Recursion operator and dispersionless rational Lax representation

K. Zheltukhin

Department of Mathematics, Faculty of Sciences, Bilkent University, 06533 Ankara, Turkey
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Abstract

We consider equations arising from dispersionless rational Lax representations. A general method to construct recursion operators for such equations is given. Several examples are given, including a degenerate bi-Hamiltonian system with a recursion operator. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently a new method of constructing a recursion operator from Lax representation was introduced in [1]. This construction depends on Lax representation of a given system of PDEs. Let

\[ L_t = [A, L] \]  \hspace{1cm} \text{(1)}

be Lax representation of an integrable nonlinear system of PDEs. Then a hierarchy of symmetries can be given by

\[ L_{t_n} = [A_n, L], \quad n = 0, 1, 2, \ldots, \]  \hspace{1cm} \text{(2)}

where \( t_0 = t \), \( A_0 = A \) and \( A_n, n = 0, 1, 2, \ldots, \) are Gel’fand–Dikkii operators given in terms of \( L \). The recursion relation between symmetries can be written as

\[ L_{t_{n+1}} = LL_{t_n} + [R_n, L], \quad n = 0, 1, 2, \ldots, \]  \hspace{1cm} \text{(3)}

where \( R_n \) is an operator such that \( \text{ord } R_n = \text{ord } L \).

This symmetry relation allows us to find \( R_n \), hence \( L_{t_{n+1}} \), in terms of \( L \) and \( L_{t_n} \).

In [1,2] this method was applied to construct recursion operators for Lax equations with different classes of scalar and shift operators, corresponding to field and lattice systems, respectively. In [3] the method was applied to...
dispersionless Lax equations on a Poisson algebra of Laurent series

\[ \Lambda = \left\{ \sum_{i=-\infty}^{\infty} u_i p^i : u_i - \text{smooth functions} \right\}, \quad (4) \]

with a polynomial Lax function. The present work is a continuation of [3]. Here we consider a dispersionless Lax equation on the Poisson algebra \( \Lambda \) with a rational Lax function. Such equations one can find in context of topological field theories (see [4,5]).

We have a Lax function

\[ L = \frac{\Delta_1}{\Delta_2}, \quad (5) \]

where \( \Delta_1, \Delta_2 \) are polynomials of degree \( N \) and \( M \), respectively, and \( N > M \). The dispersionless Lax equation is

\[ \frac{\partial L}{\partial t_n} = \left\{ \left( \left( L^{n-n} \right)_{n \geq 0}, L \right), n = 0, 1, 2, \ldots \right\}, \quad (6) \]

where the Poisson bracket is given by

\[ \{ f, g \} = p \left( \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \right). \]

Eq. (6) is of hydrodynamic type. There are several methods for construction of a recursion operator for some equations of hydrodynamic type (see [6–8]). Also a recursion operator can be found with the help of two compatible Hamiltonian formulations of a given equation. For Hamiltonian formulations of equations of hydrodynamic type see Refs. [9,10] and for Hamiltonian formulations of the equations admitting a dispersionless Lax representation see Refs. [11–15].

We construct a recursion operator for a hierarchy of symmetries (6), using a dispersionless Lax representation. First we study the symmetry relation (3) for the rational Lax function. Then we give some examples of calculation of a recursion operator. In particular, we find a recursion operator \( \mathcal{R} \) for Eq. (6) with the Lax function

\[ L = p + S + \frac{P}{p + Q}, \quad (7) \]

which leads to the system [11]

\[ S_t = P_x, \quad P_t = PS_x - QP_x - P Q_x, \quad Q_t = QS_x - Q Q_x. \quad (8) \]

The recursion operator is given by

\[ \mathcal{R} = \begin{pmatrix} S & 1 & PQ^{-1} + P_x D_x^{-1} \cdot Q \\ 2P & S - Q & -2P + (PS_x - (P Q_x) D_x^{-1} \cdot Q \end{pmatrix}, \quad (9) \]

In [11] bi-Hamiltonian representation of this equation was constructed with Hamiltonian operators

\[ \mathcal{D}_1 = \begin{pmatrix} 0 & P & Q \\ P & -2P Q & -Q^2 \\ Q & -Q^2 & 0 \end{pmatrix} D_x + \begin{pmatrix} 0 & P_x \\ 0 & -(P Q_x) \\ 0 & -(Q Q_x) \end{pmatrix}, \quad (10) \]

and

\[ \mathcal{D}_2 = \begin{pmatrix} 2P & P(S - 3Q) & Q(S - Q) \\ P(S - Q) & 2P - 2SQ + 4Q^2 & Q(2P - SQ + Q^2) \\ Q(S - Q) & Q(2P - SQ + Q^2) & 2Q^2 \end{pmatrix} D_x. \]
These Hamiltonian operators are degenerate, so, one cannot use them to find a recursion operator. But it turns out that they are related to the recursion operator $\mathcal{R}$. One can easily check that the following equality holds

$$\mathcal{R}D_1 = D_2.$$  \hfill (12)

We observe that the degeneracy in the bi-Hamiltonian operators is due to the following fact. Let $p' = p + F$ then the Lax function becomes

$$L = p' + G + \frac{P}{p'}.$$  \hfill (13)

This means that we have two independent variables $P$ and $G$, where $G = S - F$. The equation corresponding to the Lax function (13) has been studied in [3].

To remove degeneracy one can take the Lax function as

$$L = p + S + \frac{P}{p} + \sum_{i=1}^{m} \frac{Q_i}{p + F_i}.$$  \hfill (14)

As an example we shall consider the Eq. (6) with the Lax function

$$L = p + S + \frac{P}{p} + \frac{Q}{p + F}.$$  \hfill (15)

2. Symmetry relation for rational dispersionless Lax representation

Following [1] we consider the hierarchy of symmetries for the dispersionless Lax equation (6) with the Lax function (5)

$$\frac{\partial L}{\partial t_n} = \left\{ \left( \frac{L}{L_1 N - M} + n \right)_{\geq 0}, L \right\}, \quad n = 0, 1, 2, \ldots.$$  \hfill (16)

**Lemma 1.** For any $n = 0, 1, 2, \ldots$,

$$\frac{\partial L}{\partial t_n} = L \frac{\partial L}{\partial t_{n-1}} + [R_n, L].$$  \hfill (17)

Function $R_n$ has a form

$$R_n = A + \frac{B}{\Delta z},$$  \hfill (18)

where $A$ is a polynomial of degree $(N - M)$ and $B$ is a polynomial of degree $(M - 1)$.

**Proof.** We have

$$L_{N-M+n} \geq 0 = \left[ L \left( L_{N-M+(n-1)} \geq 0 \right) + L \left( L_{N-M+(n-1)} \leq 0 \right) \right]_{\geq 0}.$$  

So,

$$L_{N-M+n} \geq 0 = L \left( L_{N-M+(n-1)} \geq 0 \right) \geq 0 + \left[ L \left( L_{N-M+(n-1)} \leq 0 \right) \geq 0 - \left( L \left( L_{N-M+(n-1)} \leq 0 \right) \geq 0 \right) \right].$$
If we take
\[ R_n = \left( L \left( L \frac{1}{n-M} + (n-1) \right) < 0 \right) \geq 0 - \left( L \left( L \frac{1}{n-M} + (n-1) \right) \geq 0 \right) < 0, \] (19)
then
\[ \left( L \frac{1}{n-M} + n \right) \geq 0 = L \left( L \frac{1}{n-M} + (n-1) \right) \geq 0 + R_n. \]
Hence,
\[ \partial L \partial t_n = \left\{ \left( L \frac{1}{n-M} + n \right) \geq 0 \right\} L \left( L \frac{1}{n-M} + (n-1) \right) \geq 0 + R_n, \]
and (17) is satisfied. The remainder \( R_n \) has form (18). Indeed in (19) we set
\[ A = \left( L \left( L \frac{1}{n-M} + (n-1) \right) < 0 \right) \geq 0, \] and
\[ B = \Delta_2 \left( L \left( L \frac{1}{n-M} + (n-1) \right) \geq 0 \right) < 0. \]
Then \( A \) is a polynomial of degree \((N - M - 1)\) and \( B \) is a polynomial of degree \((M - 1)\). \( \square \)

Now we can apply the Lemma 1 to find recursion operators.

3. Examples

**Example 2.** Let us consider the Eq. (8) given in introduction.

**Lemma 3.** A recursion operator for (8) is given by (9).

**Proof.** Using (18) for \( R_n \), we have \( R_n = A + \frac{B}{p+Q} \). So, the symmetry relation (17) is
\[ \partial L \partial t_n = \left\{ \left( L \frac{1}{n-M} + n \right) \geq 0 \right\} \left( L \left( L \frac{1}{n-M} + (n-1) \right) \geq 0 + R_n, L \right) = L \frac{\partial L}{\partial t_n} + \{ R_n, L \}, \]
and (17) is satisfied. The remainder \( R_n \) has form (18). Indeed in (19) we set
\[ A = \left( L \left( L \frac{1}{n-M} + (n-1) \right) < 0 \right) \geq 0, \]
and
\[ B = \Delta_2 \left( L \left( L \frac{1}{n-M} + (n-1) \right) \geq 0 \right) < 0. \]
Then \( A \) is a polynomial of degree \((N - M - 1)\) and \( B \) is a polynomial of degree \((M - 1)\). \( \square \)

Now we can apply the Lemma 1 to find recursion operators.

**Example 4.** The dispersionless Lax equation (6) with the Lax function (15), for \( n = 1 \), gives the following system
\[ S_t = P_x + Q_x, \quad P_t = PS_x, \quad Q_t = QS_x - FQ_x - QF_x, \quad F_t = FS_x - FF_x. \] (20)
Lemma 5. A recursion operator for (20) is given by

\[
\begin{pmatrix}
S & 2 + P_x D_x^{-1} \cdot P^{-1} & 1 \\
2P & S + QF^{-1} + PS_x D_x^{-1} \cdot P^{-1} & PF^{-1} \\
2Q & -QF^{-1} \quad -PF^{-1}(Q_x - QF^{-1}F_x)D_x^{-1} \cdot P^{-1} & S - F - PF^{-1} \\
F & 1 + (P_x - PF^{-1}F_x)D_x^{-1} \cdot P^{-1} & -1 \\
\end{pmatrix}
\begin{pmatrix}
QF^{-1} + Q_x D_x^{-1} \cdot F^{-1} \\
-2PQF^{-2} \\
-2PQF^{-2} - 2Q \\
-PF^{-1}F_x \\
\end{pmatrix}.
\]

(21)

Proof. Using (18) for \( R_n \), we have \( R_n = C + \frac{A}{p} + \frac{B}{p + F} \). So, the symmetry relation (17) is

\[
\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n} \cdot \frac{1}{p} + \frac{\partial Q}{\partial t_n} \cdot \frac{1}{p} + \frac{\partial F}{\partial t_n} \cdot \frac{-Q}{(p + F)^2} = \left( p + S + \frac{Q}{p + F} \right) \left( \frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}} \cdot \frac{1}{p} + \frac{\partial Q}{\partial t_{n-1}} \cdot \frac{1}{p + F} + \frac{\partial F}{\partial t_{n-1}} \cdot \frac{-Q}{(p + F)^2} \right)
\]

\[
+ p \left( \frac{B}{p^2} + \frac{-C}{(p + F)^2} \right) \left( S_x + \frac{P_x}{p} + \frac{Q_x}{(p + F)} + \frac{-QF_x}{(p + F)^2} \right)
\]

\[
- p \left( A_x + \frac{B_x}{p} + \frac{C_x}{(p + F) + \frac{-C}{(p + F)^2}} \right) \left( 1 + \frac{P}{p} + \frac{-Q}{(p + F)^2} \right).
\]

Therefore, the coefficients of \( p, p^{-2} \) and \( (p + F)^{-3} \) must be zero, it gives recursion relations to find \( A, B \) and \( C \). Then the coefficients of \( p^0, p^{-1}, (p + F)^{-1}\) and \( (p + F)^{-2} \), give expressions for \( \frac{\partial S}{\partial t_n}, \frac{\partial P}{\partial t_n}, \frac{\partial Q}{\partial t_n} \) and \( \frac{\partial F}{\partial t_n} \). \( \square \)

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References