Classical and Quantum Gravity

Accelerated charge Kerr–Schild metrics in *D* dimensions

To cite this article: Metin Gürses and Özgür Sarioglu 2002 *Class. Quantum Grav.* **19** 4249

INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **19** (2002) 4249–4261 PII: S0264-9381(02)35477-7

Accelerated charge Kerr–Schild metrics in *D* dimensions

Metin Gürses[1] **and Özgür Sarıoğlu**[2]

[1] Department of Mathematics, Faculty of Sciences, Bilkent University, 06533 Ankara, Turkey
[2] Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06531 Ankara, Turkey

Received 15 April 2002
Published 30 July 2002
Online at stacks.iop.org/CQG/19/4249

Abstract

We consider the D-dimensional Einstein–Maxwell theory with a null fluid in Kerr–Schild geometry. We obtain a complete set of differential conditions that are necessary for finding the solutions. We examine the case of vanishing pressure and cosmological constant in detail. For this specific case, we give the metric, the electromagnetic vector potential and the fluid energy density. This is, in fact, the generalization of the well-known Bonnor–Vaidya solution to arbitrary D dimensions. We show that due to the acceleration of charged sources, there is an energy flux in $D \geqslant 4$ dimensions and we give the explicit form of this energy flux formula.

PACS numbers: 0420J, 0240, 4160

1. Introduction

Radiation, and hence energy loss due to the acceleration of an electron is a well-known phenomenon in classical electromagnetism. An exact solution describing this phenomenon in general relativity is given by Bonnor and Vaidya [1], where the metric is in Kerr–Schild form [2–4]. An acceleration parameter has also been considered in Robinson–Trautman metrics [5–7]. The energy-loss formula turns out to be exactly the same as the one obtained from classical electromagnetism. When the acceleration vanishes, Bonnor–Vaidya metrics reduce to the Reissner–Nordström (RN) metric.

In this work, our main motivation is to generalize Bonnor–Vaidya and photon-rocket solutions of $D = 4$ general relativity. For this purpose, we consider the D-dimensional Kerr–Schild metric with an appropriate vector potential and a fluid velocity vector and derive a complete set of conditions for the Einstein–Maxwell theory with a null perfect fluid. We find the expressions for the pressure and the mass density of the fluid. We classify our solutions under some certain assumptions. The field equations are highly complex and with some nontrivial equation of state it is quite difficult to solve the corresponding equations.

To this end, we assume vanishing pressure and a cosmological constant. We give the complete solution for an arbitrary dimension D. This generalizes the Bonnor–Vaidya solution. We obtain the energy flux formula depending on dimension D.

Our conventions are similar to those of Hawking–Ellis [8]. This means that the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$, the Ricci tensor $R_{\alpha\beta}$, the Ricci scalar R_s and the Einstein tensor $G_{\mu\nu}$ are defined by

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\gamma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\nu\gamma}\Gamma^{\gamma}{}_{\beta\mu}, \tag{1}$$

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}, \qquad R_s = R^{\alpha}{}_{\alpha}, \qquad G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R_s. \tag{2}$$

Here $g_{\mu\nu}$ is the D-dimensional metric tensor with signature $(-, +, +, \ldots, +)$. The source of the Einstein equations is composed of the electromagnetic field with the vector potential field A_μ and a perfect fluid with a velocity vector field u^μ, energy density ρ and pressure p. Their energy–momentum tensors are, respectively, given by

$$T^{e}_{\mu\nu} = F_{\mu}{}^{\alpha} F_{\nu\alpha} - \tfrac{1}{4} F^2 g_{\mu\nu}, \qquad F^2 \equiv F^{\alpha\beta} F_{\alpha\beta}, \tag{3}$$

$$T^{f}_{\mu\nu} = (p + \rho) u_\mu u_\nu + p g_{\mu\nu}. \tag{4}$$

Then, the Einstein equations are given by

$$G_{\mu\nu} = \kappa T_{\mu\nu} = \kappa \left(T^{e}_{\mu\nu} + T^{f}_{\mu\nu}\right) + \Lambda g_{\mu\nu}, \tag{5}$$

$$(p + \rho) u^\nu u^\mu{}_{;\nu} = -u^\nu (\rho u^\mu)_{;\nu} + p_{,\nu}(g^{\mu\nu} + u^\mu u^\nu) + F^{\mu}{}_{\nu} J^\nu, \tag{6}$$

$$F^{\mu\nu}{}_{;\nu} = J^\mu. \tag{7}$$

In the following section, we develop the kinematics of a curve C in the D-dimensional Minkowski manifold M_D. We construct solutions of the electromagnetic vector field due to the acceleration of charged particles in four and six dimensions. We then find the energy flux due to acceleration. In section 3, we give a detailed study of Kerr–Schild geometry under certain assumptions. In section 4, we find the solution of the D-dimensional Einstein–Maxwell field equations with a null fluid. We also obtain the generalization of Bonnor–Vaidya metrics to D dimensions. We derive the energy flux formula depending on dimension D and discuss its finiteness in section 5. We state our conclusion in section 6. Finally, in the appendix we give some well-known formulae that are needed in the text.

2. Radiation due to acceleration: Maxwell theory

Let $z^\mu(\tau)$ describe a smooth curve C defined by $z : I \subset \mathbf{R} \to M_D$. Here $\tau \in I$, I is an interval on the real line and M_D is the D-dimensional Minkowski manifold. From an arbitrary point x^μ outside the curve, there are two null lines intersecting the curve C. These points are called the retarded and the advanced times. Let Ω be the distance between points x^μ and $z^\mu(\tau)$, so by definition it is given by

$$\Omega = \eta_{\mu\nu}(x^\mu - z^\mu(\tau))(x^\nu - z^\nu(\tau)). \tag{8}$$

Hence $\Omega = 0$ for two values of τ for a non-spacelike curve. Let us denote these as τ_0 (retarded) and τ_1 (advanced). We shall focus ourselves on the retarded case only. The main reason for this is that Green's function for the vector potential chooses this point on the curve C [9, 10].

If we differentiate Ω with respect to x^μ and let $\tau = \tau_0$, then we get

$$\lambda_\mu \equiv \tau_{,\mu} = \frac{x_\mu - z_\mu(\tau_0)}{R}, \tag{9}$$

where λ_μ is a null vector and R is the retarded distance defined by

$$R \equiv \dot{z}^\mu(\tau_0)(x_\mu - z_\mu(\tau_0)). \tag{10}$$

Here a dot over a letter denotes differentiation with respect to τ_0. We now list some properties of R and λ_μ

$$\lambda_{\mu,\nu} = \frac{1}{R}[\eta_{\mu\nu} - \dot{z}_\mu \lambda_\nu - \dot{z}_\nu \lambda_\mu - (A - \epsilon)\lambda_\mu \lambda_\nu], \tag{11}$$

$$R_{,\mu} = (A - \epsilon)\lambda_\mu + \dot{z}_\mu, \tag{12}$$

where

$$A = \ddot{z}^\mu (x_\mu - z_\mu), \qquad \dot{z}^\mu \dot{z}_\mu = \epsilon = 0, \pm 1. \tag{13}$$

Here $\epsilon = -1$ for timelike curves and vanishes for null curves. Furthermore, we have

$$\lambda_\mu \dot{z}^\mu = 1, \qquad \lambda^\mu R_{,\mu} = 1. \tag{14}$$

Let $a = A/R$. Then it is easy to prove that

$$a_{,\mu}\lambda^\mu = 0. \tag{15}$$

Similar to a, we have other scalars satisfying the same property (15) obeyed by a.

Lemma 1. *Let*

$$a_k = \lambda_\mu \frac{\mathrm{d}^k \ddot{z}^\mu}{\mathrm{d}\tau_0^k}, \qquad k = 1, 2, \ldots, n, \tag{16}$$

then

$$a_{k,\alpha}\lambda^\alpha = 0, \tag{17}$$

for all k. Furthermore, if $A_k = Ra_k$ is constant for a fixed k at all points then $A_i = 0$ for all $i \geqslant k$ and $\mathrm{d}^m \ddot{z}^\mu / \mathrm{d}\tau_0^m = 0$ for all $m \geqslant k$. Here n is an arbitrary positive integer.

Proof. The proof of this lemma depends on the following formula for the derivative of A_k:

$$A_{k,\alpha} = \frac{\mathrm{d}^k \ddot{z}_\alpha}{\mathrm{d}\tau_0^k} + \left[A_{k+1} - \left(\mathrm{d}^k \ddot{z}^\beta / \mathrm{d}\tau_0^k\right) \dot{z}_\beta\right] \lambda_\alpha. \tag{18}$$

In the following, for the sake of simplicity, we shall use τ instead of τ_0. The first part of the lemma can be proved by contracting the above formula (18) by λ^α. One obtains

$$\lambda^\alpha A_{k,\alpha} = a_k. \tag{19}$$

This implies that $\lambda^\alpha a_{k,\alpha} = 0$. For the second part of the lemma, contracting the above formula (18) by $\dot{z}^\alpha$ one obtains

$$A_{k,\alpha}\dot{z}^\alpha = A_{k+1}, \tag{20}$$

which implies that $A_m = 0$ for $m > k$. From (19) we have also $A_k = 0$. With these results (18) reduces to

$$\frac{\mathrm{d}^k \ddot{z}_\alpha}{\mathrm{d}\tau^k} = \mu\lambda_\alpha, \qquad \mu = \left(\frac{\mathrm{d}^k \ddot{z}^\beta}{\mathrm{d}\tau^k}\right) \dot{z}_\beta. \tag{21}$$

Differentiating this equation with respect to x^β and contracting the resulting equation by λ^β we get $\mu = 0$. This completes the proof of the lemma. □

In (16), n depends on the dimension D of the manifold M_D. The scalars (a, a_k), are related to the curvature scalars of the curve C in M_D. The number of such scalars is $D - 1$ [11, 12]. Hence we let $n = D - 1$.

Before moving on to the main subject of this work, i.e. examining the Einstein–Maxwell theory within Kerr–Schild geometry, we briefly study the Maxwell theory in flat (even)

D-dimensional Minkowski space. This should at least serve as a reminder of the well-known basics regarding the usual $D = 4$ Maxwell theory.

By using the above curve C and its kinematics, we can construct divergence-free (Lorentz gauge) vector fields A_α satisfying the wave equation $\eta^{\mu\nu}\partial_\mu\partial_\nu A_\alpha = 0$ outside curve C in any even dimension D. For instance, in the cases $D = 4$ and 6 we have, respectively [13, 14],

$$A_\mu = \begin{cases} \dfrac{e}{4\pi}\dfrac{\dot{z}_\mu}{R}, & (D=4) \\ \dfrac{e}{4\pi^2}\left[\dfrac{\ddot{z}_\mu - a\dot{z}_\mu}{R^2} + \epsilon\dfrac{\dot{z}_\mu}{R^3}\right] & (D=6). \end{cases} \tag{22}$$

These vector fields represent, respectively, the electromagnetic vector potentials of an accelerated charge in four- and six-dimensional Maxwell theory of electromagnetism. The flux of electromagnetic energy is then given by [9]

$$\mathrm{d}E = -\int_S \dot{z}_\mu T^{\mu\nu}\,\mathrm{d}S_\nu, \tag{23}$$

where $T_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}F^2\eta_{\mu\nu}$ is the Maxwell energy–momentum tensor, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field tensor and $F^2 = F^{\alpha\beta}F_{\alpha\beta}$. The surface element $\mathrm{d}S_\mu$ on S is given by

$$\mathrm{d}S_\mu = n_\mu R\,\mathrm{d}\tau\,\mathrm{d}\Omega, \tag{24}$$

where n_ν is orthogonal to the velocity vector field $\dot{z}_\mu$ which is defined through

$$\lambda_\mu = \epsilon\dot{z}_\mu + \epsilon_1\frac{n_\mu}{R}, \qquad n^\mu n_\mu = -\epsilon R^2. \tag{25}$$

Here $\epsilon_1 = \pm 1$. For the remaining part of this section, we shall assume $\epsilon = -1$ (C is a timelike curve). One can consider S in the rest frame as a sphere of radius R. Here $\mathrm{d}\Omega$ is a solid angle. Letting $\mathrm{d}E/\mathrm{d}\tau = N_e$ [1], we have

$$N_e = -\int_S \dot{z}_\mu T^{\mu\nu} n_\nu R\,\mathrm{d}\Omega. \tag{26}$$

At very large values of R we obtain for $D = 4$

$$N_e = \left(\frac{e}{4\pi}\right)^2 \epsilon_1 \int (-\ddot{z}_\mu\ddot{z}^\mu + a^2)\,\mathrm{d}\Omega \tag{27}$$

$$= -\epsilon_1\left(\frac{e}{4\pi}\right)^2 (\ddot{z}^\mu\ddot{z}_\mu)\int (1-\cos^2\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{28}$$

$$= -\epsilon_1\left(\frac{e^2}{4\pi}\right)\frac{2}{3}(\ddot{z}^\mu\ddot{z}_\mu) \tag{29}$$

and for $D = 6$ at very large values of R we have

$$N_e = -\int_S \dot{z}_\mu T^{\mu\nu} n_\nu R^3\,\mathrm{d}\Omega, \tag{30}$$

$$= -\left(\frac{e}{4\pi^2}\right)^2 \epsilon_1 \int \xi_\mu\xi^\mu\,\mathrm{d}\Omega, \tag{31}$$

where

$$\xi^\mu = \frac{\mathrm{d}^3 z^\mu}{\mathrm{d}\tau^3} - 3a\frac{\mathrm{d}^2 z^\mu}{\mathrm{d}\tau^2} + (-a_1 + 3a^2)\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau} \tag{32}$$

so that $\lambda_\mu \xi^\mu = 0$. For a charge e with acceleration $|\ddot{z}_\mu| = \kappa_1$ we have (for $D = 6$)

$$N_e = -\left(\frac{e}{4\pi^2}\right)^2 \frac{32\pi^2\epsilon_1}{15}\left(\dot{\kappa}_1^2 + \kappa_1^2\kappa_2^2 + \frac{9}{7}\kappa_1^4\right). \tag{33}$$

Remark 1. To be compatible with the classical results [9, 10], we take $\epsilon_1 = -1$. We also conjecture that the sign of the energy flux will be the same for all even dimensions.

3. Accelerated Kerr–Schild metrics in *D* dimensions

We now consider the Einstein–Maxwell field equations with a perfect fluid distribution in D dimensions. Here we make some assumptions. First of all, we assume that the metric of the D-dimensional spacetime is the Kerr–Schild metric. Furthermore, we take the null vector λ_μ in the metric to be the same as the null vector defined in (9). With these assumptions the Ricci tensor takes a special form.

Proposition 2. *Let $g_{\mu\nu} = \eta_{\mu\nu} - 2V\lambda_\mu\lambda_\nu$ and λ_μ be the null vector defined in (9) and let V be a differentiable function, then the Ricci tensor and the Ricci scalar are given by*

$$R^\alpha{}_\beta = \zeta_\beta\lambda^\alpha + \zeta^\alpha\lambda_\beta + r\delta^\alpha{}_\beta + q\lambda_\beta\lambda^\alpha, \tag{34}$$

$$R_s = -2\lambda^\alpha K_{,\alpha} - 4\theta K - \frac{2V}{R^2}(D-2)(D-3), \tag{35}$$

where

$$r = \frac{2(-D+3)V}{R^2} - \frac{2K}{R}, \tag{36}$$

$$q = \eta^{\alpha\beta}V_{,\alpha\beta} + \epsilon r + \frac{2A}{R}(K + \theta V) - \frac{4}{R}(\dot{z}^\mu V_{,\mu} + AK - \epsilon K), \tag{37}$$

$$\zeta_\alpha = -K_{,\alpha} - \frac{D-4}{R}V_{,\alpha} + \frac{2V}{R^2}(D-3)\dot{z}_\alpha, \tag{38}$$

where $K \equiv \lambda^\alpha V_{,\alpha}$ and $\theta \equiv \lambda^\alpha{}_{,\alpha} = (D-2)/R$.

Let us further assume that the electromagnetic vector potential A_μ is given by $A_\mu = H\lambda_\mu$, where H is a differentiable function. Let p and ρ be the pressure and the energy density of a perfect fluid distribution with a velocity vector field λ_μ. Then the difference tensor $\mathcal{T}^\mu{}_\nu = G^\mu{}_\nu - \kappa T^\mu{}_\nu$ is given by the following proposition.

Proposition 3. *Let $g_{\mu\nu} = \eta_{\mu\nu} - 2V\lambda_\mu\lambda_\nu$, $A_\mu = H\lambda_\mu$, where λ_μ is given in (9), V and H are differentiable functions. Let p and ρ be the pressure and energy density of a perfect fluid with velocity vector field λ_μ. Then the difference tensor becomes*

$$\mathcal{T}^\alpha{}_\beta = \lambda^\alpha W_\beta + \lambda_\beta W^\alpha + \mathcal{P}\delta^\alpha{}_\beta + \mathcal{Q}\lambda^\alpha\lambda_\beta, \tag{39}$$

where

$$\mathcal{P} = r - \tfrac{1}{2}R_s - \tfrac{1}{2}\kappa(\lambda^\mu H_{,\mu})^2 - (\kappa p + \Lambda), \tag{40}$$

$$\mathcal{Q} = q - \kappa(p + \rho) - \kappa(\eta^{\alpha\beta}H_{,\alpha}H_{,\beta}), \tag{41}$$

$$W_\alpha = \zeta_\alpha + \kappa(\lambda^\mu H_{,\mu})H_{,\alpha}, \tag{42}$$

where ζ_α, r, R_s and q are given in (38), (36), (35) and (37), respectively.

The vanishing of the difference tensor $\mathcal{T}^{\alpha}{}_{\beta}$ in proposition 3 implies that $\mathcal{P} = 0$ and $W_\alpha = -\frac{1}{2}\mathcal{Q}\lambda_\alpha$. Hence the following corollary gives a set of equations that are equivalent to the Einstein equations under the assumptions of proposition 3.

Corollary 4. *With the assumptions of proposition 3, the Einstein equations (5) imply that*

$$\kappa p + \Lambda = \frac{1}{2}\kappa(\lambda^\alpha H_{,\alpha})^2 + \frac{D-2}{R}K + \frac{(D-2)(D-3)}{R^2}V, \tag{43}$$

$$\kappa(p+\rho) = q - \kappa\eta^{\alpha\beta}H_{,\alpha}H_{,\beta} + 2w, \tag{44}$$

$$W_\alpha = w\lambda_\alpha, \tag{45}$$

where $w = W^\alpha \dot{z}_\alpha$ and Λ is the cosmological constant. Here q and W_α are respectively given in (37) and (42).

We shall now assume that the functions V and H depend on R and on some R-independent functions c_i $(i = 1, 2, \ldots)$ such that

$$c_{i,\alpha}\lambda^\alpha = 0, \tag{46}$$

for all i. It is clear that due to the properties (15) and (17) of a_k, all of these functions (c_i) are functions of the scalars a and a_k $(k = 1, 2, \ldots, D-1)$.

Note that there are in fact $D+2$ equations contained in (43)–(45). In particular, using the vector equation (45) we can produce other scalar equations by contracting it with the vectors λ^α and $\dot{z}^\alpha$. We summarize the results in the following proposition.

Proposition 5. *Let V and H depend on R and functions c_i $(i = 1, 2, \ldots)$ that satisfy (46), then the Einstein equations given in proposition 4 reduce to the following set of equations:*

$$\kappa p + \Lambda = \frac{1}{2}V'' + \frac{3D-8}{2R}V' + \frac{(D-3)^2}{R^2}V, \tag{47}$$

$$\kappa(H')^2 = V'' + \frac{D-4}{R}V' - \frac{2V}{R^2}(D-3), \tag{48}$$

$$\kappa(p+\rho) = q - \kappa\eta^{\alpha\beta}H_{,\alpha}H_{,\beta} + 2\left[\frac{2(A-\epsilon)(D-3)V}{R^2} - \sum_{i=1}(w_i c_{i,\alpha}\dot{z}^\alpha)\right], \tag{49}$$

$$\sum_{i=1} w_i c_{i,\alpha} = \left[\sum_{i=1}(w_i c_{i,\beta}\dot{z}^\beta)\right]\lambda_\alpha, \tag{50}$$

where

$$w_i = V'_{,c_i} + \frac{D-4}{R}V_{,c_i} - \kappa H' H_{,c_i}, \tag{51}$$

and the prime over a letter denotes partial differentiation with respect to R. Equation (6) is satisfied identically with the electromagnetic current vector

$$J_\mu = \sum_{i=1}\left[\frac{1}{R}(4-D)H_{,c_i} - (H')_{,c_i}\right]c_{i,\mu} + \left[-H'' + \frac{1}{R}(2-D)H'\right]\dot{z}_\mu$$
$$+ \left[\sum_{i=1}\left(2(H')_{,c_i}(c_{i,\alpha}\dot{z}^\alpha) + H_{,c_i}\left(c_{i,\alpha}{}^{,\alpha}\right) - \frac{2}{R}H_{,c_i}(c_{i,\alpha}\dot{z}^\alpha)\right) + AH''\right]\lambda_\mu. \tag{52}$$

Note that (48) is obtained by contracting (45) with the vector λ^α. The above equations can be described as follows. Equations (47) and (49) define the pressure and the mass density of the perfect fluid distribution with null velocity λ_μ. Equation (48) gives a relation between the electromagnetic and gravitational potentials H and V. This relation is quite simple, given one of them one can easily solve the other. Equation (50) implies that there are some functions c_i $(i = 1, 2, \ldots)$ where this equation is satisfied. The functions c_i $(i = 1, 2, \ldots)$ arise as integration constants (with respect to the variable R) while determining the R dependence of the functions V and H. Assuming the existence of such c_i the above equations give the most general solution of the Einstein–Maxwell field equations with a null perfect fluid distribution under the assumptions of proposition 3.

4. Null-dust solutions in D dimensions

In this section, we give a class of new exact solutions in the Kerr–Schild geometry. Assuming that the null fluid has no pressure, the cosmological constant vanishes and using proposition 5, we have the following result.

Theorem 6. *Let $p = \Lambda = 0$. Then*

$$V = \begin{cases} \dfrac{\kappa e^2(D-3)}{2(D-2)} R^{-2D+6} + mR^{-D+3} & (D \geqslant 4) \\ -\dfrac{\kappa}{2} e^2 \ln R + m & (D = 3) \end{cases} \tag{53}$$

$$H = \begin{cases} c + \epsilon e R^{-D+3} & (D \geqslant 4) \\ c + \epsilon e \ln R & (D = 3), \end{cases} \tag{54}$$

where for $D > 3$

$$\begin{aligned} \rho = {} & -(c_{,\alpha} c^{,\alpha}) - \epsilon(3-D)eR^{3-D}\left(c^{,\alpha}{}_{,\alpha}\right) - \epsilon 2(3-D)(2-D)e(c_{,\alpha}\dot{z}^\alpha)R^{2-D} \\ & + \frac{1}{\kappa} a(2-D)(1-D)MR^{2-D} + (2-D)(3-D)ae^2R^{5-2D} \\ & - \epsilon(2-D)(1-D)(3-D)aceR^{2-D} + \frac{1}{\kappa}\dot{M}(2-D)R^{2-D} \\ & - \epsilon(3-D)(2-D)c\dot{e}R^{2-D} + (3-D)e\dot{e}R^{5-2D}, \end{aligned} \tag{55}$$

$$J_\mu = \frac{1}{R}(4-D)c_{,\mu} + \left[c^{,\alpha}{}_{,\alpha} - \frac{2}{R}(c_{,\alpha}\dot{z}^\alpha) + \epsilon(3-D)\dot{e}R^{2-D} + \epsilon(3-D)(2-D)eaR^{2-D}\right]\lambda_\mu \tag{56}$$

and for $D = 3$

$$\begin{aligned} \rho = {} & -(c_{,\alpha}c^{,\alpha}) - \epsilon e(c_{,\alpha}{}^{,\alpha}) + \epsilon\frac{2e}{R}(c_{,\alpha}\dot{z}^\alpha) - \frac{ae^2}{2R} + \frac{2Ma}{\kappa R} - \frac{ae^2}{R}\ln R - \frac{\dot{M}}{\kappa R} + \frac{e\dot{e}}{R} \\ & + \epsilon\frac{c\dot{e}}{R} + \frac{e\dot{e}}{R}\ln R - 2\epsilon ec\frac{a}{R}, \end{aligned} \tag{57}$$

$$J_\mu = \frac{1}{R}c_{,\mu} + \left[c^{,\alpha}{}_{,\alpha} - \frac{2}{R}(c_{,\alpha}\dot{z}^\alpha) + \frac{\epsilon}{R}\dot{e} - \epsilon e\frac{a}{R}\right]\lambda_\mu. \tag{58}$$

Here $M = m + \epsilon\kappa(3-D)ec$ for $D \geqslant 4$ and $M = m + \frac{\kappa}{2}e^2 + \epsilon\kappa ec$ for $D = 3$. In all cases, e is assumed to be a function of τ only but the functions m and c which are related through the arbitrary function $M(\tau)$ (depends on τ only) depend on the scalars a and a_k $(k \geqslant 1)$.

Equation (47) with zero pressure and (48) determine the R dependence of the potentials V and H completely. Using proposition 5 we have chosen the integration constants (R-independent functions) as the functions c_i $(i = 1, 2, 3)$ so that $c_1 = m, c_2 = e, c_3 = c$ and

$$c = c(\tau, a, a_k), \qquad e = e(\tau),$$

$$m = \begin{cases} M(\tau) + \epsilon\kappa(D-3)ec & (D \geqslant 4) \\ M(\tau) - \dfrac{\kappa}{2}e^2 - \epsilon\kappa ec & (D = 3), \end{cases}$$

where a_k are defined in lemma 1.

Remark 2. The curve C is a geodesic of the curved geometry with the metric $g_{\mu\nu} = \eta_{\mu\nu} - 2V\lambda_\mu\lambda_\nu$ if and only if it is a straight line in M_D.

Remark 3. When $D = 4$, we obtain the Bonnor–Vaidya solutions with one essential difference. In the Bonnor–Vaidya solutions the parameters m and c (which are related through (4)) depend on τ and a only. In our solution, these parameters depend not only on τ and a but also on all other scalars a_k $(k \geqslant 1)$.

Remark 4. The electromagnetic vector potentials in flat space (see section 2) and in curved space (see section 3) are different. It is known that in four dimensions they are gauge equivalent [1]. It is this equivalence that led Bonnor and Vaidya to choose the function $c(\tau, a, a_k) = -ea$. All other choices of c do not have flat space limits. In higher dimensions $(D > 4)$ such an equivalence cannot be established. As an example, let

$$A_f^\mu = \frac{\ddot{z}^\mu - a\dot{z}^\mu}{R^2} + \epsilon\frac{\dot{z}^\mu}{R^3}$$

be the vector potential (22) in six-dimensional flat space. Let $A_\mu = H\lambda_\mu = \Phi_{,\mu} + A_{f\mu}$, where Φ is the gauge potential to be determined. Here A_μ is the vector potential (54) in six-dimensional curved spacetime. Except for the static case, it is a simple calculation to show that no Φ exists to establish such a gauge equivalence. Hence our higher-dimensional solutions have no flat space counterparts. In the static case (the curve C is a straight line) our solutions are, for all D, gauge equivalent to the Tangherlini [15] solutions (see also [16]). Because of this reason the energy flux expressions in six dimensions (see the following section) in flat and curved spacetimes are different. To obtain gauge equivalent solutions we conjecture that the Kerr–Schild ansatz has to be abandoned.

Remark 5.

(a) It is easy to prove that $\rho = 0$ only when the curve C is a straight line in M_D (static case). This means that there are no accelerated vacuum and electro-vacuum solutions.

(b) We can have pure fluid solutions when $e = c = 0$. In this case, we have

$$V = mR^{3-D}, \qquad \rho = \frac{2-D}{\kappa}[a(1-D)M + \dot{M}]R^{2-D}, \tag{59}$$

for $D \geqslant 4$ and

$$V = m, \qquad \rho = \frac{2Ma - \dot{M}}{\kappa R}, \tag{60}$$

for $D = 3$. Such solutions are usually called *photon rocket* solutions [17–20]. Here we give the D-dimensional generalizations of this type of metric as well.

5. Radiation due to acceleration

In this section, we give a detailed analysis of the energy flux due to the acceleration of charged sources in the case of the solution given in theorem 6. For the $D > 3$ case, the solution described by the functions c, e and M give the energy density given in (55). Remember that at this point, $c = c(\tau, a, a_k)$ and arbitrary. Choosing $e =$ constant, $c = -ea$ as was done by Bonnor and Vaidya [1], the expression for ρ simplifies and one obtains

$$\rho = \frac{1}{\kappa}(2-D)\dot{M}R^{2-D} + \frac{a}{\kappa}(2-D)(1-D)MR^{2-D} - \frac{e^2}{R^2}(\dddot{z}^{\alpha}\ddot{z}_{\alpha} + \epsilon a^2) + (3-D)(2-D)e^2\left[\epsilon a_1 R^{2-D} + aR^{5-2D} - aR^{1-D} - \epsilon D a^2 R^{2-D}\right]. \tag{61}$$

The flux of null fluid energy is then given by

$$N_f = -\int_{S^{D-2}} T_f{}^{\alpha}{}_{\beta}\dot{z}_{\alpha}n^{\beta}R^{D-3}\,\mathrm{d}\Omega \tag{62}$$

and since $T_f{}^{\alpha}{}_{\beta} = \rho\lambda^{\alpha}\lambda_{\beta}$ for the special case $p = \Lambda = 0$ that we are examining, one finds that

$$N_f = \epsilon\epsilon_1 \int_{S^{D-2}} \rho R^{D-2}\,\mathrm{d}\Omega, \tag{63}$$

where ρ is given in (61). The flux of electromagnetic energy is similarly given by

$$N_e = -\int_{S^{D-2}} T_e{}^{\alpha}{}_{\beta}\dot{z}_{\alpha}n^{\beta}R^{D-3}\,\mathrm{d}\Omega \tag{64}$$

and for the solution we are examining, one finds that

$$-T_e{}^{\alpha}{}_{\beta}\dot{z}_{\alpha}n^{\beta} = \epsilon_1\epsilon(3-D)e[(3-D)e(A-\epsilon)R^{5-2D} + \dot{e}R^{6-2D} + \epsilon(c_{,\alpha}\dot{z}^{\alpha})R^{3-D}] + \epsilon\epsilon_1 R(c_{,\alpha}c^{,\alpha}) + (3-D)^2e^2(R_{,\alpha}n^{\alpha})R^{4-2D} + (3-D)eR^{5-2D}\dot{e}(\lambda_{\alpha}n^{\alpha}) + \epsilon(3-D)eR^{2-D}(c_{,\alpha}n^{\alpha}) \tag{65}$$

for the case $c = c(\tau, a, a_k)$ and arbitrary. Taking the special case $e =$ constant, $c = -ea$ of Bonnor–Vaidya [1], this simplifies and hence (64) becomes

$$N_e = \epsilon_1 e^2 \int_{S^{D-2}} \mathrm{d}\Omega\,[a^2 + \epsilon(\dddot{z}^{\alpha}\ddot{z}_{\alpha})]R^{D-4}. \tag{66}$$

The total energy flux is given by

$$N = N_e + N_f = \epsilon_1 \int_{S^{D-2}} \left[\frac{(2-D)\epsilon}{\kappa}\dot{M} + \frac{\epsilon}{\kappa}a(2-D)(1-D)M + (3-D)(2-D)e^2 a_1 - D(2-D)(3-D)e^2a^2\right] \tag{67}$$

for large enough R. For a charge with acceleration $|\ddot{z}_{\alpha}| = \kappa_1$, we have (see the appendix)

$$N = \frac{1}{2}(2-D)\left\{\dot{M}\epsilon + 2(3-D)e^2(\kappa_1)^2\Omega_{D-2} - D(3-D)e^2(\kappa_1)^2\Omega_{D-3}\Gamma\left(\frac{D-2}{2}\right)\gamma_D\right\}\epsilon_1, \tag{68}$$

where

$$\gamma_D = \frac{-2\sqrt{\pi}}{\Gamma((D-1)/2)} + 2^D\frac{\Gamma((D+2)/2)}{\Gamma(D)}.$$

Note that when $D = 4$, this reduces to the result of Bonnor–Vaidya in [1]. We also give the $D = 6$ case as another example.

$$N = \begin{cases} \epsilon_1 \left[-\epsilon \dot{M} - \frac{8\pi}{3} e^2 (\kappa_1)^2 \right] & (D = 4) \\ \epsilon_1 \left[-2\epsilon \dot{M} - \frac{96\pi^2}{15} e^2 (\kappa_1)^2 \right] & (D = 6). \end{cases} \tag{69}$$

Remark 6. To be consistent with Bonnor and Vaidya, we take $\epsilon_1 = -1$ which is also consistent with our remark 1. Electromagnetic energy flux (66) is finite only when $D = 4$, but the total energy flux (68) is finite, for all D (due to the cancellation of divergent terms in N_e and N_f).

Remark 7. The difference between the energy flux expressions in the $D = 4$ flat space (29) with that obtained above (69) is due to the scaling factor $(1/4\pi)$. Apart from this difference, the energy flux expressions obtained from classical electromagnetism and general relativity are exactly the same for the special case $c = -ea$ and $M =$ constant. For other choices we obtain different expressions for the energy flux. For $D > 4$, since we do not have gauge equivalent solutions (see remark 4), the energy flux expressions obtained from classical electromagnetism and general relativity are different.

For the $D = 3$ case, substituting the special choice $e =$ constant, $c = -ea$ of [1], the expression for ρ given in (57) reduces to

$$\rho = -\frac{\dot{M}}{\kappa R} + \frac{2Ma}{\kappa R} - \frac{e^2}{R^2}(\ddot{z}^{\alpha}\ddot{z}_{\alpha}) - \epsilon\frac{e^2 a_1}{R} - \epsilon\frac{e^2 a^2}{R^2} - \frac{e^2 a}{R}\ln R + \frac{e^2 a}{R}\left[3\epsilon a + \frac{1}{R} - \frac{1}{2}\right]. \tag{70}$$

Similarly, the energy flux due to the fluid is found as

$$N_f = \epsilon\epsilon_1 \int_0^{2\pi} \rho R \,\mathrm{d}\theta,$$

where ρ is given in (70). For the energy flux due to the electromagnetic field, one finds

$$\begin{aligned} -T_e{}^{\alpha}{}_{\beta}\dot{z}_{\alpha}n^{\beta} &= \epsilon_1\epsilon\frac{e^2}{R}(A - \epsilon) + \epsilon\epsilon_1 e\dot{e}\ln R + \epsilon_1 e(c_{,\alpha}\dot{z}^{\alpha}) + \epsilon\epsilon_1 R(c_{,\alpha}c^{,\alpha}) \\ &\quad + \frac{e^2}{R^2}(R_{,\alpha}n^{\alpha}) + \frac{e\dot{e}}{R}\ln R(\lambda_{\alpha}n^{\alpha}) + \epsilon\frac{e}{R}(c_{,\alpha}n^{\alpha}). \end{aligned} \tag{71}$$

Here $c = c(\tau, a, a_k)$ is an arbitrary function of its arguments. For the special case $e =$ constant, $c = -ea$, energy flux due to the electromagnetic field becomes

$$N_e = \frac{e^2}{R}\epsilon_1 \int_0^{2\pi} \mathrm{d}\theta \,[a^2 + \epsilon(\ddot{z}^{\alpha}\ddot{z}_{\alpha})].$$

The total energy loss is given by

$$N = N_e + N_f = \epsilon_1 \int_0^{2\pi} \mathrm{d}\theta \left[-\frac{\epsilon\dot{M}}{\kappa} + \frac{2\epsilon Ma}{\kappa} - e^2 a_1 - \epsilon a e^2 \ln R + 3e^2 a^2 + \frac{\epsilon}{R}e^2 a - \frac{\epsilon}{2}e^2 a\right]. \tag{72}$$

Assuming that the curve C is timelike ($\epsilon = -1$), performing the angular integration in (72), and then taking R large enough we get

$$N = \left[\tfrac{1}{2}\dot{M} + \pi e^2 \kappa_1^2\right]\epsilon_1.$$

Remark 8. The sign of the energy flux expression in three dimensions is opposite to the one obtained in other dimensions ($D > 3$).

6. Conclusion

We have found exact solutions for the D-dimensional Einstein–Maxwell field equations with a null perfect fluid source. Physically, these solutions describe the electromagnetic and gravitational fields of a charged particle moving on an arbitrary curve C in a D-dimensional manifold. The metric and the electromagnetic vector potential arbitrarily depend on a scalar, $c(\tau_0, a, a_k)$ which can be related to the curvatures of the curve C. In four dimensions with a special choice of this scalar, our solution matches with the Bonnor–Vaidya metric [1]. For other choices we have different solutions. In higher dimensions, our solutions given in theorem 6, for all D, can be considered as the accelerated Tangherlini [15] solutions.

We have also studied the flux of electromagnetic energy due to the acceleration of charged particles. We observed that the energy flux formula, for all dimensions, depends on the choice of the scalar c in terms of the functions a, a_k (or the curvature scalars of the curve C). In dimensions $D > 4$, electromagnetic and fluid energy fluxes diverge for large values of R but the total energy flux is finite. We obtained the energy flux expression corresponding to a special choice, $c = -ea$, for all dimensions.

Acknowledgment

We would like to thank the referees for their constructive criticisms. This work is partially supported by the Scientific and Technical Research Council of Turkey and the Turkish Academy of Sciences.

Appendix. Serret–Frenet frames

In this appendix, we shall give the Serret–Frenet frame in four dimensions which can be easily extended to any arbitrary dimension D. The curve C described in section 2 has the tangent vector $T^\mu = \dot{z}^\mu$. Starting from this tangent vector, by repeated differentiation with respect to the arclength parameter τ_0, one can generate an orthonormal frame $\{T^\mu, N_1^\mu, N_2^\mu, N_3^\mu\}$, the *Serret–Frenet frame*:

$$\dot{T}^\mu = \kappa_1 N_1^\mu, \tag{A1}$$

$$\dot{N}_1^\mu = \kappa_1 T^\mu - \kappa_2 N_2^\mu, \tag{A2}$$

$$\dot{N}_2^\mu = \kappa_2 N_1^\mu - \kappa_3 N_3^\mu, \tag{A3}$$

$$\dot{N}_3^\mu = \kappa_3 N_2^\mu. \tag{A4}$$

Here κ_i $(i = 1, 2, 3)$ are the curvatures of the curve C at the point $z^\mu(\tau_0)$. The normal vectors N_i $(i = 1, 2, 3)$ are spacelike unit vectors. Hence at the point $z^\mu(\tau_0)$ on the curve we have an orthonormal frame which can be used as a basis of the tangent space at this point. In section 2, we also defined some scalars

$$a_k = \frac{\mathrm{d}^k \ddot{z}_\mu}{\mathrm{d}\tau_0^k} \lambda^\mu,$$

where $\lambda_\mu = \epsilon T^\mu + \epsilon_1 (n^\mu / R)$. Here n^μ is a spacelike vector orthogonal to T^μ. Hence we let

$$n^\mu = \alpha N_1^\mu + \beta N_2^\mu + \gamma N_3^\mu$$

with $\alpha^2 + \beta^2 + \gamma^2 = R^2$. One can choose the spherical angles $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$ such that

$$\alpha = R\cos\theta, \qquad \beta = R\sin\theta\cos\phi \qquad \gamma = R\sin\theta\sin\phi.$$

Hence we can calculate the scalars a_k in terms of the curvatures and the spherical coordinates (θ, ϕ) at the point $z^{\mu}(\tau_0)$. These expressions are quite useful in the energy flux formulae. As an example we give a and a_1:

$$a = -\epsilon\epsilon_1\kappa_1 \cos\theta, \qquad a_1 = (\kappa_1)^2 - \epsilon\epsilon_1\dot{\kappa}_1 \cos\theta + \epsilon\epsilon_1\kappa_1\kappa_2 \sin\theta\cos\phi. \tag{A5}$$

All other a_k can be found similarly. Hence for all k, these scalars depend on the curvatures and the spherical angles.

In four dimensions, the spacelike vector n is given by (we shall omit the indices on the vectors)

$$n = R[N_1 \cos\theta + N_2 \sin\theta\cos\phi + N_3 \sin\theta\sin\phi],$$

where $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$. The line element on S^2 is

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2.$$

The solid angle integral is

$$\int_{S^2} \mathrm{d}\Omega = 4\pi.$$

In six dimensions, we have

$$\begin{aligned} n = R[&N_1 \cos\theta + N_2 \sin\theta\cos\phi_1 + N_3 \sin\theta\sin\phi_1\cos\phi_2 + N_4 \sin\theta\sin\phi_1\sin\phi_2\cos\phi_3 \\ &+ N_5 \sin\theta\sin\phi_1\sin\phi_2\sin\phi_3], \end{aligned} \tag{A6}$$

where $\theta, \phi_1, \phi_2 \in (0, \pi)$ and $\phi_3 \in (0, 2\pi)$ and

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi_1^2 + \sin^2\theta\sin^2\phi_1\,\mathrm{d}\phi_2^2 + \sin^2\theta\sin^2\phi_1\sin^2\phi_2\,\mathrm{d}\phi_3^2.$$

The solid angle integral is

$$\int_{S^4} \mathrm{d}\Omega = \frac{8\pi^2}{3}.$$

The solid angle integral in D dimensions is

$$\Omega_{D-2} \equiv \int_{S^{D-2}} \mathrm{d}\Omega = \frac{(D-1)\pi^{(D-1)/2}}{\Gamma((D+1)/2)}.$$

References

[1] Bonnor W B and Vaidya P C 1972 *General Relativity* ed L O'Raifeartaigh (Dublin: Dublin Institute for Advanced Studies) p 119 (papers in honor of J L Synge)

[2] Kerr R and Schild A 1965 Applications of nonlinear partial differential equations in mathematical physics *Proc. Symp. Applied Mathematics* (Providence, RI: American Mathematical Society) vol 17, p 199

[3] Gürses M and Gürsey F 1975 *J. Math. Phys.* **16** 2385

[4] Kramer D, Stephani H, Mac Callum M A H and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)

[5] Robinson I and Trautman A 1962 *Proc. R. Soc.* A **265** 463

[6] Newman E T 1974 *J. Math. Phys.* **15** 44

[7] Newman E T and Unti T W J 1963 *J. Math. Phys.* **12** 1467

[8] Hawking S W and Ellis G F R 1977 *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press)

[9] Barut A O 1980 *Electrodynamics and Classical Theory of Fields and Particles* (New York: Dover)

[10] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)

[11] Spivak M 1979 *A Comprehensive Introduction to Differential Geometry* (Boston, MA: Publish or Perish)

[12] Arrega G, Capovilla R and Güven J 2001 *Class. Quantum Grav.* **18** 5065

[13] Gürses M and Sarıoğlu Ö Radiation due to acceleration in *D*-dimensional Kerr–Schild geometry (in preparation)

[14] Gürses M and Sarıoğlu Ö Accelerated Born–Infeld metrics in Kerr–Schild geometry (submitted)

[15] Tangherlini F R 1963 *Nuovo Cimento* **77** 636
[16] Myers R C and Perry M J 1986 *Ann. Phys., NY* **172** 304
[17] Kinnersley W 1969 *Phys. Rev.* **186** 1353
[18] Bonnor W B 1994 *Class. Quantum Grav.* **11** 2007
[19] Damour T 1995 *Class. Quantum Grav.* **12** 725
[20] Dain S, Moreschi O M and Gleiser R J 2002 Photon rockets and the Robinson–Trautman geometries *Preprint* gr-qc/0203064