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Polarization of radiation in multipole Jaynes–Cummings model

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Abstract. We discuss the spatial properties of quantum radiation emitted by a multipole transition in a single atom. It is shown that the polarization of multipole radiation and quantum fluctuations of polarization change with distance from the source. In the case of a transition specified by a given quantum number m, the quantum noise of polarization contains contributions coming from the modes with $m' \neq m$ as well.

1. Introduction

It is well known that the Jaynes–Cummings model [1] plays an important role in the investigation of the interaction between atoms and the quantum radiation field (e.g. see [2–5]). The point is that the model describes fairly well the physical processes in the system and, at the same time, allows an exact solution.

In the usual formulation of the Jaynes–Cummings model [1–5], the atom is considered as though it consists of two or very few non-degenerated levels. In fact, the radiative transitions in real atoms occur between the states with given angular momentum $j \geqslant 1$ and its projection $m = -j, \ldots, j$ (e.g. see [6]). This means that even in the case of only two levels, the degeneration with respect to the quantum number m taking $(2j+1)$ different values should be considered. The simplest example is provided by a dipole interaction between the states $|j = 1, m = 0, \pm 1\rangle$ and $|j' = 0, m' = 0\rangle$ when the excited atomic state is a triply degenerated one (see figure 1).

Let us stress one more important difference. The radiation field in the conventional Jaynes–Cummings model is represented by the plane waves of photons with given linear momentum and polarization. At the same time, the multipole transitions in real atoms emit the multipole photons represented by the quantized spherical waves with given angular momentum and parity [7, 8]. Although there is no principle difference between the plane and spherical waves within the classical domain, since both represent the complete orthogonal sets of solutions of the homogeneous wave equation [9] and can be re-expanded with respect to each other, the quantum counterpart of these two representations are non-equivalent because they describe the physical quantities (the linear and angular momenta, respectively), which cannot be measured at once.

The multipole generalization of the Jaynes–Cummings model has been discussed in [10, 11]. Let us stress that similar models have been used in different

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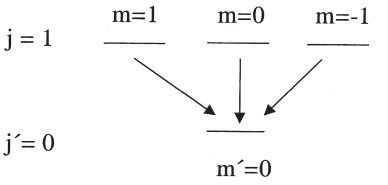

Figure 1. Energy diagram of triple degenerated excited and ground states of a dipole transition $j = 1 \leftrightarrow j' = 0$.

problems for the interaction of quantum light with matter (e.g. see [12–15] and references therein).

The main objective of this paper is to examine the quantum polarization properties of light emitted by a dipole atom at different distances from the source, depending on the boundary conditions. The paper is arranged as follows. In section 2 we review the properties of a quantum multipole field in comparison with those of plane waves of photons and briefly discuss the multipole Jaynes–Cummings model. In section 3 we consider the polarization of multipole radiation and introduce a novel general polarization matrix. This matrix permits us to take into account the spatial anisotropy of both the electric and magnetic fields at once. In section 4 we examine the spatial properties of multipole photons emitted by an atom in an ideal spherical cavity as well as in empty space. In particular, we show that the polarization properties of quantum multipole radiation changes with distance from the atom. In section 5 we briefly discuss the results obtained.

2. Multipole Jaynes–Cummings model

Following [7, 8, 16], we list below some important formulas describing the quantum multipole field. It is usually considered in the so-called helicity basis [16]

$$\boldsymbol{\chi}_{\pm} = \mp \frac{\mathbf{e}_x \pm \mathrm{i}\mathbf{e}_y}{2^{1/2}}, \quad \boldsymbol{\chi}_0 = \mathbf{e}_z. \tag{1}$$

It is clear that $\{\boldsymbol{\chi}_\mu\}$ formally coincides with the three eigenstates of spin 1 of a photon. Since the polarization is defined to be the spin state of photons [17], one can choose to interpret $\boldsymbol{\chi}_\pm$ as the unit vector of circular polarization with either positive or negative helicity, while $\boldsymbol{\chi}_0$ gives the linear polarization in the z direction. To within the sign at $\boldsymbol{\chi}_\pm$, the helicity basis (1) coincides with the so-called circular polarization basis widely used in optics [18]. In the basis (1), the positive-frequency part of the operator vector potential of a multipole field can be expanded as follows [7, 16]

$$\mathbf{A}(\mathbf{r}) = \sum_k \sum_\lambda \sum_{\mu=-1}^{1} \sum_{j=1}^{\infty} \sum_{m=-j}^{j} (-1)^\mu \boldsymbol{\chi}_{-\mu} V_{\lambda kjm\mu}(\mathbf{r}) a_{\lambda kjm}, \tag{2}$$

where $a_{\ldots}$ is the photon annihilation operator which obeys the following commutation relation

$$[a_{\lambda kjm}, a^+_{\lambda' k' j' m'}] = \delta_{\lambda\lambda'}\delta_{kk'}\delta_{jj'}\delta_{mm'}.$$

Here k is the wave number, $\lambda = E, M$ denotes the type of radiation (parity), index $j \geqslant 1$ gives the angular momentum, and index $m = -j, \ldots, j$. The mode functions $V\ldots(\mathbf{r})$ are represented in the following way

$$\begin{aligned} V_{Ekjm\mu}(\mathbf{r}) &= \gamma_{Ekj}[j^{1/2} f_{j+1}(r)\langle 1, j+1, \mu, m-\mu | jm\rangle\, Y_{j+1,m-\mu}(\theta,\phi) \\ &\quad - (j+1)^{1/2} f_{j-1}(r)\langle 1, j-1, \mu, m-\mu | jm\rangle\, Y_{j-1,m-\mu}(\theta,\phi)], \\ V_{Mkjm\mu}(r) &= \gamma_{Mkj} f_j(r)\langle 1, j, \mu, m-\mu | jm\rangle\, Y_{j,m-\mu}(\theta,\phi) \end{aligned} \tag{3}$$

for the electric and magnetic multipole radiation, respectively. Here

$$\gamma_{Ekj} = \frac{\gamma_{Mkj}}{(2j+1)^{1/2}}, \quad \gamma_{Mkj} = \left(\frac{2\pi\hbar c}{k\mathcal{V}}\right)^{1/2}$$

are the normalization constants, $\mathcal{V}$ is the volume of quantization, $\langle \cdots | \cdots \rangle$ denotes the Clebsch–Gordan coefficient of vector addition of the spin and orbital parts of the angular momentum of a multipole photon and $Y_{\ell m}$ is the spherical harmonics, describing the angular dependence. The radial contribution into the mode functions (3) depends on the boundary conditions. In the standard case for quantization of spherical waves in an ideal spherical cavity [7, 16], we have

$$f_\ell(r) = j_\ell(kr) \equiv \left(\frac{\pi}{2kr}\right)^{1/2} J_{\ell+1/2}(kr), \tag{4}$$

where $j_\ell(x)$ is the spherical Bessel function. The positive-frequency parts of the operator field strengths obey the following symmetry relations

$$\begin{aligned} E_{Ekjm\mu} &= \mathrm{i}kV_{Ekjm\mu}(\mathbf{r})a_{Ekjm}, \quad B_{Ekjm\mu} = -\mathrm{i}kV_{Mkjm\mu}(\mathbf{r})a_{Ekjm}, \\ E_{Mkjm\mu} &= \mathrm{i}kV_{Mkjm\mu}(\mathbf{r})a_{Mkjm}, \quad B_{Mkjm\mu} = \mathrm{i}kV_{Ekjm\mu}(\mathbf{r})a_{Mkjm}. \end{aligned} \tag{5}$$

It can be easily seen from (2) and (5) that the electric multipole field always has the longitudinal component of the electric field strength in addition to the two transversal components, while it is completely transversal with respect to the magnetic induction. At the same time, the magnetic multipole field has all three spatial components of magnetic induction and only two transversal components of the electric field strength.

The position dependence of the mode functions (3) is not an unusual fact. In reality, the mode functions of the plane waves also depend on position:

$$\mathbf{A}^{(\mathrm{plane})}(\mathbf{r}) = \sum_k \left(\frac{2\pi\hbar c}{k\mathcal{V}}\right)^{1/2} \sum_{\mu=\pm 1} (-1)^\mu \boldsymbol{\chi}_{-\mu} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\mathbf{a}_{k\mu}. \tag{6}$$

Here we choose the basis (1) with $\boldsymbol{\chi}_0 = \mathbf{k}/k$ and $\mathbf{a}\ldots$ denotes the photon annihilation operator, corresponding to the states with given linear momentum (direction of propagation) and transversal polarization with either helicity. The third projection of the photon spin is forbidden in this case [17].

The interaction of the quantum multipole field (2) with an atom can be described in a standard way [8, 16]. As an example of some considerable interest, we now discuss a two-level atom with the electric dipole transition $j = 1 \to j' = 0$. The coupling constant of the atom–field interaction can be found by calculating the matrix element [8, 16]

$$g = -\frac{e}{2m_e c}\langle 0,0|\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}|1,m\rangle = \mathrm{i}k_0\langle 0,0|\mathbf{d}\cdot\mathbf{A}|1,m\rangle, \tag{7}$$

obtained from the standard expression $(\mathbf{p} - e\mathbf{A}/c)^2/2m_e$, describing the interaction between the atomic electron with linear momentum $\mathbf{p}$, charge $-e$, and mass m_e and radiation field specified by the vector potential $\mathbf{A}$. Here $\mathbf{d} \equiv e\mathbf{r}$ is the dipole moment of the atomic transition with the resonance frequency $\omega_0 = ck_0$. Assuming the central symmetry of an atomic field and taking into account the fact that the spin state of an atom does not change under the electric dipole transition, we can represent the atomic states in (7) as follows

$$|1,m\rangle = \mathcal{R}_e(kr)\,Y_{1m}(\theta,\phi), \quad |0,0\rangle = \mathcal{R}_g(kr)\,Y_{00}(\theta,\phi),$$

where $\mathcal{R}_{\cdot}$ is the radial part of the atomic wave function of either excited or ground state.

Expanding the dipole momentum $\mathbf{d}$ over the helicity basis (1), substituting (2), and carrying out the calculation of spatial integrals in (7) over a small volume occupied by the atom, we get

$$g = \frac{k_0 c}{(kc)^{1/2}}D, \tag{8}$$

where D is the effective dipole factor that, by construction, is independent of the quantum number m. Thus, the coupling constant (8) has the same value for the transitions $|1,m\rangle \leftrightarrow |0,0\rangle$ at any m.

Taking into account the properties of the Clebsch–Gordan coefficients and spherical harmonics, for the position-dependent mode function in (3) we get

$$\lim_{kr\to 0} V_{Ek1m\mu} \sim \delta_{m\mu}.$$

This means that the electric dipole transition $|1,m\rangle \to |0,0\rangle$ at any given m creates a photon with polarization $\mu = m = 0, \pm 1$.

Finally, the Jaynes–Cummings Hamiltonian of the electric dipole transition in the rotating-wave approximation [19] takes the form [10, 11]:

$$\begin{aligned} \hbar^{-1}H &= H_0 + H_{\mathrm{int}}, \\ H_0 &= \sum_{m=-1}^{1}(\omega a_m^+ a_m + \omega_0 R_{mm}), \\ H_{\mathrm{int}} &= g\sum_{m=-1}^{1}(R_{mg}a_m + a_m^+ R_{gm}). \end{aligned} \tag{9}$$

To simplify the notations, hereafter we omit insignificant indices. Here the atomic operators are defined as usual [19] in terms of the projections on the atomic states:

$$R_{mg} = |1,m\rangle\langle 0,0|, \quad R_{mm'} = |1,m\rangle\langle 1,m'|.$$

The Hamiltonian (9) describes the creation and absorption of the single cavity-mode photons *at the atom location*. Everywhere in the surrounding space, we have to take into account the spatial dependence of the radiation field described by the vector potential (2) and mode functions (3). In particular, although the atom emits the photon with given polarization (at $kr \to 0$), the polarization can change with the distance from the atom.

A similar model can be constructed in the case of magnetic dipole radiation as well as in the case of high-order atomic multipoles.

3. Polarization of multipole radiation

By definition, the polarization determines the direction of oscillations of the field strengths. Within the classical picture based on the consideration of plane waves, the polarization is defined to be the measure of transversal anisotropy of the electric field strength [18]. In turn, the quantum mechanics interprets the polarization as a given spin state of photons [17]. In the usual approach, the quantitative description of polarization is based either on the Hermitian polarization matrix or on the equivalent set of real Stokes parameters. In the standard case of plane waves, we get the (2×2) polarization matrix and the four Stokes parameters [18], while the description of multipole radiation requires the (3×3) polarization matrix and nine Stokes parameters [20, 21]. Moreover, the electric- and magnetic-type radiation fields should be described in terms of different polarization matrices, taking into account the spatial anisotropy of corresponding field strengths [22].

Here we construct a more general object, describing in a unique way the polarization properties of multipole radiation of either type, both classical and quantum as well as those of the plane waves and other forms of electromagnetic radiation (e.g. of cylindrical waves).

In general, the field is described in terms of the field-strength tensor that can be chosen as follows [23]

$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{10}$$

It seems to be tempting to introduce the general quantitative description of polarization using (10). Since the polarization is specified by the intensities of different spatial components of the radiation field and by the phase differences between the components [18], it should be described in terms of a bilinear form in the field strengths. Assume that the elements in (10) are the positive-frequency parts of the field strengths. Then, the simplest bilinear form defined in terms of (10) is

$$R = F^{+}F, \tag{11}$$

which differs from the energy-momentum tensor by a scalar. In some sense, (11) is similar to the Ricci tensor considered in general relativity [24]. It is easily seen that (11) has the following block structure

$$R = \begin{pmatrix} W_E & \mathbf{S} \\ \mathbf{S}^{+} & P \end{pmatrix},$$

where $W_E = \mathbf{E}^{+} \cdot \mathbf{E}$ is a scalar, $\mathbf{S}$, apart from an unimportant factor, coincides with the Pointing vector, and P is the Hermitian (3×3) matrix of the form

$$P = P_E + P_B. \tag{12}$$

Here

$$P_E = \begin{pmatrix} E_x^+ E_x & E_x^+ E_y & E_x^+ E_z \\ E_y^+ E_x & E_y^+ E_y & E_y^+ E_z \\ E_z^+ E_x & E_z^+ E_y & E_z^+ E_z \end{pmatrix} \tag{13}$$

and

$$P_B = \begin{pmatrix} \mathbf{B}^+ \cdot \mathbf{B} - B_x^+ B_x & -B_y^+ B_x & -B_z^+ B_x \\ -B_x^+ B_y & \mathbf{B}^+ \cdot \mathbf{B} - B_y^+ B_y & -B_z^+ B_y \\ -B_x^+ B_z & -B_y^+ B_z & \mathbf{B}^+ \cdot \mathbf{B} - B_z^+ B_z \end{pmatrix}. \tag{14}$$

We note here that the matrix (13) has been proposed in [20] in order to describe the spatial anisotropy of the electric dipole radiation, while (14) is similar to that discussed in [21, 22] in the case of magnetic dipole radiation.

We choose to interpret (12) as the general polarization matrix, in which the terms (13) and (14) give the electric and magnetic contribution, respectively.

To justify this statement, consider first the case of plane waves propagating in the z direction. In this case, $E_z = B_z = 0$ and $B_x = -E_y$, $B_y = E_x$. Then, the matrix (13) takes the form

$$P_E^{(\text{plane})} = \begin{pmatrix} E_x^+ E_x & E_x^+ E_y & 0 \\ E_y^+ E_x & E_y^+ E_y & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

It is seen that the non-zero submatrix in (15) coincides with the conventional (2×2) polarization matrix of plane waves [18]. In turn, (14) takes the form

$$P_B^{(\text{plane})} = \begin{pmatrix} E_x^+ E_x & E_x^+ E_y & 0 \\ E_y^+ E_x & E_y^+ E_y & 0 \\ 0 & 0 & W_E \end{pmatrix}, \tag{16}$$

where the (2×2) submatrix in the top left corner coincides with that in (15). Thus, the general polarization matrix (12) describes the polarization of plane waves adequately.

Consider now the multipole radiation. In the case of electric-type radiation when $B_z = 0$ everywhere, the matrix (13), in general, contains all elements, while the magnetic polarization matrix (14) in view of (5) is reduced to

$$P_B = \begin{pmatrix} B_y^+ B_y & -B_y^+ B_x & 0 \\ -B_x^+ B_y & B_x^+ B_x & 0 \\ 0 & 0 & \mathbf{B}^+ \cdot \mathbf{B} \end{pmatrix}. \tag{17}$$

Therefore, the spatial anisotropy of the field oscillations is determined by (13), while (17) describes the magnetic field contribution into the transversal anisotropy. Conversely, the spatial anisotropy of magnetic-type radiation is described by (14) with $B_z \neq 0$, while (13) gives the transversal anisotropy of the electric field and

coincides with (15). In this case, (14) coincides, to within the transposition of lines and columns, with the polarization matrix considered in [22].

It is natural that the general polarization matrix (12) reflects the three-dimensional structure of the radiation field. The diagonal terms in (13) and (14) give the radiation intensities. Their angular and radial dependence corresponds to the radiation patterns of the multipole field. The off-diagonal terms give the phase information as in the case of plane waves [18]. In contrast to the standard case of plane waves, there are two independent phase differences $\Delta_{ij} \equiv \arg E_i - \arg E_j$ instead of only one phase difference because

$$\Delta_{xy} + \Delta_{yz} + \Delta_{zx} = 0.$$

Since $\mathbf{E}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) = 0$ at any point, the magnetic part (14) of the general polarization matrix (12) contains the same phase differences as (13).

The polarization matrix (12) can also be expressed in the helicity basis (1). For example, the electric-field contribution (13) takes the form

$$P_E = \begin{pmatrix} E_+^+ E_+ & -E_+^+ E_0 & E_+^+ E_- \\ -E_0^+ E_+ & E_0^+ E_0 & -E_0^+ E_- \\ E_-^+ E_+ & -E_-^+ E_0 & E_-^+ E^- \end{pmatrix}. \tag{18}$$

The quantum counterpart of (12) can be obtained by formal substitution of the operators instead of the classical field strengths (see [25, 26]). Averaging of the corresponding operator matrix over a given state of the radiation field then gives the polarization matrix. By construction, the operator matrices (12)–(18) correspond to the normal ordering in the creation and annihilation operators:

$$P = P(a^+ a).$$

In addition, one can define the anti-normal operator polarization matrix

$$P^{(\mathrm{an})} = P(aa^+)$$

by a simple change of the order of the field strengths in all elements of the matrices (12)–(18) before quantization. It is then clear that the matrix

$$P^{(\mathrm{an})} - P = P([a, a^+]) = \langle 0|P^{(\mathrm{an})}|0\rangle \equiv P^{(\mathrm{vac})} \tag{19}$$

determines the zero-point (vacuum) contribution into the polarization. By construction, the elements of P_{vac} are the position-dependent c-numbers. It is a straightforward matter to show that the vacuum polarization of plane waves of photons is uniform in space, while the multipole vacuum polarization concentrates near the origin [27, 28].

4. Polarization of a single-atom radiation

It was shown in section 2 that an atomic electric dipole transition with given m emits the photon with polarization $\mu = m$. We now examine the spatial properties of polarization. We show that the polarization is not a global property of the multipole field, while changing from point to point.

Consider the single atom electric-type dipole radiation. In view of the equations (5), the operator polarization matrix (13) can be considered as having elements

$$P_{E;\mu\mu'} = (k\gamma_{Ek})^2 \sum_{m,m'=-1}^{1} V^*_{m\mu}(\mathbf{r}) V_{m'\mu'}(\mathbf{r}) a^+_m a_m. \tag{20}$$

As above, we drop here unimportant subscripts. The polarization matrix can be obtained from (20) by averaging over a given state of the radiation field. Assume that the atom emits the electric dipole photon with $m = +1$, i.e. the circularly polarized photon with positive helicity. Consider the polar direction ($\theta = 0$), corresponding to the maximum of the radiation pattern in this case [9]. Then, matrix (20) averaged over the photon state $|1_+\rangle$ takes the form

$$\overline{P}_E = \frac{\hbar\omega}{3\mathcal{V}} \begin{pmatrix} [\frac{1}{2} j_2(kr) - j_0(kr)]^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

Thus, there is only one polarization in the polar direction.

Consider now the variance of (20), describing the quantum noise of polarization. It is easy to show that the averaging over state $|1_m\rangle$ yields

$$\overline{(\Delta P_{E;\mu\mu'})^2} = \overline{P}_{E;\mu\mu'} (k\gamma_{Ek})^2 \sum_{\sigma=-1}^{1} \sum_{m' \neq m} |V_{m'\sigma}|^2. \tag{22}$$

It is seen that the vacuum modes of the cavity field with $m' \neq m$ also contribute to the quantum fluctuations of polarization of the photon state with given m. Taking into account (21) and definition of the mode functions (3) and (4), we can conclude that the polarization in the polar direction (21) does not manifest any quantum noise in the case of radiation in the photon number state $|1_+\rangle$.

In the less probable case of the radiation in the equatorial direction ($\theta = \pi/2$), from (20) we get

$$\overline{P}_E = \frac{\hbar\omega}{3\mathcal{V}} \begin{pmatrix} [\frac{1}{4} j_2(kr) + j_0(kr)]^2 & 0 & -\frac{3}{4} j_2(kr)[\frac{1}{4} j_2(kr) + j_0(kr)] \exp(2i\phi) \\ 0 & 0 & 0 \\ -\frac{3}{4} j_2(kr)[\frac{1}{4} j_2(kr) + j_0(kr)] \exp(-2i\phi) & 0 & \frac{9}{16} [j_2(kr)]^2 \end{pmatrix} \tag{23}$$

so that there are the two circularly polarized components with opposite helicities. Comparing of intensities of these two components shows that the positive helicity dominates at short distances $kr \leqslant 3$, that is at $r \leqslant \lambda/2$, where λ is the wavelength, while both components contribute equally at far distances ($kr \gg 1$) (see figure 2). In view of (21) and (23), one can conclude that any deviation from the polar direction leads to the creation of polarizations additional to $\mu = +1$. Thus, the polarization of radiation under consideration strongly depends on the direction and distance from the source. A similar picture can be obtained for polarization of photons with $m = -1$ and $m = 0$. It is also seen from (23) that the phase difference between the components with different helicity $\Delta_{+1-1} = 2\phi$ is the classical quantity, having a simple geometrical nature. Unlike the case of the polar direction, the

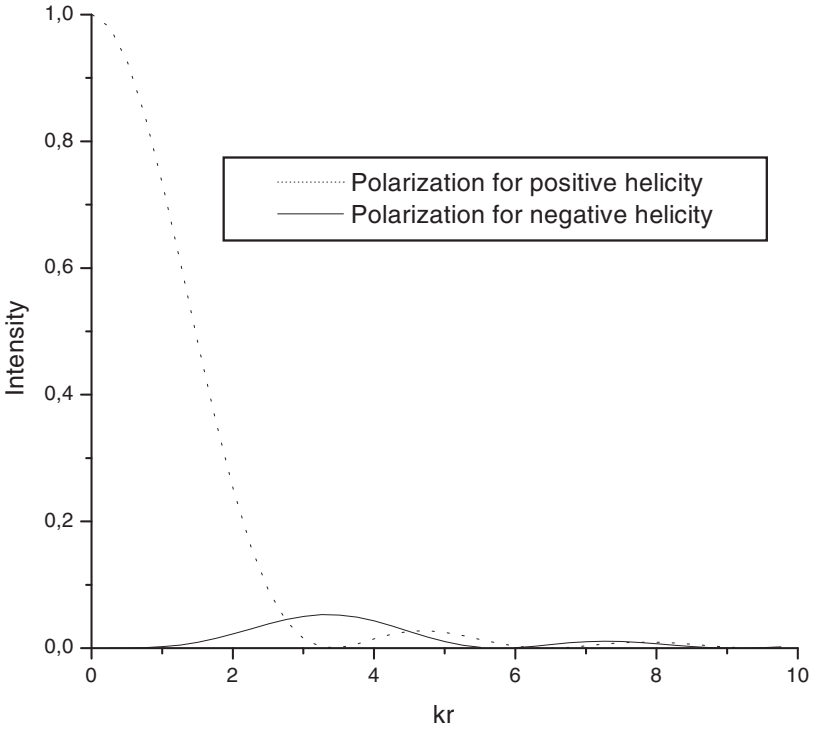

Figure 2. Distance dependence of intensity of the multipole radiation generated by the atomic transition $|j = 1, m = +1\rangle \rightarrow |j' = 0, m' = 0\rangle$ in the equatorial direction.

polarization in the equatorial direction undergoes quantum fluctuations. The quantum noise of polarizations in (23) is shown in figure 3. It is seen that fluctuations of polarization are very strong in spite of the fact that the radiation is in the single photon state.

The above results were obtained for the case of standing waves in an ideal spherical cavity when the radial dependence of the mode functions (3) is specified by equation (4). In particular, the spatial oscillations of polarization in figure 2 are caused by the properties of the spherical Bessel functions. In this case, the radiation field is subjected to the Rabi oscillations that can be described through the use of the steady-state time-dependent wave function of the system with Hamiltonian (9):

$$|\psi(t)\rangle = \exp(-\mathrm{i}Ht)|e_m; 0\rangle = \tfrac{1}{2}\sum_{\ell=\pm 1} \exp(-\mathrm{i}\ell g t)(|e_m; 0\rangle + \ell|g; 1_m\rangle), \qquad (24)$$

where we choose the initial state as the vacuum state of the cavity field and excited state of the atomic sublevel m. Then, the elements of the polarization matrices (21) and (23) should be multiplied by an additional factor of $(1 - \cos 2gt)$, describing the steady-state evolution of polarization.

Consider now the radiation by a single atom in empty space. Then, Hamiltonian (9) should be generalized as follows

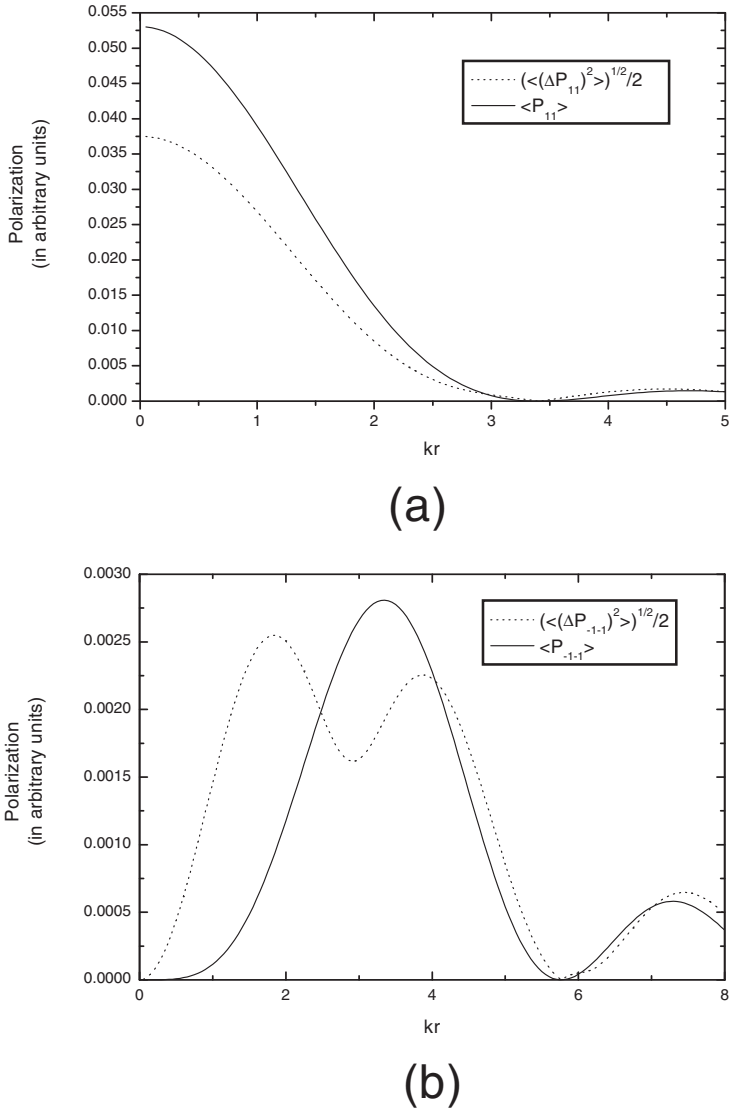

Figure 3. Quantum fluctuations of polarization as a function of distance kr. The solid and dotted lines show the polarizations and corresponding fluctuations for (a) $\mu = +1$ and (b) $\mu = -1$ cases in the equatorial direction.

$$\begin{aligned} \hbar^{-1}H &= H_0 + H_{\text{int}}, \\ H_0 &= \sum_{m=-1}^{1}\left(\sum_k \omega_k a_{km}^{+} a_{km} + \omega_0 R_{mm}\right), \\ H_{\text{int}} &= \sum_{m=-1}^{1}\sum_k g_k (R_{mg} a_{km} + a_{km}^{+} R_{gm}), \end{aligned} \tag{25}$$

to take into account the k dependence of the radiation field. Let us again choose the initial state as the vacuum state of photons and excited atomic state with given m:

$$|\psi_0\rangle = |e_m\rangle \otimes \left[\bigotimes_k |0_k\rangle\right]. \tag{26}$$

It is then clear that the radiation should be represented by the outgoing spherical waves of photons which causes the choice of the radial dependence in (3) in terms of the spherical Hankel functions of the first kind

$$f_\ell(kr) = h_\ell^{(1)}(kr) = j_\ell(kr) + \mathrm{i}(-1)^{\ell+1} j_{-\ell-1}(kr) \tag{27}$$

instead of (4) [29]. This choice assumes that the atom occupies a small but finite spherical volume of radius r_a at the origin to avoid the divergence at $kr \to 0$.

Using the method proposed in [30], we can calculate the elements of the polarization matrix (18) in the equatorial direction as follows

$$\begin{aligned}
\overline{E_+^+ E_+} &= [\Gamma_+(k_0 r)]^2 + [\Gamma_-(k_0 r)]^2, \\
\overline{E_-^+ E_-} &= \tfrac{9}{16}\{[j_2(k_0 r)]^2 + [j_{-3}(k_0 r)]^2\}, \\
\overline{E_+^+ E_-} &= -\tfrac{3}{4}\{[j_2(k_0 r)\Gamma_+(k_0 r) + j_{-3}(k_0 r)\Gamma_-(k_0 r)]^2 \\
&\quad + [j_2(k_0 r)\Gamma_-(k_0 r) - j_{-3}(k_0 r)\Gamma_+(k_0 r)]^2\}^{1/2} \exp(\mathrm{i}\varphi),
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\Gamma_+(k_0 r) &= \tfrac{1}{4} j_2(k_0 r) + j_0(k_0 r), \\
\Gamma_-(k_0 r) &= \tfrac{1}{4} j_{-3}(k_0 r) + j_{-1}(k_0 r).
\end{aligned}$$

All elements containing E_0 are equal to zero. It is seen that, unlike the case for radiation in an ideal cavity, the phase difference between the components with opposite helicities

$$\varphi = 2\phi + \tan^{-1}\left(\frac{\Gamma_- j_2 - \Gamma_+ j_{-3}}{\Gamma_+ j_2 + \Gamma_- j_{-3}}\right),$$

depends on the distance from the source. To take into account the time evolution, we have to multiply the elements of the polarization matrix (28) by the following factor [30]

$$1 + \exp(-\eta t) - 2\exp(-\eta t/2)\cos(\Delta\omega t),$$

where

$$\eta = \frac{\pi}{2}\sum_k g_k^2 \delta(\omega_k - \omega_0), \quad \Delta\omega = \frac{\mathcal{P}}{\omega_0 - \omega_k}$$

and $\mathcal{P}$ denotes the principal value of corresponding potential.

For the polarization matrix in the polar direction, we again get only one non-zero element

$$\overline{E_+^+ E_+} = [\tfrac{1}{2} J_2(k_0 r) - j_0(k_0 r)]^2 + [\tfrac{1}{2} j_{-3}(k_0 r) - j_{-1}(k_0 r)]^2.$$

It is again seen that any deviation from the polar direction leads to the creation of additional polarization.

5. Conclusion

Let us briefly discuss the results obtained. In this paper we have concentrated on the description of spatial properties of polarization of multipole radiation by a single atom. The consideration is based on the sequential use of the representation of multipole photons corresponding to the radiation of a real atomic transition.

To describe the polarization of the multipole field, we propose in section 3 a new definition of the polarization matrix (12) based on the bilinear form in the field-strength tensor. The generalized polarization matrix (12) is additive with respect to the contributions coming from the electric field strength and magnetic induction. The structure of (12) reflects the three-dimensional nature of polarization caused by the three possible states of spin 1 of a photon. In the special case of plane waves, when the third spin state is forbidden, (12) reduces to the conventional (2×2) polarization matrix. In the case of multipole radiation of either type, (12) combines the objects considered earlier in the case of pure E- and M-type multipole radiation [21, 22].

The proposed generalization of the polarization matrix can be used to investigate the quantum polarization properties of the radiation field under different boundary conditions. For example, the case of cylindrical geometry corresponding to waveguides and optical fibres can be examined in the same way as above. Details of this investigation will be discussed in a forthcoming article.

By construction, the generalized polarization matrix (12) is a local object in spite of the global nature of the photon operators of creation and annihilation. The spatial properties of polarization are specified by the mode functions and change with distance and direction from the source. The phase differences between the components with different polarization are the classical quantities. In the case of radiation by an atom in empty space, the phase differences change with distance from the source although, in the case of an ideal cavity, they are specified by the polar angle ϕ only.

The polarization undergoes quantum fluctuations. It is interesting that, in the case of radiation in a cavity emitted by the atomic sublevel with given m, the quantum noise of polarization contains contributions coming from the vacuum fluctuations of the modes with $m' \neq m$. Owing to the specific spatial behaviour of the mode functions, these quantum fluctuations are strong enough in a small vicinity of the atom of the order of $\lambda/3$, where λ denotes the wavelength. Since the quantum noise defines the quantum limit for the precision of the measurements [31], this fact can be important for the polarization measurements in traps where the interatomic distances usually correspond to the intermediate zone [32]. In particular, it can be important for the consideration of the polarization entanglement in the system of two atoms in a cavity similar to that examined in [33].

Let us stress that the precision of a real measurement of polarization should depend on the distance and direction from the source. In fact, any real measurement of intensity assumes the finite aperture of a detecting device [26]. Thus, the spatial variation of the polarization discussed in section 4 should worsen the precision of measurement together with the quantum noise.

Although the above results were obtained in the case of the electric dipole transition, in general, they are valid for an arbitrary multipole transition as well.

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