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Relaxation of multidimensional variational problems with constraints of general form

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This paper is devoted to further developement of an idea of a well-known theorem of Bogolubov [2]. Here we construct a relaxation of multidimensional variational problems with constraints of rather general form on gradients of admissible functions; it is assumed that the gradient of an admissible function belongs to an arbitrary bounded set. This relaxation involves as a class of admissible functions the closure of the class of admissible functions of the original problem in the topology of uniform convergence, and uses a theorem characterizing this closure, which is proved in [15]. The case when the gradient of an admissible function is constrained within a bounded closed convex body is studied in the works [13,15,19].

The study of multidimensional variational problems was started in 1970s by Ekeland and Temam [13]. The existing literature on relaxation of variational problems, including two monographs by Buttazzo [3] and Dacorogna [9], and the review paper by Marcellini [18] containing a considerable list of references, is quite rich. However, the author failed to find a setting similar to that of the paper. For the most recent results on relaxation and related topics see [1,4–8,11,14].

This paper deals with the case where an integrand depends on a scalar function of several variables. At the end of the paper we will make a conjecture on generalization of the main relaxation result of the paper to the case of an integrand depending on a vector function of several variables. We also make a conjecture on generalization of

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the above-mentioned theorem on characterization of the closure, which is an important tool in the proof of the main result, for the vectorial case.

R^n will stand for n-dimensional Euclidean space of points $t=(t_1,\dots,t_n)$. Let Ω be an arbitrary bounded open set in R^n. Denote by $C(\overline{\Omega})$ the space of all real continuous functions on $\overline{\Omega}$ with the norm

$$\|x(\cdot)\|_{C(\overline{\Omega})}=\max_{t\in\overline{\Omega}}|x(t)|.$$

Denote by $W^1_\infty(\Omega)$ the Sobolev space of all essentially bounded measurable functions on Ω, with essentially bounded first generalized partial derivatives. It is well known that a function $x(\cdot)$ from $W^1_\infty(\Omega)$ is continuous on Ω and possesses the ordinary first derivatives $\partial x/\partial t_i$ $(i=1,\dots,n)$ almost everywhere (a.e.) on Ω (see [13,20]). If domain Ω satisfies additional conditions (e.g., if Ω is Lipschitzian), then $W^1_\infty(\Omega)\subset C(\overline{\Omega})$. Let $\overline{W}^1_\infty(\Omega)=W^1_\infty(\Omega)\cap C(\overline{\Omega})$. So, if Ω is sufficiently regular, then $\overline{W}^1_\infty(\Omega)=W^1_\infty(\Omega)$. Denote by $B_r(0)$ a ball in R^n with the center at the origin and radius r. Given a set $V\subset R^n$ and a positive number r let $V_r=\{v\in V\colon \operatorname{dist}(v,\partial V)\geq r\}$, where ∂V is the boundary of V.

Recall that function $x(\cdot):\overline{\Omega}\to R$ is said to be piece-wise affine, if it is continuous and there exists a partition of $\overline{\Omega}$ into a subset of measure zero and a finite number of open sets, on which $x(\cdot)$ is affine. A continuous function on $\overline{\Omega}$ is said to be almost piece-wise affine, if its restriction to an arbitrary strict interior subdomain of Ω is piece-wise affine.

Let X, Y be topological spaces, and I, J be functionals defined on X and Y, respectively. The variational problem $\inf\{J(y)\colon y\in Y\}$ is said to be a relaxation of the problem $\inf\{I(x)\colon x\in X\}$, if there exists a continuous mapping $i:X\to Y$, such that: (i) $i(X)$ is dense in Y, (ii) $J(i(x))\leq I(x)$ for each $x\in X$, and (iii) for an arbitrary $y\in Y$ there exists a sequence $x_k\in X$ $(k\in N)$ such that $i(x_k)\to y$ and $J(y)\geq\lim_{k\to\infty}I(x_k)$. Moreover, if functional J is lower semicontinuous, then a relaxation is called a lower semicontinuous relaxation (see [16]).

Let $f:\overline{\Omega}\times R\times R^n\to R$ be a continuous function, U be an arbitrary bounded set in R^n with an affine hull R^n, $\Gamma\subset\partial\Omega$ and $\phi:\Gamma\to R$ be some fixed function. Consider the following problem of multidimensional variational calculus, which we will refer to as problem (P):

$$J(x(\cdot))=\int_\Omega f(t,x(t),\operatorname{grad}x(t))d(t)\to\inf, \tag{1}$$

$$\operatorname{grad}x(t)\in U\quad\text{a.e. in }\Omega, \tag{2}$$

$$x(t)=\phi(t)\quad\text{for } t\in\Gamma, \tag{3}$$

where $x(\cdot)\in\overline{W}^1_\infty(\Omega)$. The case when $\Gamma=\emptyset$, i.e., when the boundary condition (3) is absent, will be referred to as problem (P_0).

A function $x(\cdot)\in\overline{W}^1_\infty(\Omega)$ is called admissible in problem (P)((P_0)), if it satisfies conditions (2), (3) ((2)). The set of all admissible functions in problem (P)((P_0)) will be denoted by $E(U,\phi)(E(U))$. Thus

$$E(U) = \{x(\cdot) \in \overline{W}^1_\infty(\Omega)\text{: } \operatorname{grad} x(t) \in U \text{ a.e. in } \Omega\},$$

$$E(U,\phi) = \{x(\cdot) \in E(U)\text{: } x(\cdot)|_\Gamma = \phi\}.$$

The space $\overline{W}^1_\infty(\Omega)$ and its subsets $E(U)$, $E(U,\phi)$ will be considered with the metric of uniform convergence.

Along with problem (P) we consider the following problem (problem (PR)):

$$J_R(x(\cdot)) = \int_\Omega f_U^{**}(t, x(t), \operatorname{grad} x(t)) d(t) \to \inf, \tag{1'}$$

$$x(t) = \phi(t) \quad \text{for } t \in \Gamma, \tag{3'}$$

where $\overline{co}\, U$ is the closed convex hull of U and $f_U^{**}(t,x,\cdot) = (f(t,x,\cdot) + \delta(\cdot|U))^{**}$. Here

$$\delta(u|U) = \begin{cases} 0 & \text{for } u \in U, \\ +\infty & \text{for } u \in R^n \setminus U \end{cases}$$

is the indicator function of U, and ** designates the operation of taking second Young–Fenchel conjugate (see [17, p. 183]). In case of $\Gamma = \emptyset$ problem (PR) will be denoted as (P_0R).

The above-mentioned assertion on closure consists of the following:

$$\overline{E(U)} = E(\overline{co}\, U),$$

i.e. the closure in the uniform metric of a class of functions continuous on Ω with gradients from the bounded set U coincides with the class of functions continuous on $\overline{\Omega}$ and with gradients from the closed convex hull of U. Moreover, if condition (4) of Theorem 1 below is satisfied, then Theorem 1′ from Hüsseinov [15] implies the following coincidence

$$\overline{E(U,\phi)} = E(\overline{co}\, U, \phi).$$

Theorem 1. *Let $U \subset R^n$ be an arbitrary bounded set in R^n with the affine hull R^n. Suppose that there exists an admissible function $y_0(\cdot) \in E(\overline{co}\, U, \phi)$ in problem* (PR) *such that*

$$\operatorname{grad} y_0(t) \in U_0 \quad \text{a.e. in } \Omega, \tag{4}$$

where U_0 is a closed set contained in the interior of $\overline{co}\, U$. Then, for an arbitrary function $x(\cdot) \in E(\overline{co}\, U, \phi)$ admissible in problem (PR), *there exists a sequence of functions $x_k(\cdot)$ $(k \in N)$, admissible in problem* (P), *uniformly converging to $x(\cdot)$, and such that*

$$\lim_{k\to\infty} J(x_k(\cdot)) = J_R(x(\cdot)).$$

In particular, when the boundary condition (3) is absent, i.e. for problem (P_0), condition (4) in Theorem 1 is satisfied automatically.

The following lemma will be used in the proof of Theorem 1.

Lemma. *Let T be a topological space, U be an arbitrary bounded set in $R^n, U_0 \subset U$ be a compact set contained in the interior of $\overline{co}\, U$ or a segment, and $f : T \times R^n \to R$ be a continuous function. Then a restriction of function $f_U^{**}(\tau, u)$ to $T \times U_0$ is continuous.*

Proof. Since $f_U^{**} = f_{\overline{U}}^{**}$, we suppose, without loss of generality, that U is closed. Fix a point $(\tau_0, u_0) \in T \times U_0$ and a positive number ε. It is easily seen that, there exists a neighborhood $S(\tau_0)$ of point τ_0 such that

$$|f(\tau, u) - f(\tau_0, u)| < \varepsilon \quad \text{for } \tau \in S(\tau_0), \quad u \in \overline{co}\,\overline{U}. \tag{5}$$

It is well known that

$$f_U^{**} = \min\left\{\sum_{i=1}^{n+1} \lambda_i f(\tau, u_i)\colon \sum_{i=1}^{n+1} \lambda_i u_i = u,\ u_i \in U,\ \sum_{i=1}^{n+1} \lambda_i = 1,\ \lambda_i \geq 0\right\}.$$

From this and (5) we obtain that

$$f_U^{**}(\tau, u) = \sum_{i=1}^{n+1} \bar{\lambda}_i f(\tau, \bar{u}_i) \geq \sum_{i=1}^{n+1} \bar{\lambda}_i f(\tau_0, \bar{u}_i) - \varepsilon \geq f_U^{**}(\tau_0, u) - \varepsilon.$$

Symmetrically,

$$f_U^{**}(\tau_0, u) \geq f_U^{**}(\tau, u) - \varepsilon.$$

Consequently,

$$|f_U^{**}(\tau_0, u) - f_U^{**}(\tau_0, u)| < \varepsilon \quad \text{for } \tau \in S(\tau_0) \quad u \in \overline{co}\, U.$$

Since $f_U^{**}(\tau_0, \cdot)$ is a convex and lower semicontinuous it is continuous on U (in both the cases stipulated in the lemma). Therefore, there exists a number $\delta > 0$ such that

$$|f_U^{**}(\tau, u) - f_U^{**}(\tau_0, u_0)| < \varepsilon \quad \text{for } u \in U_0, \quad \|u - u_0\| < \delta.$$

The last two inequalities imply that

$$|f_U^{**}(\tau, u) - f_U^{**}(\tau_0, u_0)| < 2\varepsilon$$

for $\tau \in S(\tau_0)$, $\|u - u_0\| < \delta$. Therefore, function $f_U^{**}|_{T \times U_0}$ is continuous at the point (τ_0, u_0).

Proof of Theorem 1. Let $x(\cdot) \in E(\overline{co}\,\overline{U}, \phi)$ be an admissible function in problem (PR) and $\varepsilon > 0$. Consider the sequence of functions $x_k(t) = ((k-1)/k)x(t) + (1/k)y_0(t)$ $(k \in N)$. Clearly, $x_k(\cdot) \in E(\overline{co}\, U, \phi)$ and

$$x_k(\cdot) \to_k x(\cdot) \quad \text{uniformly on } \Omega, \tag{6}$$

$$\operatorname{grad} x_k(t) \to_k \operatorname{grad} x(t) \quad \text{for a.a. } t \in \Omega, \tag{7}$$

$$\operatorname{grad} x_k(t) + B_{r_k}(0) \subset U \quad \text{for a.a. } t \in \Omega, \tag{8}$$

where r_k, $(k \in N)$ are positive numbers.

It follows from relations (6), (8) and the lemma that

$$\|x_k(\cdot) - x(\cdot)\|_{C(\overline{\Omega})} < \frac{\varepsilon}{4},$$
$$|J_R(x_k(\cdot)) - J_R(x(\cdot))| < \frac{\varepsilon}{4} \tag{9}$$

for sufficiently large indices k.

Let k_0 be such that (9) holds for k_0. Let $\bar{x}(\cdot) = x_{k_0}(\cdot)$, $r = r_{k_0}/2$. By Theorem 1′ from Hüsseinov [15], there exists a sequence of almost piece-wise affine functions $y_k(\cdot) \in E(\overline{co}\, U, \phi)$ uniformly converging to $x(\cdot)$. Then the sequence of vector functions $\operatorname{grad} y_k(\cdot)$ $(k \in N)$ weakly converges to vector function $\operatorname{grad} \bar{x}(\cdot)$ in Banach space $L_1^n(\Omega)$ of summable n-vector functions on domain Ω. By Mazur's Theorem (Corollary 3.14 from Dunford and Schwartz [12, p. 457]) it follows that there exist convex combinations $z_m(\cdot) = \sum_{k=N_m+1}^{N_{m+1}} \alpha_k^m y_k\ (\cdot)$ $(m \in N)$ of functions $y_k(\cdot)$ $(k \in N)$, where $\alpha_k \geq 0$, $\sum_{k=N_m+1}^{N_{m+1}} \alpha_k^{(m)} = 1$ and N_m $(m \in N)$ is a strictly increasing sequence of integers such that

$$\operatorname{grad} z_m(t) \to \operatorname{grad} \bar{x}(t) \quad \text{for a.a. } t \in \Omega. \tag{10}$$

Thus, the functions $z_m(\cdot)$ are almost piece-wise affine, $z_m(\cdot) \in E((\overline{co}\, U)_r, \phi)$ $(m = 1, 2, \ldots)$, the sequence $z_m(\cdot)$ $(m \in N)$ uniformly converges to $\bar{x}(\cdot)$, and condition (10) is satisfied. From that we obtain

$$\|z_m(\cdot) - \bar{x}(\cdot)\|_{C(\overline{\Omega})} < \frac{\varepsilon}{4},$$
$$|J_R(z_m(\cdot)) - J_R(\bar{x}(\cdot))| < \frac{\varepsilon}{4} \tag{11}$$

for sufficiently large m. Fix one of such indices m_0 and denote $\bar{z}(\cdot) = z_{m_0}(\cdot)$. We obtain from relations (9) with $k = k_0$ and (11) with $m = m_0$

$$\|\bar{z}(\cdot) - x(\cdot)\|_{C(\Omega)} < \frac{\varepsilon}{2},$$
$$|J_R(\bar{z}(\cdot)) - J_R(x(\cdot))| < \frac{\varepsilon}{2}. \tag{12}$$

So, function $\bar{z}(\cdot)$ is almost piece-wise affine, $\bar{z}(\cdot) \in E((\overline{co}\, U)_r, \phi)$ and satisfies relations (12).

Denote $M = 1 + \max |x(t)|$. Since integrand f is continuous on compact $K = \overline{\Omega} \times [-M, M] \times \overline{U}$, there exists a positive number $\delta_0' < \varepsilon/2$ such that

$$|f(t_1, x_1, u_1) - f(t_2, x_2, u_2)| < \frac{\varepsilon}{2} \tag{13}$$

for $(t_1, x_1, u_1), (t_2, x_2, u_2) \in K$, $\|t_1 - t_2\| < \delta_0'$, $\|u_1 - u_2\| < \delta_0'$.

In sequel, we shall omit the index U in notation f_U^{**}. By the lemma function f^{**} is continuous on compact $K_r = \overline{\Omega} \times [-M, M] \times (\overline{co}\, U)_r$. Hence, there exists $\delta_0 \in (0, \delta_0')$ such that

$$|f^{**}(t_1, x_1, u_1) - f^{**}(t_1, x_1, u_1)| < \frac{\varepsilon}{2} \tag{14}$$

for $(t_1,x_1,u_1),(t_2,x_2,u_2)\in K_r$'$\|t_1-t_2\|<\delta_0$, $\|u_1-u_2\|<\delta_0$. Since the functions $x(\cdot)$ and $\bar z(\cdot)$ are continuous on Ω, there exists $\delta\in(0,\delta_0/2)$ such that

$$|x(t_1)-x(t_2)|<\delta_0,\quad |\bar z(t_1)-\bar z(t_2)|<\frac{\delta_0}{2}\quad\text{for } \|t_1-t_2\|<\delta. \tag{15}$$

Denote by Δ_j $(j\in N)$ the simplices of affineness of function $\bar z(\cdot)$, $a_j=\operatorname{grad} z(t)$ for $t\in \operatorname{int}\Delta_j$ $(j\in N)$. Without loss of generality, we assume that $\operatorname{diam}\Delta_j<\delta$ $(j\in N)$. Fix $t_j\in\Delta_j$ $(j\in N)$. It is well known that

$$f^{**}(t_j,\bar z(t_j),a_j)$$
$$=\inf\left\{\sum_{i=1}^{n+1}\alpha_i^j f(t_j,\bar z(t_j),v_i^j)\colon \sum_{i=1}^{n+1}\alpha_i^j v_i^j=a_j,\ v_i^j\in U,\ \sum_{i=1}^{n+1}\alpha_i^j=1,\ \alpha_i^j\geq 0\right\}.$$

Then for some numbers $\alpha_i^j>0$ $(i=1,2,\ldots,n+1)$, $\sum_{i=1}^{n+1}\alpha_i^j=1$ and affinely independent vectors v_i^j $(i=1,2,\ldots,n+1)$ from U

$$\left|f^{**}(t_j,\bar z(t_j),a_j)-\sum_{i=1}^{n+1}\alpha_i^j f(t_j,\bar z(t_j),v_i^j)\right|<\frac{\varepsilon}{2},$$

$$\sum_{i=1}^{n+1}\alpha_i^j v_i^j=a_j. \tag{16}$$

Put $u_i^j=v_i^j-a_j$ $(i=1,2,\ldots,n+1)$ and denote $\sum_j=\operatorname{co}\{u_1^j,\ldots,u_{n+1}^j\}$. Since, vectors u_i^j $(i=1,2,\ldots,n+1)$ are affinely independent and $\sum_{i=1}^{n+1}\alpha_i^j v_i^j=0$, where $\alpha_i^j>0$ $(i=1,2,\ldots,n+1)$ then $\sum_j$ is an n-dimensional simplex with the interior containing zero. Denote $D_j=\sum_j^0$ polar of the simplex $\sum_j$, $s_j(\cdot)$ – support function of set $\{u_1^j,\ldots,u_{n+1}^j\}$.

Partition simplex Δ_j into at most countably many simplices $\Delta_1^j,\Delta_2^j,\ldots,$ homothetic to D_j and such that $\operatorname{diam}\Delta_k^j<\delta\operatorname{diam}D_j$. Denote by d_k^j the similarity coefficients of simplices Δ_k^j and D_j and put

$$s_k^j(t)=\begin{cases} s(t-t_k^j)-d_k^j & \text{for } t\in\Delta_k^j,\\ 0 & \text{for } t\in\overline{\Omega}\setminus\Delta_k^j\end{cases}$$

and $\sigma_i(\Delta_k^j)=\{t\in\Delta_k^j\colon s_k^j(t)=\langle t-t_k^j,u_k^j\rangle-d_k^j\}$ $(i=1,2,\ldots,n+1)$, for arbitrary indices j,k, where $t_k^J\in\Delta_k^j$ is the image of the origin under the homothety $D_j\to\Delta_k^j$. Obviously, function $s_k^j(\cdot)$ is piece-wise affine and

$$-\delta\leq s_k^j(t)\leq 0. \tag{17}$$

Put

$$s(t)=\sum_{j,k}s_k^j(t)\quad\text{and}\quad z(t)=\bar z(t)+s(t).$$

Since

$$\operatorname{grad} z(t)=\operatorname{grad}\bar z(t)+u_i^j=a_j+u_i^j=v_i^j\in U\quad\text{for } t\in\sigma_i(\Delta_k^j)$$

and simplices $\sigma_i(\Delta_k^j)$ $(i=1,2,\ldots,n+1;\ j,k\in N)$ cover domain Ω, then function $z(\cdot)$ is admissible in problem (P), i.e. $z(\cdot)\in E(U,\phi)$.

Utilizing inequalities (15)–(17) and Proposition 2 from Hüsseinov [15] we estimate the difference

$$\left|\int_{\Delta_k^j} f^{**}(t,\bar z(t),\operatorname{grad}\bar z(t))\,\mathrm{d}(t)-\int_{\Delta_k^j} f(t,z(t),\operatorname{grad} z(t))\,\mathrm{d}t)\right|$$

$$=\left|\int_{\Delta_k^j} f^{**}(t,\bar z(t),\operatorname{grad}\bar z(t))\,\mathrm{d}(t)-\sum_{i+1}^{n+1}\int_{\sigma_i(\Delta_k^j)} f(t,\bar z(t)+s_k^j(t),v_i^j)\,\mathrm{d}(t)\right|$$

$$\leq\left|mes(\Delta_k^j)f^{**}(t_j,\bar z(t_j),a_j)-\sum_{i=1}^{n+1}\alpha_i^j\,mes(\Delta_k^j)f(t_j,\bar z(t_j),v_i^j)\right|+\varepsilon\,mes(\Delta_k^j)$$

$$=mes(\Delta_k^j)\left[\left|f^{**}(t_j,\bar z(t_j),a_j)-\sum_{i=1}^{n+1}\alpha_i^j f(t_j,\bar z(t_j),v_i^j)\right|\right]\leq 2\varepsilon\,mes(\Delta_k^j). \tag{18}$$

Summing up inequalities (18) by j,k we obtain

$$|J_{f^{**}}(\bar z(\cdot))-J(\bar z(\cdot))|<2\varepsilon\,mes(\Omega). \tag{19}$$

It is clear from (17) that

$$\|\bar z(\cdot)-z(\cdot)\|_{C(\overline{\Omega})}<\frac{\varepsilon}{2}.$$

From this and from the first of inequalities (12) it follows that

$$\|z(\cdot)-x(\cdot)\|_{C(\overline{\Omega})}<\varepsilon,$$

and from (19) and from the second of inequalities (12) that

$$|J_R(x(\cdot))-J(z(\cdot))|<\varepsilon[1+2\,mes(\Omega)].$$

The theorem is proved. □

Theorem 1 and Lemma 4 from Hüsseinov [15] imply the following result.

Theorem 2. *Let U be a bounded set in R^n with an affine hull R^n, and assumption* (4) *of Theorem* 1 *be satisfied. Then problem* (PR) *is a lower semicontinuous relaxation of problem* (P).

For $U\subset R^{m\times n}$ the closure of the quasiconvex hull is defined as (see [10, Definition 2.2]):

$$\overline{Qco}\,U=\{\xi\in R^{m\times n}\colon f(\xi)\leq 0,\ \forall f: R^{m\times n}\to R,\ \text{quasiconvex and } f|_U=0\}.$$

We denote for $U\subset R^{m\times n}$

$$E(U)=\{x(\cdot)\in W^1_\infty(\Omega;R^m)\colon Dx(t)\in U \text{ a.e. in } \Omega\},$$

where $Dx(t)$ denotes the Jacobi matrix of $x(\cdot)$ at t. We conjecture the following coincidence: $\overline{E(U)} = E(\overline{Qco}\,U)$, where $\overline{E(U)}$ denotes the closure of $E(U)$ in uniform metric of $\overline{W}^1_\infty(\Omega;R^m)$.

Consider the following two variational problems. The first is the problem $(\mathscr{P})$ obtained from (P) by treating f as a function $R^{m\times n} \to R$, $\operatorname{grad} x(t)$ replaced by $Dx(t)$ the Jacobi matrix of $x(\cdot):\Omega \to R^m$ at t, and $\phi(\cdot):\Gamma \to R^m$. The second problem is

$$J_R(x(\cdot)) = \int_\Omega Qf_U(t,x(t),Dx(t))\,\mathrm{d}t \to \inf,$$

$$x(t) = \phi(t) \quad \text{for } t \in \Gamma,$$

where $Qf_U(t,x,\cdot)$ is the quasiconvex envelope (i.e. the maximal quasiconvex function not exceeding f) of the function $f(t,x,\cdot)+\delta(\cdot|U)$, $\delta(\cdot|U)$ being the indicator function of U.

Conjecture. Let $U \subset R^{m\times n}$ be an arbitrary bounded set with $\overline{Qco}\,U$ having an interior point. Suppose that there exists an admissible function $y_0(\cdot) \in E(\overline{Qco}\,U,\varphi)$ in problem $(\mathscr{P}R)$ such that $Dy_0(t) \in U_0$ a.e. in Ω, where U_0 is a closed set contained in the interior of $\overline{Qco}\,U$, then for an arbitrary vector function $x(\cdot) \in E(\overline{Qco}\,U,\varphi)$ admissible in problem $(\mathscr{P}R)$, there exists a sequence of vector-functions $x_k(\cdot)$ $(k \in N)$ admissible in problem $(\mathscr{P})$, uniformly converging to $x(\cdot)$ and such that

$$\lim J(x_k(\cdot)) = J_R(x(\cdot)).$$

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