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Continuum Quantum Systems as Limits of Discrete Quantum Systems, I: State Vectors

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Dynamical systems on "continuum" Hilbert spaces may be realized as limits of dynamical systems on "discrete" (possibly finite-dimensional) Hilbert spaces. In this first of four papers on the topic, the "continuum" and "discrete" spaces are interfaced to one another algebraically, convergence of vectors is defined in such a way as to preserve inner products, and a necessary and sufficient coordinate-wise criterion for convergence is proved.

1. INTRODUCTION

Motives for seeking discrete versions of Heisenberg–Weyl phase space come from engineering and physics as much as from fundamental mathematics. The discrete scenario introduced by von Neumann [15], and systematically developed by Weil [21], has attracted attention in number theory, as in Lion and Vergne [14]; signal analysis, as in Richman *et al.* [17]; and quantum physics, as in Ruzzi and Galetti [19]. Much of the progress toward linking this discrete scenario with the familiar continuum scenario (see the references in [19]) entails modifying the discrete scenario in such a way as to lose the very close algebraic analogy that is manifest in Weil's unified treatment of the two scenarios.

Meanwhile, the discrete scenario originating in Stratonovich [20], and pursued in the context of optics by Atakishiyev and Wolf [3] and Atakishiyev *et al.* [1] has the group SU(2) playing the role of the continuum Heisenberg–Weyl group. Or rather, SU(2) corresponds to the group of Euclidian canonical transforms, which is generated by the continuum Heisenberg–Weyl group and the continuum fractional Fourier transforms. The correspondence is an *algebraic analogy*, and at the same time, it can be characterized in terms of *limits*.

In this paper and its first two sequels [6, 7], we provide a mathematical prerequisite for some of the limiting techniques that are currently being

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used in a heuristic fashion. The definitions and results will then be applied in the fourth paper [8] of the series, where we show how the continuum Heisenberg translates, the continuum fractional Fourier transforms, and the more general continuum complex Fourier transforms may be realized, via limits, in terms of finite-dimensional representations of SU(2) and SL(2, $\mathbb{C}$). Let us emphasize that we are *supporting* the existing heuristic techniques; only in the fourth paper do we we turn to an application where "dead reckoning" and formal transcription of symbols are truly inappropriate.

If our outlook is a little biased toward quantum physics, it may be because, in that sphere of application, finite-dimensional analogy and approximation of continuum systems is already a venerable topic. Nevertheless, the mathematical objects involved are of broad relevance. For instance, one-parameter families of signal transforms such as fractional Fourier transform, chirping, and dialation are, from a mathematical point of view, quantum dynamical systems.

Eckmann [13] has remarked on the peculiar absence of any known "reduction principle" from which certain "continuum" theorems perhaps ought to be obtainable from some analogous and comparatively shallow "linear" theorems. Bearing in mind that "continuum" function spaces are inevitably infinite-dimensional, and that "linear," here, pertains to what are actually finite-dimensional (hence discrete) function spaces, we are led to ask whether or not such a "reduction principle" might possibly be arrived at using something of the approach adopted in the material below.

Digernes *et al.* [12] have established a general (and rigorous) theory of correspondences between discrete and continuum quantum systems whose Hamiltonians are Schrödinger operators. They showed how the spectrum of a discrete system may be related to the spectrum of the corresponding continuum system. The emphasis of *our* concern is directed rather more toward results on preservation of algebraic structure. We seek correspondences that satisfy:

Analogy requirement. Algebraic constructions appearing in the continuum scenario (such as time-evolution, symmetries of a system, symplectic transforms, the Heisenberg–Weyl group) are to have corresponding analogues in the discrete scenario.

Approximation requirement. Algebraic constructions appearing in the continuum scenario are to be realizable as limits of their discrete analogues.

Let $\mathscr{L}_\infty$ be a Hilbert space. Let $\mathscr{N}$ be a directed set (such as the set of positive integers) and for each $n \in \mathscr{N}$, let $\mathscr{L}_n$ be a Hilbert space. In regard to applications, we are to interpret $\mathscr{L}_\infty$ as a "continuum" space; a space of continuous functions, for instance, $L^2(\mathbb{R})$. We are to interpret each $\mathscr{L}_n$ as a "discrete" space, for instance, a finite-dimensional inner product space, or a space, such as ℓ^2, with an explicitly specified complete orthonormal set. The analogy requirement: an object $\mathcal{O}_\infty$ associated with $\mathcal{L}_\infty$ is to be an algebraic analogue of objects $\mathcal{O}_n$ associated with the spaces $\mathcal{L}_n$. The approximation requirement: the equation

$$\mathcal{O}_\infty = \lim_{n \in \mathcal{N}} O_n$$

must hold (and must be meaningful). It should be possible to take limits of sequences (or nets) of vectors, operators, quantum dynamical systems, and perhaps, group representations. The convergence must respect all additive and multiplicative structure.

From a theoretical point of view, it helps to dismiss the jargon "continuum" and "discrete." The space $\mathcal{L}_\infty$ and the spaces $\mathcal{L}_n$ are better perceived simply as abstract Hilbert spaces. The interfacing of $\mathcal{L}_\infty$ with the spaces $\mathcal{L}_n$ is to be a purely algebraic condition, quite independent of any interpretation of the vectors as functions. (This point is especially important in connection with phase space techniques, where the use of canonical transforms to effect "change of variables" or "change of operators" is a distinctive feature. The criteria for convergence must not be tied to, say, the description of the state space whereby the state vector is expressed as a function of the position variable.)

In Section 2, the interfacing of $\mathcal{L}_\infty$ and the spaces $\mathcal{L}_n$ is expressed by the notion of an *inductive resolution*. There are three different ways of characterizing convergence of vectors. An abstract criterion, which we adopt as our definition, is introduced in Section 2. Proof of the necessity and sufficiency of a coordinate-wise criterion—Theorem 3.4—is the goal of the present paper. A point-wise characterization, applicable only in the presence of function-space representations of the state spaces, is discussed in the second paper [6]. The point-wise criterion, in heuristic form, is already in frequent use. We shall review the (well-known) realization of the Hermite–Gaussians as limits of the Kravchuk functions. The third paper [7] introduces convergence of operators. Using the coordinate-wise criterion for convergence of vectors, we shall prove the sufficiency of a coordinate-wise criterion for convergence of operators. In particular, we shall review the (well-known) realizations of the continuum fractional Fourier transform as the limit of the Kravchuk function FRFT introduced by [3], and also as the limit of the Harper function FRFT introduced by Pei–Yeh [16]. The case of the Harper function FRFT has already been discussed in [5] and in [9]. The former of these works contains some ideas that are developed more systematically below; the latter invokes the criterion for operator convergence proved in [7]. The examples of convergence discussed in [6] and [7] could be—and indeed, have been—established merely by "dead reckoning" (in some cases supported by numerical evidence). In so far as concerns applications external to fundamental mathematics, the results in the fourth part [8] comprise the main motive for the material in the previous three parts of this work.

2. INDUCTIVE RESOLUTIONS

All our Hilbert spaces are to be complex and separable, possibly finite-dimensional. Let $\mathscr{L}_\infty$ be a Hilbert space. Let $\mathscr{N}$ be a directed set, and for each $n \in \mathscr{N}$, let $\mathscr{L}_n$ be a Hilbert space. Given a net $(x_n)_{n \in \mathscr{N}}$ in some Hausdorff space, we may ask whether the net is convergent, and when it is, we can write the limit as $\lim_{n \in \mathscr{N}} x_n$. Consider vectors $\psi_\infty \in \mathscr{L}_\infty$ and $\psi_n \in \mathscr{L}_n$ for each $n \in \mathscr{N}$. We still speak of $(\psi_n)_{n \in \mathscr{N}}$ as a net, and we wish to be able to ask whether ψ_∞ is the limit.

Let $\mathscr{S}$ be a dense subspace of $\mathscr{L}_\infty$, and for each $n \in \mathscr{N}$, let res_n be a linear map $\mathscr{S} \to \mathscr{L}_n$ such that the inner product of any two elements $\phi, \chi \in \mathscr{S}$ satisfies

$$\langle \phi \mid \chi \rangle = \lim_{n \in \mathscr{N}} \langle \mathrm{res}_n(\phi) \mid \mathrm{res}_n(\chi) \rangle.$$

The maps res_n are called *restriction maps*. The net of Hilbert spaces $(\mathscr{L}_n)_n$ (abusing language) together with the net of linear maps $(\mathrm{res}_n)_n$, is called an *inductive resolution* of $\mathscr{L}_\infty$.

The net $(\psi_n)_n$ is said to *converge* to ψ_∞ provided the norms $\|\psi_n\|$ are essentially bounded (bounded for sufficiently large n) and

$$\langle \phi \mid \psi_\infty \rangle = \lim_{n \in \mathscr{N}} \langle \mathrm{res}_n(\phi) \mid \psi_n \rangle$$

for all $\phi \in \mathscr{S}$. By the Riesz Representation Theorem, and the denseness of $\mathscr{S}$ in $\mathscr{L}_\infty$, the net $(\psi_n)_n$ cannot converge to two distinct vectors in $\mathscr{L}_\infty$. When $(\psi_n)_n$ converges ψ_∞, we call ψ_∞ the *limit* of $(\psi_n)_n$, and we write

$$\psi_\infty = \lim_{n \in \mathscr{N}} \psi_n.$$

These definitions are still applicable when the vectors ψ_n are given not necessarily for all n, but only for sufficiently large n in $\mathscr{N}$. (For example, it is shown in [6] that the Hermite–Gaussian h_s of degree s is the limit of the Kravchuk functions $h_{s,n}$ of degree s; the functions $h_{s,n}$ exist only when $s < n$.)

The clause requiring the norms $\|\psi_n\|$ to be essentially bounded is a technicality, inserted for the good enough reason that the statements of many of our results below would otherwise be false. The author settled upon this clause after experimenting with several other definitions of convergence. If $\mathcal{N} \subset \mathbb{N}$ then, of course, the norms are essentially bounded if and only if they are bounded.)

In [9] and [5] some alternative terminology was used: instead of saying that $(\psi_n)_n$ *converges* to ψ_∞, it was said that $(\psi_n)_n$ *induces* ψ_∞. When those papers were written, the author was not aware of [6, Theorem 3.1], which shows that the above definition of convergence is compatible with some customary heuristic use of limiting arguments in discrete quantum mechanics.

Remark 2.1. Given any $\phi \in \mathcal{S}$, then $\phi = \lim_n \operatorname{res}_n(\phi)$.

Proof. Since $\|\phi\| = \lim_n \|\operatorname{res}_n(\phi)\|$, the norms $\|\operatorname{res}_n(\phi)\|$ are essentially bounded. ∎

Remark 2.2. Given vectors $\psi_\infty = \lim_{n \in \mathcal{N}} \psi_n$ and $\theta_\infty = \lim_{n \in \mathcal{N}} \theta_n$ in $\mathcal{L}_\infty$, and given complex numbers λ and μ, then $\lambda\psi_\infty + \mu\theta_\infty = \lim_n (\lambda\psi_n + \mu\theta_n)$.

Proof. This is obvious. ∎

Remark 2.3. Let $\psi_\infty \in \mathcal{L}_\infty$, and for each $n \in \mathcal{N}$, let $\psi_n, \theta_n \in \mathcal{L}_n$. Suppose that $\psi_\infty = \lim_n \psi_n$ and $\lim_n \|\psi_n - \theta_n\| = 0$. Then $\psi_\infty = \lim_n \theta_n$.

Proof. Clearly, the norms $\|\theta_n\|$ are essentially bounded. For any $\phi \in \mathcal{S}$, we have

$$|\langle \operatorname{res}_n(\phi) \mid \psi_n \rangle - \langle \operatorname{res}_n(\phi) \mid \theta_n \rangle| \leqslant \|\operatorname{res}_n(\phi)\| \cdot \|\psi_n - \theta_n\|. \quad ∎$$

Every vector in $\mathcal{L}_\infty$ is the limit of a net of vectors in the spaces $\mathcal{L}_n$. More precisely:

THEOREM 2.4. *Given any vector $\psi_\infty \in \mathcal{L}_\infty$, then there exist vectors $\psi_n \in \mathcal{L}_n$ such that $\psi_\infty = \lim_n \psi_n$, and $\|\psi_n\| = \|\psi_\infty\|$ for each $n \in \mathcal{N}$.*

Proof. Throughout the argument, we may assume that ψ_∞ is normalized; $\|\psi_\infty\| = 1$. For the moment, let us assume also that $\psi_\infty \in \mathcal{S}$. Since the norms $\|\operatorname{res}_n(\psi_\infty)\|$ converge to unity, $\operatorname{res}_n(\psi_\infty) \neq 0$ for sufficiently large n. When $\operatorname{res}_n(\psi_\infty) \neq 0$, put $\psi_n = \operatorname{res}_n(\psi_\infty)/\|\operatorname{res}_n(\psi_n)\|$, otherwise, choose ψ_n to be any normalized vector in $\mathcal{L}_n$. The assertion is now clear in the case where $\psi_\infty \in \mathcal{S}$.

Now let the normalized vector $\psi_\infty \in \mathcal{L}_\infty$ be arbitrary. Since $\mathcal{S}$ is dense in $\mathcal{L}_\infty$, there exist vectors $\phi_m \in \mathcal{S}$ such that $\|\psi_\infty - \phi_m\| < 2^{-m}$ for each integer $m \geqslant 1$. Letting $\theta_1 = \phi_1$, and $\theta_{m+1} = \phi_{m+1} - \phi_m$, then $\|\theta_m\| < 2^{2-m}$ for all $m \geqslant 1$. We have $\psi_\infty = \sum_{m=1}^\infty \theta_m$. By the previous paragraph, we can write $\theta_m = \lim_n \theta_{m,n}$, where $\theta_{m,n} \in \mathcal{L}_n$ and $\|\theta_{m,n}\| = \|\theta_m\|$. Since each $\mathcal{L}_n$ is complete, and the series $\sum_{m=1}^{\infty} \|\theta_{m,n}\|$ converges, we can define $\psi_n := \sum_{m=1}^{\infty} \theta_{m,n}$ in each $\mathscr{L}_n$. We will show that $\psi_\infty = \lim_n \psi_n$.

Let $\phi \in \mathscr{S}$. Choose a real A such that $\|\phi\| < A \geqslant \|\mathrm{res}_n(\phi)\|$ for sufficiently large n. Let $\varepsilon > 0$. Choose a positive integer M such that $2^{3-M} A \leqslant \varepsilon$. If n is sufficiently large, then $|\langle \phi \mid \theta_m \rangle - \langle \mathrm{res}_n(\phi) \mid \theta_{m,n} \rangle| < \varepsilon/2M$ for all $m \leqslant M$, whereupon

$$|\langle \phi \mid \psi_\infty \rangle - \langle \mathrm{res}_n(\phi) \mid \psi_n \rangle| \leqslant \sum_{m=1}^{\infty} |\langle \phi \mid \theta_m \rangle - \langle \mathrm{res}_n(\phi) \mid \theta_{m,n} \rangle|$$

$$< \sum_{m=1}^{M} \frac{\varepsilon}{2M} + \sum_{m=M+1}^{\infty} A 2^{2-m} < \varepsilon.$$

The norms $\|\psi_n\|$ are bounded by $\sum_{m=1}^{\infty} 2^{2-m} = 4$. We have shown that $\psi_\infty = \lim_{n \in \mathscr{N}} \psi_n$.

As in the first paragraph of the argument, we may demand that the vectors $\theta_{m,n}$ be chosen such that $\theta_{m,n}$ is a scalar multiple of $\mathrm{res}_n(\theta_m)$ unless $\mathrm{res}_n(\theta_m) = 0$. Our demand granted,

$$\lim_{n \in \mathscr{N}} \left\| \sum_{j=1}^{m} \theta_{j,n} - \sum_{j=1}^{m} \mathrm{res}_n(\theta_j) \right\| = 0$$

for each m. Since $\|\psi_\infty\| = 1$, and $\|\psi_\infty - \phi_m\| \leqslant 2^{-m}$, and

$$\lim_{n \in \mathscr{N}} \left\| \sum_{j=1}^{m} \mathrm{res}_n(\theta_j) \right\| = \left\| \sum_{j=1}^{m} \theta_j \right\| = \|\phi\|$$

we deduce that if n is sufficiently large then $\|\sum_{j=1}^{m} \theta_{j,n}\|$ differs from unity by at most 2^{1-m}. On the other hand,

$$\left\| \psi_m - \sum_{j=1}^{m} \theta_{j,n} \right\| \leqslant \sum_{j=m+1}^{\infty} \|\theta_{j,n}\| \leqslant \sum_{j=m+1}^{\infty} 2^{1-j} = 2^{1-m}.$$

So if n is sufficiently large, then $\|\psi_n\|$ differs from unity by at most 2^{2-m}. Since m is aribitrary, $\lim_n \|\psi_n\| = 1$. Normalizing the vectors ψ_n as in the first paragraph, we obtain the required conclusion. ∎

If each $\mathscr{L}_n = \mathscr{L}_\infty$, and each res_n is the inclusion $\mathscr{S} \hookrightarrow \mathscr{L}_n$, then $\psi_\infty = \lim_n \psi_n$ if and only if the net $(\psi_n)_n$ weakly converges to ψ_∞. Thence, we see that the condition $\psi_\infty = \lim_n \psi_n$ does not imply that $\|\psi_\infty\| = \lim_n \|\psi_n\|$, and we also see that the converse to Remark 2.3 is false.

EXAMPLE 2.A. Let r be a positive integer, let $\mathscr{L}_\infty$ be the Hilbert space $L^2(\mathbb{R}^r)$, and let $\mathscr{S}$ be the Schwartz subspace $\mathscr{S}(\mathbb{R}^r)$. Let $\mathscr{N}$ be the set of positive integers. For each $n \in \mathcal{N}$, let $\mathcal{X}_n$ be a countable set (with the discrete measure) and let $\mathcal{L}_n = \ell^2(\mathcal{X}_n)$. Let σ_n be a function $\mathcal{X}_n \to \mathbb{R}^r$, and let $v(n)$ be a positive real number. We impose the hypothesis that, for every bounded convex subset U of $\mathbb{R}^r$, the preimage $\mathcal{X}_n(U) := \sigma_n^{-1}(U)$ is finite, and the sequence $(|\mathcal{X}_n(U)|/v(n)^2)_n$ converges to the volume (the measure) of U. The hypothesis is equivalent to the condition that

$$\int_V \bar{\phi}(x)\, \chi(x)\, dx = \lim_{n \in \mathcal{N}} v(n)^{-2} \sum_{X \in \sigma_n^{-1}(V)} \bar{\phi}(\sigma_n(X))\, \chi(\sigma_n(X))$$

for all convex $V \subseteq \mathbb{R}^r$ (not necessarily bounded), and all $\phi, \chi \in \mathcal{S}$. The elements X of $\mathcal{X}_n$ are to be regarded as indices corresponding to sample-points $\sigma_n(X)$ in $\mathbb{R}^r$. We define

$$\mathrm{res}_n(\phi)(X) = \phi(\sigma_n(X))/v(n).$$

The "Riemann sum" $\langle \mathrm{res}_n(\phi) \,|\, \mathrm{res}_n(\chi) \rangle$ converges to $\langle \psi \,|\, \chi \rangle$. We have realised $(\mathcal{L}_n)_n$ as an inductive resolution of $\mathcal{L}_\infty$. The interpretation of this rather abstract inductive resolution is clarified in the following three Examples, which are special cases.

Example 2.B. In the notation of Example 2.A, let $r = 1$, and let $\mathcal{X}_n$ be the set of integers X in the range $-n/2 < X \leqslant n/2$. Let c be any positive real number. We complete the specification of the inductive resolution $(\mathcal{L}_n)_n$ by putting $v(n) = (cn)^{1/4}$ and $\sigma_n(X) = X/v(n)^2$. In the particular case $c = 1$, the restriction maps res_n are as in [6B, Section 2]. In the case $c = 1/2\pi$, the inductive resolution $(\mathcal{L}_n)_n$ pertains to the Harper functions and the Harper function fractional Fourier transform; see [9], [6]. It is straightforward to extend this Example to the case where the positive integer r is arbitrary, and $\mathcal{X}_n$ consists of the r-tuples $(X_1, ..., X_r)$ with each integer X_j in the range $-n/2 < X_j \leqslant n/2$.

Example 2.C. This example is virtually the same as the previous example, but it is worth recording separately because much use of it will be made in the three sequels. Let $r = 1$. Given an element $n \in \mathcal{N}$, write $n = 2\ell + 1$, and let $\mathcal{X}_n$ be the set of rational numbers X such that $X + \ell$ is an integer, and $-\ell \leqslant X \leqslant \ell$. Thus, if n is odd, then $\mathcal{X}_n$ is the same as it was in Example 2.B, otherwise $\mathcal{X}_n$ is a set of halves of odd integers. Either way, $|\mathcal{X}_n| = n$. As before, let c be a positive real number. This time, we complete the specification of the inductive resolution $(\mathcal{L}_n)_n$ by putting $v(n) = (c\ell)^{1/4}$ and $\sigma_n(X) = X/v(n)^2$. In the case $c = 1$, the inductive resolution here pertains to the Kravchuk functions and to the Kravchuk function fractional

Fourier transform; see [6] and Atakishiyev and Wolf [3]. Again, generalization to the case of arbitrary r is straightforward.

EXAMPLE 2.D. In the case $r=2$ of Example 2.A, Atakishiyev *et al.* [2] have proposed, for applications to circularly symmetric systems, various polar distributions of sample-points, some of which have the uniform density property that we demanded. Of course, $\mathbb{R}^r$ can be replaced by another manifold. Particularly, periodic lattices of sample-points in an r-torus have received much recent attention in quantum physics; see Athanasiu *et al.* [4], Bars and Minic [10], Bouzouina and De Bièvre [11], and Rivas and Ozorio de Almeida [18].

EXAMPLE 2.E. Let $\mathscr{L}_\infty$ be a Hilbert space, let $\mathscr{S} \subseteq \mathscr{L}_\infty$ be a dense subspace, and let $\mathscr{N}$ be a directed set of subspaces of $\mathscr{L}_\infty$, the direction being inclusion. We insist that $\bigcup \mathscr{N}$ is dense in $\mathscr{L}_\infty$. For each $n \in \mathscr{N}$, let $\mathscr{L}_n = n$. We make the net $(\mathscr{L}_n)_n = (n)_n$ become an inductive resolution of $\mathscr{L}_\infty$ by letting each res_n be the orthogonal projection $\mathscr{S} \to \mathscr{L}_n$. Given vectors $\psi_\infty \in \mathscr{L}_\infty$ and $\psi_n \in \mathscr{L}_n$ for each n, then $\psi_\infty = \lim_n \psi_n$ if and only if the norms $\|\psi_n\|$ are essentially bounded, and $(\psi_n)_n$ weakly converges to ψ_∞. Once again, we have defined a rather general kind of inductive resolution. The last two Examples in this Section are (in essence) special cases.

EXAMPLE 2.F. Let $\mathscr{S}$ be a dense subspace of an infinite-dimensional Hilbert space $\mathscr{L}_\infty$, let $\mathscr{B}$ be a complete orthonormal set in $\mathscr{L}_\infty$, and let $\mathscr{N}$ be a directed set of subsets of $\mathscr{B}$, directed by inclusion, with $\bigcup \mathscr{N} = \mathscr{B}$. Taking res_n to be the orthogonal projection $\mathscr{S} \to \mathscr{L}_n$, then $(\mathscr{L}_n)_n$ becomes an inductive resolution of $\mathscr{L}_\infty$. One "applicable" special case is where $\mathscr{B}$ is enumerated $\{\beta_0, \beta_1, ...\}$ such that, the coordinates have (quickly) decreasing significance in the application. Changing notation, and now writing $\mathscr{N} = \mathbb{N}$ and $\mathscr{L}_n = \langle \beta_0, ..., \beta_n \rangle$, then restriction res_n is truncation to the $n+1$ most significant coordinates.

EXAMPLE 2.G. Recall that a *wavelet* (in the simplest scenario) is a square-integrable function $\theta_{0,0} \to \mathbb{C}$ such that, writing $\theta_{j,k}(x) = 2^j \theta_{0,0}(2^j x - k)$, then the set $\Theta := \{\theta_{j,k} : j, k \in \mathbb{Z}\}$ is a complete orthonormal set in $L^2(\mathbb{R})$. Without forcing a well-ordering on Θ, let us construct, from Θ, two fairly natural inductive resolutions of $L^2(\mathbb{R})$. Let $\mathscr{S} = \mathscr{L}_\infty = L^2(\mathbb{R})$. For the first construction, we let $\mathscr{N} = \mathbb{Z}$, and for each $n \in \mathscr{N}$, we let $\mathscr{L}_n$ be the Hilbert subspace of $\mathscr{L}_\infty$ with a complete orthonormal set consisting of the vectors $\theta_{j,k}$ such that $j \leqslant n$. In the terminology of Wojtaszczyk [22, Definition 2.2], the net $(\mathscr{L}_n)_n$ is a *multiresolution analysis*. Letting res_n be the orthogonal projection to $\mathscr{L}_n$, then $(\mathscr{L}_n)_n$ becomes an inductive resolution of $\mathscr{L}_\infty$. For the second construction, let $\mathscr{N}$ be the set of quadruples

(a, α, b, β) of integers with $a \leqslant \alpha$ and $b \leqslant \beta$. We make $\mathcal{N}$ become a directed set with direction $\leqslant$ such that $(a, \alpha, b, \beta) \leqslant (a', \alpha', b', \beta')$ provided $a \geqslant a'$ and $\alpha \leqslant \alpha'$ and $b \geqslant b'$ and $\beta \leqslant \beta'$. For an element $n = (a, \alpha, b, \beta)$ of $\mathcal{N}$, let $\mathcal{L}_n$ be the finite-dimensional subspace of $\mathcal{L}_\infty$ with a basis consisting of the vectors $\theta_{j,k}$ such that $a \leqslant j \leqslant \alpha$ and $b \leqslant k \leqslant \beta$. Again, we let res_n be the orthogonal projection to $\mathcal{L}_n$. The point of this inductive resolution (in accordance with the principal motivation for wavelets in the first place) is that projection to $\mathcal{L}_n$ is a way of zooming in on a particular window of time (the home variable) and logarithmic frequency.

3. CONVERGENCE OF COORDINATES

Throughout this Section, we consider the general case of an inductive resolution $(\mathcal{L}_n)_n$ of a Hilbert space $\mathcal{L}_\infty$. As before, we write the restriction maps as $\text{res}_n : \mathcal{S} \to \mathcal{L}_n$. We shall explain how the spaces $\mathcal{L}_n$ admit systems of coordinates that are, in an appropriate sense, compatible with any given system of coordinates of the space $\mathcal{L}_\infty$. Our aim is to show that a given net $(\psi_n)_n$ of vectors $\psi_n \in \mathcal{L}_n$ converges to a given vector $\psi_\infty \in \mathcal{L}_\infty$ if and only if the norms $\|\psi_n\|$ are essentially bounded, and the coordinates of the vectors ψ_n converge to the coordinates of ψ_∞. This criterion for convergence of vectors will be needed in [7].

By an *enumerated set*, we mean a set $\{\mathcal{O}_j : j \in J\}$ equipped with a bijection $j \mapsto \mathcal{O}_j$, where either $J = \mathbb{N}$ or else $J = \{0, 1, \ldots, d-1\}$ for some natural number d. Given an enumerated orthonormal set of vectors $\mathcal{A}_\infty = \{\alpha_{j,\infty} : j \in J_\infty\}$ in $\mathcal{L}_\infty$, and an enumerated orthonormal set of vectors $\mathcal{A}_n = \{\alpha_{j,n} : j \in J_n\}$ in each $\mathcal{L}_n$, we say that the net $(\mathcal{A}_n)_{n \in \mathcal{N}}$ *converges* to $\mathcal{A}_\infty$ provided $\alpha_{j,\infty} = \lim_{n \in \mathcal{N}} \alpha_{j,n}$ for each $j \in J_\infty$.

THEOREM 3.1. *Any enumerated orthonormal set in $\mathcal{L}_\infty$ is the limit of a net of enumerated orthonormal sets in the spaces $\mathcal{L}_n$.*

Proof. Let $\mathcal{A}_\infty = \{\alpha_{j,\infty} : j \in J_\infty\}$ be an enumerated orthonormal set in $\mathcal{L}_\infty$. Theorem 2.4 guarentees that, for each $j \in J_\infty$, there exist vectors $\gamma_{j,n} \in \mathcal{L}_n$ such that $\|\gamma_{j,n}\| = 1$ and $\alpha_{j,\infty} = \lim_n \gamma_{j,n}$. Given $\varepsilon > 0$, we can choose vectors $\phi_j \in \mathcal{S}$ such that $\|\phi_j - \beta_{j,\infty}\| \leqslant \varepsilon$ for all $j \in J_\infty$. For fixed $j, k \in J_\infty$, we have

$$\lim_{n \in \mathcal{N}} \langle \text{res}_n(\phi_j) \mid \gamma_{k,n} \rangle = \langle \phi_j \mid \alpha_{k,\infty} \rangle = \delta_{j,k} + O(\varepsilon)$$

But $\lim_n \|\text{res}_n(\phi_j)\| = \|\phi_j\| = 1 + O(\varepsilon)$ and $\|\gamma_{k,n}\| = 1$, so

$$\langle \gamma_{j,n} \mid \gamma_{k,n} \rangle = \delta_{j,k} + O(\varepsilon)$$

for sufficiently large n. Let $\mathscr{A}_n = \{\alpha_{j,n} : j \in J_n\}$ be the enumerated orthonormal set constructed from $\{\gamma_{j,n} : j \in J_\infty\}$ by the Gram–Schmidt process. Thus $\alpha_{0,n} = \gamma_{0,n}/\|\gamma_{0,n}\|$, and generally, each $\alpha_{j,n}$ is the normalized orthogonal projection of $\gamma_{j,n}$ to the orthogonal complement of the span of $\{\gamma_{0,n}, ..., \gamma_{j-1,n}\}$. For given n, the set $\mathscr{A}_n$ is finite if and only if the process terminates after finitely many steps; the set $\mathscr{A}_n$ has finite size d provided d is minimal such that either $d = |J_\infty|$ or else $\gamma_{d,n}$ is a linear combination of the vectors $\gamma_{0,n}, ..., \gamma_{d-1,n}$. The approximate orthonormality condition for the vectors $\gamma_{j,n}$ ensures that, given any $j \in J_\infty$ then, for sufficiently large n, the set $\{\gamma_{0,n}, ..., \gamma_{j,n}\}$ is linearly independent, hence $\alpha_{j,n}$ is defined. The same condition ensures that $\lim_n \|\gamma_{j,n} - \alpha_{j,n}\| = 0$. By Remark 2.3, $\alpha_{j,\infty} = \lim_n \alpha_{j,n}$. ∎

For the rest of this paper, we let $\mathscr{B}_\infty = \{\beta_{j,\infty} : j \in J_\infty\}$ be an enumerated complete orthonormal set for $\mathscr{L}_\infty$, and for each $n \in \mathscr{N}$, we let $\mathscr{B}_n = \{\beta_{j,n} : j \in J_n\}$ be an enumerated orthonormal set for $\mathscr{L}_n$ such that the net $(\mathscr{B}_n)_n$ converges to $\mathscr{B}_\infty$.

COROLLARY 3.2. *If the dimension* $\dim \mathscr{L}_\infty$ *is finite, then* $\dim \mathscr{L}_\infty \leqslant \dim \mathscr{L}_n$ *for sufficiently large n. If* $\dim \mathscr{L}_\infty = \infty$, *then* $\lim_{n \in \mathscr{N}} \dim \mathscr{L}_n = \infty$.

Proof. For each $j \in J_\infty$, we have $\beta_{j,\infty} = \lim_{n \in \mathscr{N}} \beta_{j,n}$, hence $j \in J_n$ for sufficiently large n. ∎

Let us discuss a technical irritation. Is every enumerated complete orthonormal set for $\mathscr{L}_\infty$ the limit of a net of enumerated complete orthonormal sets for the spaces $\mathscr{L}_n$? The answer is negative in general, but affirmative in most cases of likely interest. Let us discuss this briefly, leaving the easy proofs of our observations as exercises. The directed set $\mathscr{N}$ has an unbounded countable subset if and only if there exists an order-preserving unbounded function $\mathscr{N} \to \mathbb{N}$. When these equivalent conditions hold, $\mathscr{N}$ is said to be *Archimedian*. If $\mathscr{N}$ is countable, then $\mathscr{N}$ is Archimedian. If $\mathscr{L}_\infty$ is infinite-dimensional and each $\mathscr{L}_n$ is finite-dimensional, then (thanks to Corollary 3.2) $\mathscr{N}$ is Archimedian. Whenever $\mathscr{N}$ is Archimedian, the answer to the above question is affirmative. Meanwhile, if $\mathscr{N}$ is the set of countable subsets of an uncountable set (directed by inclusion), then $\mathscr{N}$ is non-Archimedian. Supposing that $\mathscr{N}$ is non-Archimedian, $\mathscr{L}_\infty$ is infinite-dimensional and each res_n has domain $\mathscr{L}_\infty$ and image strictly contained in $\mathscr{L}_n$, then no complete orthonormal set in $\mathscr{L}_\infty$ is the limit of a net of complete orthonormal sets in the spaces $\mathscr{L}_n$.

To keep the notation simple, let us abuse it slightly. For a vector $\psi_\infty \in \mathscr{L}_\infty$, let us write

$$\psi_\infty = \sum_{j=0}^{\infty} c_{j,\infty} \beta_{j,\infty},$$

where $c_{j,\infty} = \langle \beta_{j,\infty} \mid \psi_\infty \rangle$ for $j \in \mathbb{N}$. The notation makes sense, granted the understanding that $\langle \beta_{j,\infty} \mid \psi_\infty \rangle := 0$ if $j \notin J_\infty$. For $\psi_n \in \mathscr{L}_n$, we write

$$\psi_n = \psi_n^\perp + \sum_{j=0}^{\infty} c_{j,n} \beta_{j,n},$$

where $c_{j,n} = \langle \beta_{j,n} \mid \psi_n \rangle$ for $j \in \mathbb{N}$. As before, it is to be understood that $\langle \beta_{j,n} \mid \psi_n \rangle := 0$ if $j \notin J_n$. Thus, the vector $\psi_n^\perp$ is the component of ψ_n orthogonal to the span of the orthonormal set $\mathscr{B}_n$. Similarly, for $\phi \in \mathscr{S}$, we write

$$\phi = \sum_{j=0}^{\infty} a_{j,\infty} \beta_{j,\infty} \quad \text{and} \quad \phi_n = \operatorname{res}_n(\phi) = \phi_n^\perp + \sum_{j=0}^{\infty} a_{j,n} \beta_{j,n}.$$

LEMMA 3.3. *Given $\phi \in \mathscr{S}$, and letting $a_{j,\infty}$ and $a_{j,n}$ be coordinates as above, then*

$$\lim_{n \in \mathscr{N}} \sum_{j=0}^{\infty} |a_{j,\infty} - a_{j,n}|^2 = 0 \qquad \text{and} \quad \lim_{n \in \mathscr{N}} \|\phi_n^\perp\| = 0.$$

Proof. We have $\sum_{j=0}^{\infty} |a_{j,\infty}|^2 = \|\phi\|^2 = \lim_{n \in \mathscr{N}} \|\phi_n\|^2 = \lim_{n \in \mathscr{N}} (\|\phi_n^\perp\|^2 + \sum_{j=0}^{\infty} |a_{j,n}|^2)$. Since $a_{j,\infty} = \overline{\langle \phi \mid \beta_{n,\infty} \rangle} = \lim_{n \in \mathscr{N}} \overline{\langle \phi_n \mid \beta_{n,j} \rangle} = \lim_{n \in \mathscr{N}} a_{j,n}$ for all $j \in J_\infty$, we have

$$\lim_{n \in \mathscr{N}} (\|\phi_n^\perp\|^2 + \lim_{j \in J_n - J_\infty} |a_{j,n}|^2) = 0.$$

The second asserted equality follows, and furthermore,

$$\sum_{j=0}^{\infty} |a_{j,\infty}|^2 = \lim_{n \in \mathscr{N}} \sum_{j=0}^{\infty} |a_{j,n}|^2.$$

Let $0 < \varepsilon < 1$. Let $n \in \mathscr{N}$, and assume that n is sufficiently large for all our purposes. Choose $m \in \mathbb{N}$ such that

$$\sum_{j=m}^{\infty} |a_{j,\infty}|^2 < \varepsilon.$$

Given $0 \leqslant j \leqslant m-1$, then $|a_{j,\infty} - a_{j,n}| < \varepsilon$, hence

$$|a_{j,\infty}|^2 - |a_{j,n}|^2 \leqslant \varepsilon(|a_{j,\infty}| + |a_{j,n}|) \leqslant \varepsilon(\|\phi\| + \|\phi_n\|) \leqslant \varepsilon(1 + 2\,\|\phi\|) = O(\varepsilon).$$

The two differences $\sum_{j=0}^{\infty} |a_{j,n}|^2 - \sum_{j=0}^{\infty} |a_{j,\infty}|^2$ and $\sum_{j=0}^{m-1} |a_{j,n}|^2 - \sum_{j=0}^{m-1} |a_{j,\infty}|^2$ both have magnitude $O(\varepsilon)$, hence

$$\sum_{j=m}^{\infty} |a_{j,n}|^2 = O(\varepsilon) + \sum_{j=m}^{\infty} |a_{j,\infty}|^2 = O(\varepsilon).$$

Therefore $\sum_{j=0}^{\infty} |a_{j,\infty} - a_{j,n}|^2 \leqslant m\varepsilon^2 + \sum_{j=m}^{\infty} |a_{j,\infty} - a_{j,n}|^2 = O(\varepsilon)$. ∎

THEOREM 3.4. *Let $\psi_\infty \in \mathscr{L}_\infty$, and let $c_{j,\infty}$ and $c_{j,n}$ be coordinates as above. Then $\psi_\infty = \lim_{n \in \mathscr{N}} \psi_n$ if and only if the norms $\|\psi_n\|$ are essentially bounded, and $c_{j,\infty} = \lim_{n \in \mathscr{N}} c_{j,n}$ for all $j \in J_\infty$.*

Proof. Throughout the argument, we may assume that the norms $\{\|\psi_n\| : n \in \mathscr{N}\}$ are essentially bounded. In fact, we may assume that the norms $\|\psi_n\|$ are bounded. Choosing an upper bound C for $\|\psi_\infty\|$ and all the norms $\|\psi_n\|$, then

$$\sum_{j=0}^{\infty} |c_{j,\infty}|^2 = \|\psi_\infty\|^2 \leqslant C^2 \geqslant \|\psi_n\|^2 = \|\psi_n^\perp\|^2 + \sum_{j=0}^{\infty} |c_{j,n}|^2.$$

Let $\phi \in \mathscr{S}$. We continue to use the notation of the proof of Lemma 3.3, and we still assume that n is sufficiently large. We have

$$\langle \phi \mid \psi_\infty \rangle = \sum_{j=0}^{\infty} \bar{a}_{j,\infty} c_{j,\infty} \qquad \text{and} \qquad \langle \phi_n \mid \psi_n \rangle = \sum_{j=0}^{\infty} \bar{a}_{j,n} c_{j,n}.$$

Assume that each $c_{j,\infty} = \lim_n c_{j,n}$. Our choice of m ensures that $\sum_{j=m}^{\infty} |\bar{a}_{j,\infty} c_{j,\infty}| \leqslant C\sqrt{\varepsilon}$. Similarly, $\sum_{j=m}^{\infty} |\bar{a}_{j,n} c_{j,n}| = O(\sqrt{\varepsilon})$. Therefore, $|\langle \phi \mid \psi_\infty \rangle - \langle \mathrm{res}_n(\phi) \mid \psi_n \rangle| = O(\sqrt{\varepsilon})$. We have deduced that $\psi_\infty = \lim_n \psi_n$.

Conversely, assume that $\psi_\infty = \lim_n \psi_n$. Fix an index $k \in J_\infty$. Since $\mathscr{S}$ is dense in $\mathscr{L}_\infty$, the vector $\phi \in \mathscr{S}$ can be chosen and fixed such that $\|\phi - \beta_{k,\infty}\| < \sqrt{\varepsilon}$, in other words

$$|a_{k,\infty} - 1|^2 + \sum_{j \neq k} |a_{j,\infty}|^2 < \varepsilon.$$

Thence $\sum_{j \neq k} = \bar{a}_{j,\infty} c_{j,\infty} < C\sqrt{\varepsilon}$.

We now impose upon m the further constraint that $m > k$. By Lemma 3.3, $a_{j,\infty} = \lim_n a_{j,n}$ for all $j \in \mathbb{N}$. So $\sum_{k \neq j \leqslant m-1} |a_{j,n}|^2 = O(\varepsilon)$. Hence, $\sum_{j \neq k} |a_{j,n}|^2 = O(\sqrt{\varepsilon})$ and

$$\sum_{j \neq k} \bar{a}_{j,\infty} c_{j,\infty} = O(\sqrt{\varepsilon})$$

Our assumption on the net $(\psi_n)_n$ yields $\sum_{j=0}^{\infty} = \bar{a}_{j,\infty} c_{j,\infty} = \lim_{n \in \mathcal{N}} \sum_{j=0}^{\infty} = \bar{a}_{j,n} c_{j,n}$. Therefore

$$|\bar{a}_{k,\infty} c_{k,\infty} - \bar{a}_{k,n} c_{k,n}| = O(\sqrt{\varepsilon}).$$

Since $|a_{k,\infty} - 1| \leqslant \sqrt{\varepsilon}$ and $a_{k,\infty} = \lim_n a_{k,n}$, we have $c_{k,\infty} = \lim_n c_{k,n}$. ∎

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