$. This is equivalent to the equation of minimal surface in $\mathbf{R^3}$ or elliptic Monge-Ampère equation ($k_1 = k_2 = -k = 1$).

(2) $k_1 > 0, k_2 < 0$. This is equivalent to the equation of minimal surface in $\mathbf{M_3}$ (3-dimensional Minkowski space with metric $(1, 1, -1)$).

(ii) $k_1k_2 < 0$. This is equivalent to the Born-Infeld equation (which is the equation of a minimal surface in a 3-dimensional Minkowski space with metric $(-1, 1, 1)$) or hyperbolic Monge-Ampère equation ($-k_1 = k_2 = k = 1$).

We have the following cases which do not arise from the embedding problem in M_3:

(iii) $k_1k_2 = 0$, but not simultaneously vanishing. This is a new type of equation.

(iv) $k_1 = k_2 = 0$. This is Bateman equation or homogeneous Monge-Ampère equation ($k_1 = k_2 = k = 0$).

The next proposition is very important since it provides the Lax pair for systems that include Equation (2):

Proposition 4.4 *Let ϕ be a differential function of t, x and let $a = \phi_t$, $b = \phi_x$. Then Equation* (31) *solves the sigma model Equation* (29)*, if k_1, k_2 not vanish simultaneously. If $k_1 = k_2 = 0$ Equation* (31) *solves the sigma model Equation* (29) *for another matrix g, namely,*

$$g_{11} = a_1 \frac{\phi_t}{\phi_x}, \quad g_{12} = g_{21} = b_1 \quad \text{and} \quad g_{22} = a_2 \frac{\phi_x}{\phi_t},$$

where a_1, a_2, b_1 are constants. The Lax pair of (31) *is then given by* (30)*.*

In the next section we will use the last Proposition to obtain the Lax representations for the Monge-Ampère equations (1). In doing so we will return to our original parameter k instead of working with the parameters k_1 and k_2. It is just a matter of scale transformation either in formula for $ds^2|_{M_3}$ or in Equation (31) (redefining x and t) to give $k_1 = \pm 1$ and $k_2 = \pm 1$. Also, we will set $\varepsilon = 1$ in the next section.

5. Matrix Lax Representation: Nonlocal Conserved Charges

Now we can write the Lax pairs for (1). First, let us give the Lax pairs for (2) more explicitly. Equation (30) can be rewritten in the form

$$\begin{aligned} \frac{\partial\psi}{\partial x} &= \frac{1}{\lambda^2+1}\left[\lambda\left(g^{11}A+g^{12}B\right)-B\right]\psi \\ \frac{\partial\psi}{\partial t} &= -\frac{1}{\lambda^2+1}\left[\lambda\left(g^{21}A+g^{22}B\right)+A\right]\psi \end{aligned} \tag{35}$$

where

$$A = g^{-1}\partial_t g\,, \quad B = g^{-1}\partial_x g \tag{36}$$

From (36) it follows the identity

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial t} - [A, B] = 0 \tag{37}$$

The integrability of (35) yields the equations

$$\det g = 1 \tag{38}$$

$$(g^{11}A - g^{12}B)_t + (g^{21}A + g^{22}B)_x = 0 \tag{39}$$

From the Proposition 4.4 we have for $k \neq 0$

$$g = \frac{1}{\sqrt{-k(1+\phi_x^2)+\phi_t^2}}\begin{pmatrix} -k+\phi_t^2 & \phi_t\phi_x \\ \phi_t\phi_x & 1+\phi_x^2 \end{pmatrix} \tag{40}$$

and for $k = 0$ (setting $a_1 = a_2 = \sqrt{2}$ and $b_1 = 1$)

$$g = \begin{pmatrix} \sqrt{2}\,\dfrac{\phi_t}{\phi_x} & 1 \\ 1 & \sqrt{2}\,\dfrac{\phi_x}{\phi_t} \end{pmatrix} \tag{41}$$

With this choice (37) and (38) are trivial identities and (39) is identical to Equation (2), i.e., to the minimal surface equation for $k = -1$, Born-Infeld equation for $k = 1$ and Bateman equation for $k = 0$.

The Bianchi transformation (16) for $k \neq 0$ assumes the form

$$\begin{aligned} \sqrt{-k}\,U_{tt} &= \frac{-k+\phi_t^2}{\sqrt{-k(1+\phi_x^2)+\phi_t^2}} \\ \sqrt{-k}\,U_{tx} &= \frac{\phi_x\phi_t}{\sqrt{-k(1+\phi_x^2)+\phi_t^2}} \\ \sqrt{-k}\,U_{xx} &= \frac{1+\phi_x^2}{\sqrt{-k(1+\phi_x^2)+\phi_t^2}} \end{aligned} \tag{42}$$

and (40) in terms of U can be written as

$$g = \sqrt{-k}\begin{pmatrix} U_{tt} & U_{tx} \\ U_{tx} & U_{xx} \end{pmatrix} \tag{43}$$

In this way (35) with (43) give us the matrix Lax representation for the hyperbolic Monge-Ampère equation ($k = 1$) and elliptic Monge-Ampère equation ($k = -1$). Let us observe that (1) for $k \neq 0$ follows from (38) while Equations (37) and (39) are trivial identities. We can also express (43) in terms of variables a and b defined in (4) by

$$g = \sqrt{-k}\begin{pmatrix} b_t & b_x \\ b_x & a_x \end{pmatrix} \tag{44}$$

and (5) follows easily since $a_t = b_x$ is a trivial identity and $\det g = -k(b_t a_x - b_x^2) = 1$. The Bianchi transformation (16) for $k = 0$ yields

$$\frac{\phi_t}{\phi_x} = \frac{U_{tt}}{U_{tx}}, \qquad \frac{\phi_x}{\phi_t} = \frac{U_{xx}}{U_{tx}} \tag{45}$$

and (41) in terms of U can be written as

$$g = \begin{pmatrix} \sqrt{2}\,\dfrac{U_{tt}}{U_{tx}} & 1 \\ 1 & \sqrt{2}\,\dfrac{U_{xx}}{U_{tx}} \end{pmatrix} \tag{46}$$

and $\det g = 1$ reproduces (1) for $k = 0$. In terms of the variables a and b we have

$$g = \begin{pmatrix} \sqrt{2}\,\dfrac{b_t}{b_x} & 1 \\ 1 & \sqrt{2}\,\dfrac{a_x}{b_x} \end{pmatrix} \tag{47}$$

which give us (5) for $k = 0$.

So, we have the following proposition:

Proposition 5.1 *The Lax pair* (35) *with* (44) *or* (47) *yields the Monge-Ampère equations as expressed in* (5) *for* $k \neq 0$ *and* $k = 0$*, respectively.*

This proposition is the second main result of our paper. This is a matrix dispersive Lax representation.

In Section 3, using the dispersionless Lax representation for the Monge-Ampère equations (1), we were able to derive two sets of infinite number of local conserved charges. Now, using (35) it will possible to find infinitely non local conserved ones. Let us denote $M = -(g^{11}A + g^{12}B)$ and $N = g^{21}A + g^{22}B$, then the Lax pair (35) can be written as

$$\begin{aligned}(\lambda^2+1)\psi_x =& -\lambda M\psi - g^{-1}g_x\psi \\ (\lambda^2+1)\psi_t =& -\lambda N\psi - g^{-1}g_t\psi\end{aligned}$$

or

$$\begin{aligned}(g\psi)_x =& -\lambda gM\psi - \lambda^2 g\,\psi_x \\ (g\psi)_t =& -\lambda gN\psi - \lambda^2 g\,\psi_t\end{aligned} \tag{48}$$

Let us assume that function ψ is analytical in the parameter λ and can be expanded as

$$\psi = \psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \cdots \tag{49}$$

Then, Equations (48) imply

$$\begin{aligned}\psi_0 =& g^{-1} \\ (g\psi_1)_x =& -gMg^{-1} \\ (g\psi_1)_t =& -gNg^{-1} \\ (g\psi_2)_x =& g_xg^{-1} + gMg^{-1}\partial_x^{-1}(gMg^{-1}) \\ (g\psi_2)_t =& g_tg^{-1} + gNg^{-1}\partial_t^{-1}(gNg^{-1}) \\ &\vdots\end{aligned} \tag{50}$$

and we have now infinitely many conserved laws in the form $(X_n)_x = (T_n)_t$ where the densities are

$$\begin{aligned}X_1 =& N \\ T_1 =& M \\ X_2 =& g^{-1}g_t + (\partial_t^{-1}N)N \\ T_2 =& g^{-1}g_x + (\partial_x^{-1}M)M \\ &\vdots\end{aligned} \tag{51}$$

Now we can use (44) and (47) to express the densities T_n in terms of variables a and b.

6. Conclusion

In this paper we have obtained the Lax representation of the Monge-Ampère equations (1). In Section 2 the Bianchi transformation relating equations (1) and (2) was given (Proposition 2.1). This transformation allowed us to translate results obtained for one equation to the other. In Section 3 the dispersionless Lax pair for (1) as well the local conserved densities were given (Proposition 3.1). In Section 4 the correspondence between sigma models and a two parameter equation for minimal surfaces was given and the matrix Lax pair for equation (2) was obtained (Proposition 4.4). A Lax representation for the system (1) as well the nonlocal conserved densities were given in Section 5 (Proposition 5.1).

The algebra of the local and nonlocal charges that follows from (27), (28) and (51) as well the multiHamiltonian formulation of the Monge-Ampère equations (1) will be the subject of a future publication. Some results on this line for the second order homogeneous Monge-Ampère equation were already obtained in [21,22]. As we have pointed, the homogeneous Monge-Ampère equation has an infinite number of inequivalent Lagrangians and somehow this should be reflected in its Lax representation. This also deserves further clarifications.

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References

1. O.I. Mokhov and Y. Nutku, Lett. Math. Phys. **32**, 121 (1994).
2. M. Born and L. Infeld, Proc. R. Soc. London **A144**, 425 (1934).
3. K. Jörgens, Math. Annal. **127**, 130 (1954).
4. E. Heinz, Nach. Akad. Wissensch. in Göttingen Mathem.-Phys. Klasse **IIa**, 51 (1952).
5. M. Gürses, Lett. Math. Phys. **44**, 1 (1998).
6. H. Bateman, Proc. R. Soc. **A125**, 598 (1929).
7. B. M. Barbishov and N. A. Chernikov, Sov. Phys. JETP **24**, 93 (1966).
8. J. C. Brunelli and A. Das, Phys. Lett. **B426**, 57 (1998).
9. M. Arik, F. Neyzi, Y. Nutku, P. J. Olver and J. Verosky, J. Math. Phys. **30**, 1338 (1988).
10. D. B. Fairlie, J. Govaerts and A. Morozov, Nucl. Phys. **B373**, 214 (1992).
11. J. C. Brunelli, Rev. Math. Phys. **8**, 1041 (1996).
12. J. C. Brunelli and A. Das, Phys. Lett. **A235**, 597 (1997).
13. M. Gürses and A. Karasu, Internat. J. Modern Phys. A. **6**, (1991)
14. M. Gürses, Lett. Math. Phys. **26**, (1992).
15. B. Dubrovin and S. Novikov, Russ. Math. Surv. **44**, 35 (1989).
16. Y. Nutku, J. Math. Phys. **26**, 1237 (1985).
17. K. P. Stanyukovich, "Unsteady Motion of Continuous Media" (Pergamon, New York, 1960), p. 137.
18. P. J. Olver and Y. Nutku, J. Math. Phys. **29**, 1610 (1988).
19. M. do Carmo, "Differential Geometry of Curves and Surfaces" (Prentice-Hall, New Jersey, 1976).
20. U. Dierken, S. Hildebrandt, A. Künster and O. Wohlrab, "Minimal Surfaces I", Grundlehren der Mathematishen Wissenschaften, No. 295 (Springer-Verlag, Berlin-Heidelberg, 1992).
21. Y. Nutku and Ö. Sarioğlu, Phys. Lett. **A173**, 270 (1993).
22. Y. Nutku, J. Phys. **29A**, 3257 (1996)