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Physica Scripta. Vol. 66, 345–353, 2002

Nonlocal, Non-Commutative Picture in Quantum Mechanics and Distinguished Continuous Canonical Maps

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Received February 2, 2002; accepted in revised form June 14, 2002

PACS Ref: 03.65.-w, 03.65.Sq, 03.65.Fd

Abstract

It is shown that continuous classical nonlinear canonical (Poisson) maps have a distinguished role in quantum mechanics. They act unitarily on the quantum phase space and generate $\hbar$-independent quantum nonlinear canonical maps. It is also shown that such maps act in the non-commutative phase space under the classical covariance. A crucial result of the work is that under the action of Poisson maps a local quantum mechanical picture is converted onto a non-local picture which is then represented in a non-local Hilbert space. On the other hand, it is known that a non-local picture is equivalent by the Weyl map to a non-commutative picture which, in the context of this work, corresponds to a phase space formulation of the theory. As a result of this equivalence, a phase space Schrödinger picture can be formulated. In particular, we obtain the $\star$-genvalue equation of Fairlie [Proc. Camb. Phil. Soc., **60**, 581 (1964)] and Curtright, Fairlie and Zachos [Phys. Rev., **D 58**, 025002 (1998)]. In a non-local picture entanglement becomes a crucial concept. The connection between the entanglement and non-locality is explored in the context of Poisson maps and specific examples of the generation of entanglement from a local wavefunction are provided by using the concept of generalized Bell states. The results obtained are also relevant for the non-commutative soliton picture in the non-commutative field theories. We elaborate on this in the context of the scalar non-commutative field theory.

1. Introduction

Recently increased activity in non-commutative (field and gauge) theories [1] and the natural observation of the non-commutative spatial coordinates in string physics [2] arose a flurry of interest in a non-standard extension of the quantum mechanics within this non-commutative picture [3]. In these theories the fundamental entity, the field, acquires an operator character through its dependence on the non-commutative coordinates (NCC). The NCC can arise in certain physical limits of a quantum theory. Recently, Bigatti and Susskind [4] have suggested a quantum model of a charged particle in the plane interacting with a perpendicular magnetic field in the limit that the field strength goes to infinity in order to observe non-commutativity in the plane coordinates. Considering that the non-commutative gauge theories are related to the ordinary ones by certain transformations [2] it is reasonable to inquire whether more general results can be obtained on the nature of the interrelationship between the non-commutative and the ordinary formalisms. For instance, in the case of gauge theories the *ordinary* and the non-commutative theories are related by a *gauge equivalence preserving* map reminiscent of a contact transformation between the fields and the gauge parameters and vice versa [2]. On the other hand, the *phase space* is spanned by the generalized coordinates and the formulation of quantum mechanics in the phase space has explicit features of non-commutativity. This fact can be used as a natural ground to understand whether general transformations can be found establishing a link between the ordinary and the non-commutative pictures. In this work we primarily address this specific issue where we demonstrate that such invertible transformations between the two pictures can be established within the context of phase space continuous nonlinear canonical transformations. In our context here the non-commutative generalized coordinates $\hat{z} = (\hat{z}_1, \hat{z}_2)$ are the standard *generalized momentum* and *position* operators respectively satisfying $[\hat{z}_j, \hat{z}_k] = i\theta_{jk}$ where $j, k = (1, 2)$ and the non commutativity parameter is $\theta_{jk} = \hbar J_{jk}$ with J_{jk} describing the symplectic matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The equivalence of the phase space in N degrees of freedom to the $2N$ dimensional non-commutative geometry approach is already known. For instance, for the Landau model with $N = 1$ this equivalence was shown in Ref. [5]. More recently it was proposed that there is an interesting connection between the quantum Hall effect and the non-commutative matrix models [6,7]. The non-commutative picture can be converted one-to-one to a bilocal coordinate picture by the Weyl map. In this bilocal representation the (non)separability becomes a crucial property. When the BLC dependence is separable the corresponding non-commutative picture reduces to the standard local (Schrödinger) picture in the Hilbert space. On the other hand, in the case of non-separability of the BLC dependence full phase space formalism is required (which may not be reducible to a pure coordinate or momentum representation as in the standard local Schrödinger picture) where the Hilbert space representations become non-local. This unified treatment of the non-commutative phase space and the corresponding non-local function space suggest an extension of the standard formulation of quantum mechanics. The second purpose of this work, in close relation with the primary goal indicated above, is to explore this extended quantum mechanical picture.

In Section 2 the basic connections between the BLC and the phase space representations are established by the Weyl map and the question of the separability versus non-separability in the BLC picture is studied. It is shown that the separable bilocal solutions are actually local and they can be fitted in the standard Schrödinger picture. The nonlocal solutions arise from the non-separable bilocal coordinate dependence and they fit in the extended picture. In the context of our primary goal we show in Section 3 that, the separable and the non-separable function spaces are disjoint under the action of the standard linear Hilbert space operators. By reaching beyond the standard Hilbert space operator methods, we find in Section 4 distinguished unitary (with respect to an inner product in the phase space) isomorphisms joining the separable and the non-separable sectors. In the view of the continuous canonical transformations, these isomorphisms are established by the continuous classical canonical (Poisson) maps. As the second goal mentioned above, it is shown that the classical Poisson transformations have unitary representations in the quantum phase space and they fit in an extended quantum mechanical phase space picture.

In any non-local picture the entanglement is a crucial concept. In Section 5 a connection between the non locality and the entanglement is examined and (non-local) *generalized Bell states* are introduced. Specific examples of the nonlinear Poisson maps are given in Section 5 generating these generalized Bell states from the local states. Recently generalized Bell states have been observed in vacuum soliton solutions of non-commutative field theories. The concept of generalized Bell states may be required in building a genuine equivalence scheme for the set of all non-commutative soliton solutions in these field theories. Section 6 is a schematical illustration and summary of the results in the work.

2. The extended quantum mechanical picture

The ordinary quantum mechanics is the sector of the field theory supporting a few degrees of freedom. We adopt an operator-like approach in this description [8,9] which will be referred to as the *waveoperator* formalism. The waveoperator is a complex functional of the non-commutative generalized coordinates which will be denoted by $\hat{\rho} = \rho(\hat{z})$. In this representation quantum mechanics is formulated by extremizing the action [8]

$$\mathcal{S} = \int \mathrm{d}t\, \mathrm{Tr}\{\mathrm{i}\hbar\, \hat{\rho}^\dagger\, \partial_t\, \hat{\rho} - \hat{\rho}^\dagger\, \hat{\mathcal{H}}\, \hat{\rho}\},$$

$$\hat{\mathcal{H}} = \mathcal{H}(\hat{z}_1, \hat{z}_2) = \frac{\hat{z}_1^2}{2m} + V(\hat{z}_2) \tag{1}$$

with respect to some coordinate space description $\rho(y, x) = \langle y|\hat{\rho}|x\rangle$. Here $\hat{\mathcal{H}}$ is a Hamiltonian operator in Hilbert space and Tr stands for the trace operation (for instance in the position basis $|x\rangle$; i.e. $\hat{z}_2\,|x\rangle = x|x\rangle$). Throughout the paper, the waveoperator is time dependent although we will not write it out explicitly [10].

Evaluating the action by considering the trace in this position basis, and minimizing it with respect to $\rho^*(x, y) = \langle y|\hat{\rho}^\dagger|x\rangle$ we obtain

$$\mathrm{i}\hbar\, \partial_t\, \rho(y, x) = \int \mathrm{d}u\, h(y, u)\, \rho(u, x) \tag{2}$$

where $\rho(y, x)$ is the representation of the waveoperator in the *bilocal coordinates* (BLC) y, x. If $\rho(y, x)$ is separable [i.e. $\rho(y, x) = \psi_1(y)\,\psi_2(x)$] it can be mapped isomorphically onto a doubled Hilbert space $H \times H$ and this procedure is known as the Gel'fand-Naimark-Segal construction. The $H \times H$ representations have been noticed recently to be of use in the phase space representations of the quantum canonical Lie algebras [11]. In this scheme $\rho(y, x)$ is referred to as the wavefunction [12] in $H \times H$. Similarly, $h(y, u) = \langle y|\mathcal{H}|u\rangle$ is the Hamiltonian in BLC. Equation (2) will be referred to as the *bilocal Schrödinger equation*.

A quick inspection reveals that (2) does not have unique solutions. The non-uniqueness arises from the indeterminacy in the x dependence. In order to show this we first assume that a particular solution is separable, e.g. $\rho(y, x) = \psi_1(y)\,\psi_2(x)$ where both ψ_k, $k = 1, 2$ have finite norm. Here we require $\psi_2(x)$ to be time independent. Inserting this in (2) and using the Hamiltonian in (1) one obtains the standard Schrödinger equation for ψ_1 whereas the ψ_2 dependence drops out from both sides. One particular question is whether (2) has non-separable solutions in terms of the BLC, e.g. $\rho(y, x) \neq \psi_1(y)\,\psi_2(x)$ are of interest.

The separable and non-separable sectors in the function space are connected to each other by certain types of *non-linear* canonical maps. This can be demonstrated by the use of the Weyl map [13] which transforms $\rho(y, x)$ into a function $\tilde{\rho}(z)$ of the phase space where $z = (z_1, z_2)$ are the generalized coordinates in the phase space $\mathcal{Z}_\star$. Note that throughout the paper a generic dependence on (z_1, z_2) will be denoted by z. The representations of functions of the BLC space in the phase space will also be denoted by a tilde. The standard approach to phase space quantum mechanics is to use the Weyl correspondence which is an analytic and invertible map from an arbitrary Hilbert space operator $\hat{O}$ to a function $\tilde{o}(z)$ in $\mathcal{Z}_\star$ which can be denoted by $\mathcal{W} : \hat{O} \leftrightarrow \tilde{o}(z)$. Moreover, since a Hilbert space operator can also be represented by a bilocal function as $\hat{O} \leftrightarrow \langle y|\hat{O}|x\rangle = o(y, x)$, the combination of these two maps is also a well-defined (Weyl) map $W : o(y, x) \leftrightarrow \tilde{o}(z)$. The Weyl map is explicitly given by

$$o(y, x) = \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \mathcal{K}_z(y, x)\, \tilde{o}(z). \tag{3}$$

Here $\mathcal{K}_z(y, x) = \mathrm{e}^{\mathrm{i}z_1(x-y)/\hbar}\, \delta(z_2 - \frac{x+y}{2})$ is an invertible Weyl kernel. The function $\tilde{o}(z)$ is the standard Weyl symbol of $\hat{O}$ in the phase space $\mathcal{Z}_\star$. In this context, o and $\tilde{o}$ are two different representations of the Hilbert space operator $\hat{O}$. We therefore have the following triangle diagram

$$\begin{array}{ccc} \hat{O} & \overset{\mathcal{W}}{\longleftrightarrow} & \tilde{o}(z) \\ \searrow\!\nwarrow & & W \swarrow\!\nearrow \\ & o(y,x) & \end{array} \tag{4}$$

Note that W in (3) is a well defined map by itself between a bilocal function $o(y, x)$ and the function $\tilde{o}(z)$. It can exist as a transformation independently from $\mathcal{W}$, a fact which we exploit later in Sections 4.3–4.5.

The non commutativity of the coordinates in $\mathcal{Z}_\star$ is encoded in the associative $\star$-product. The $\star$-product is best described by the specific Weyl map $W : [\int \mathrm{d}u\, o_1(y, u)\, o_2(u, x)] \leftrightarrow \tilde{o}_1(z) \star_z \tilde{o}_2(z)$ where

$$\begin{aligned} \tilde{o}_1 \star_z \tilde{o}_2 &= \tilde{o}_1 \exp\left\{\frac{\mathrm{i}}{2} \overleftarrow{\partial}_{z_i}\, \theta_{ij}\, \overrightarrow{\partial}_{z_j})\right\} \tilde{o}_2 \\ &\neq \tilde{o}_2 \star_z \tilde{o}_1 \end{aligned} \tag{5}$$

with the arrows indicating the direction that the partial derivatives act. The non commutativity of $\tilde{o}_1$ and $\tilde{o}_2$ in $\mathcal{Z}_\star$ is indicated by the second line in (5) which is characterized by the Moyal bracket (MB) $\{\tilde{o}_1, \tilde{o}_2\}_z^{(M)} = \tilde{o}_1 \star_z \tilde{o}_2 - \tilde{o}_2 \star_z \tilde{o}_1$. We now consider the waveoperator $\hat{\rho}$ for the arbitrary operator $\hat{O}$ in (4). Using (3) the Weyl map of (2) is then found to be

$$\mathrm{i}\hbar\, \partial_t\, \tilde{\rho}(z) = \tilde{h}(z) \star_z \tilde{\rho}(z). \tag{6}$$

Here $\tilde{h}$ is the Weyl map of the bilocal Hamiltonian h in (2). We refer to Eq. (6) as the $\star$-Schrödinger equation which is basically an operator relation manifested by the waveoperator interpretation. [Here $\hat{\rho}$ should not necessarily be confused with the density operator unless specified (see Section 5). Note that if $\hat{\rho}$ described the standard quantum mechanical density operator, the right hand side of (6) would not be given by a single $\star$-product but by a MB instead. Hence (6) would be the Moyal equation for the density operator $i\hbar\,\partial_t\,\hat{\rho} = \tilde{h}\star\tilde{\rho} - \tilde{\rho}\star\tilde{h} = \{\tilde{h},\tilde{\rho}\}^{(M)}$. See also the remark [10]].

If $\tilde{\rho}$ is an existing solution of (6), a new solution can be obtained by $\star$-multiplying (6) from the right by a *time independent* function $\tilde{\xi}(z)$ such that $-i\hbar\,\partial_t\,\tilde{\xi}(z) = \tilde{\xi}(z)\star\tilde{h} = 0$. The new solution is then represented by $\tilde{\rho}'(z) = \tilde{\rho}\star_z\tilde{\xi}$ and respects $i\hbar\,\partial_t\,\tilde{\rho}' = \{\tilde{h},\tilde{\rho}'\}^{(M)}$ of which the solution is unique (for some initial conditions).

In the separable case, Eq. (6) has a specific physical realization in terms of the generalized Wigner function. Consider the case $\hat{\rho} = |\phi_E\rangle\,\langle\chi|$ where E labels one of the basis states $|\phi_E\rangle$ which are the eigensolutions of the Schrödinger equation with some Hamiltonian. Here we consider again $|\chi\rangle$ as a time independent state under the same Hamiltonian. In this separable form $\rho(y,x) = \phi_E(y)\,\chi^*(x)$ is an eigensolution of (2) and $\tilde{\rho}(z)$ is the corresponding solution of (6). By the application of the Weyl map, it is given by

$$\tilde{\rho}(z) = \int \mathrm{d}x\,\mathrm{e}^{-i z_1 x/\hbar}\,\phi_E\Big(z_2 - \frac{x}{2}\Big)\,\chi^*\Big(z_2 + \frac{x}{2}\Big). \tag{7}$$

Equation (7) is the generalized Wigner function [14] W_{χ,ϕ_E}. If the basis states $|\phi_E\rangle$ are the energy eigenbasis of the Hamiltonian $\hat{\mathcal{H}}$ with energy E then (7) is the eigensolution of Eq. (6) [15].

Here $\rho(y,x) = \phi_E(y)\,\chi^*(x)$ is manifestly separable by choice, where the local solution is indicated by $\phi_E(y)$. In order to explore beyond the ordinary local sector we must find how to reach the non-separable sector of the doubled Hilbert space.

3. Separability and the Hilbert space operators

We look for connections between the local $H\times H$ sub-sector and the non-local sub-sector of the doubled Hilbert space. Hence the relevant question is the separability versus non-separability of the bilocal functions like $\rho(y,x)$ in this doubled Hilbert space. Within the frame of standard transformations in quantum mechanics the answer is simple. Consider the Hilbert space operator-unitary or non unitary-(speaking of the Hilbert space the standard-local-quantum mechanical Hilbert space is implied and not the doubled Hilbert space) $\hat{\Omega}$ acting on $\hat{\rho}$ as

$$\hat{\Omega} : \hat{\rho} = \hat{\rho}' = \hat{\Omega}\,\hat{\rho}\,\hat{\Omega}^\dagger. \tag{8}$$

Suppose that we start from a separable case in (8) as $\hat{\rho} = |\psi_1\rangle\,\langle\psi_2|$, e.g. $\rho(y,x) = \psi_1(y)\,\psi_2(x)$. Then $\hat{\rho}' = \hat{\Omega}\,|\psi_1\rangle\,\langle\psi_2|\,\hat{\Omega}^\dagger = |\psi_1'\rangle\,\langle\psi_2'|$, where $\|\psi_j\| = \|\psi_j'\|$. We observe that $\hat{\rho}'$ is still separable, e.g. $\rho'(y,x) = \psi_1'(y)\,\psi_2'(x)$. Likewise, the invertibility of $\hat{\Omega}$ ensures that, the non-separable $\rho(y,x)$ is transformed into a non-separable $\rho'(y,x)$. Therefore, Eq. (8) is a simple proof that the separable and the non-separable BLC representations are disjoint within the reach of standard Hilbert space transformation $\hat{\Omega}$. Our discussion here should give us the clue that in order to reach for the sub-sector of the doubled Hilbert space where non-local functions are defined one needs to reach beyond the separable sector $H\times H$. In the search for this nonlocal sector of the Hilbert space we resort to the nonlinear canonical transformations. An extended view of the unitary (canonical) nonlinear transformations in $\mathcal{Z}_\star$ will be given in the following section.

4. An extended view of the quantum canonical maps in the phase space

In this section we will examine the unitary representations of the nonlinear canonical maps in $\mathcal{Z}_\star$. Consider $\hat{z}_j$ and $\hat{Z}_j$ as the old and the new non commutative generalized coordinates. A canonical map, in the Hilbert space operator picture is defined to be $\hat{\Omega} : \hat{z}_j \mapsto \hat{Z}_j = Z_j(\hat{z}_1,\hat{z}_2)$, $(j=1,2)$ such that the canonical commutation relations are preserved

$$\hat{\Omega} : ([\hat{z}_j,\hat{z}_k] = i\,\theta_{jk}) \mapsto ([\hat{Z}_j,\hat{Z}_k] = i\,\theta_{jk}). \tag{9}$$

The corresponding relations in $\mathcal{Z}_\star$ can be obtained by the Weyl transform and are based on the *canonical* MB, where the latter is defined by $\{z_i,z_j\}^{(M)} = i\,\theta_{ij} = \{Z_i,Z_j\}^{(M)}$. The *canonical* MB is *basis independent* [see Eq. (21) below] whereas a general MB is not. Because of this property, we will choose the z basis in expressing the $\star$ products and MBs, and this will be implied by $\star = \star_z$ where the $\star$ operation is defined in (5). Here $Z_j = Z_j(z_1,z_2)$, $j=1,2$ are the Weyl symbols of the new non-commutative generalized coordinates $\hat{Z}_j$. In the following two different types of canonical maps will be examined as type-I and type-II. These two types are distinguished by their covariance properties when they act on the functions of z. We refer to Ref. [16] for a detailed comparative study of the classical and the quantum cases.

4.1. *Classically covariant canonical maps $\Omega_{\mathcal{Z}}$: type-I*

Since we are dealing with the classical case, a necessity now arises to differentiate the classical commutative space $\mathcal{Z}$ from the non commutative one $\mathcal{Z}_\star$. The commutativity in $\mathcal{Z}$ is induced by the product $\tilde{o}_1(z)\,\tilde{o}_2(z) = \tilde{o}_2(z)\,\tilde{o}_1(z)$ as opposed to the non-commutative one in $\mathcal{Z}_\star$ as expressed in (5). However there are also common features. For instance, a square integrable function in $\mathcal{Z}_\star$ is also square integrable in $\mathcal{Z}$ due to the integral property of the $\star$-product $\int \mathrm{d}^2z\,\tilde{o}_1(z)\star_z\tilde{o}_2(z) = \int \mathrm{d}^2z\,\tilde{o}_1(z)\,\tilde{o}_2(z)$. This allows us to refer to the functions of z without referring to the underlying commutative or non-commutative space. The representations of the operators, however, generally differ due to the different covariance properties of them in $\mathcal{Z}$ or $\mathcal{Z}_\star$.

With this in mind, we assume that an infinitesimal generator $\mathcal{G}(z)$ of the classical canonical map exists generating a first order infinitesimal change in the transformation of the function $\tilde{o}(z)$, i.e. $\tilde{o}\to\tilde{o}+\delta\,\tilde{o}$. This is given by the classical textbook formula, $\delta\,\tilde{o}(z) = \epsilon\,\{\mathcal{G},\tilde{o}\}^{(P)}$ where (P) stands for the Poisson Bracket (PB). A general PB is given between two such phase space functions $\tilde{o}_1$ and $\tilde{o}_2$ by $\{\tilde{o}_1,\tilde{o}_2\}^{(P)} \equiv J_{jk}\,(\partial_{z_j}\tilde{o}_1)\,(\partial_{z_k}\tilde{o}_2)$. According to standard phase space analytical mechanics, $\mathcal{G}$ is transformed into a Hamiltonian vector field as $X_{\mathcal{G}} = J_{jk}\,(\partial_{z_j}\mathcal{G})\,(\partial_{z_k})$. Here $X_{\mathcal{G}}$ is the infinitesimal generator of which action is defined by the Lie bracket $\delta\,\tilde{o}(z) = \epsilon\,[X_{\mathcal{G}},\tilde{o}] = \epsilon\,(X_{\mathcal{G}}\,\tilde{o} - \tilde{o}\,X_{\mathcal{G}})$.

In the standard Lie group theoretical formulation of the classical canonical transformations the continuous classical canonical maps (which we denote as $\Omega_{\mathcal{Z}}$) are finite transformations in $\mathcal{Z}$ obtained by exponentiating the classical generators as

$$\Omega_{\mathcal{Z}} = \mathrm{e}^{\epsilon\,X_{\mathcal{G}}}. \tag{10}$$

The finite action of $\Omega_{\mathcal{Z}}$ on the functions of z is then given by

$$\Omega_{\mathcal{Z}} : \tilde{o} = \tilde{o} + \epsilon [X_{\mathcal{G}}, \tilde{o}] + \cdots + \frac{\epsilon^n}{n!} \underbrace{[X_{\mathcal{G}}, \ldots [X_{\mathcal{G}}}_{n}, [\tilde{o}, X_{\mathcal{G}}] \ldots] + \cdots . \quad (11)$$

The action of $\Omega_{\mathcal{Z}}$ on $\mathcal{Z}$ has the manifest covariance property

$$\Omega_{\mathcal{Z}} : \tilde{o}(z) = \tilde{o}(\Omega_{\mathcal{Z}} \, : \, z) = \tilde{o}(Z). \quad (12)$$

For two such phase space functions $\tilde{o}_1(z)$ and $\tilde{o}_2(z)$ Eq. (12) implies

$$[\Omega_{\mathcal{Z}} : \tilde{o}_1(z)\, \tilde{o}_2(z)] = \tilde{o}'_1(z)\, \tilde{o}'_2(z) = \tilde{o}_1(Z)\, \tilde{o}_2(Z). \quad (13)$$

Equation (13) will be a useful relation to facilitate the comparison between the actions of the continuous classical and the quantum nonlinear transformations. Eqs (10)–(13) are well-known results and they can be found in standard textbooks on Lie algebraic techniques in analytical mechanics [17]. In (13) the square brackets indicate that there are no other operators to the left or right acted upon by the transformation and the primes denote the transformed functions of z. We remark that Eq. (13) defines the *classical covariance* as used in the context of this work (see also Ref. [16]). We can also define the action of the canonical map $\Omega_{\mathcal{Z}}$ on the Poisson Bracket (PB) similarly. Denoting the latter by $\{\tilde{o}_1, \tilde{o}_2\}^{(P)} \equiv J_{jk}\, (\partial_{z_j} \tilde{o}_1)\, (\partial_{z_k} \tilde{o}_2)$, the classical covariance implies

$$[\Omega_{\mathcal{Z}} : \{\tilde{o}_1(z), \tilde{o}_2(z)\}^{(P)}] = \{[\Omega_{\mathcal{Z}} : \tilde{o}_1], [\Omega_{\mathcal{Z}} : \tilde{o}_2]\}^{(P)} = \{\tilde{o}_1(Z), \tilde{o}_2(Z)\}^{(P)}. \quad (14)$$

We now examine the generators of infinitesimal canonical transformations in the quantum phase space $\mathcal{Z}_\star$.

4.2. *⋆-covariant canonical maps $\Omega_{\mathcal{Z}_\star}$: type-II*

Consider a unitary Hilbert space operator $\hat{\Omega} = \mathrm{e}^{\mathrm{i}\epsilon \hat{G}}$ with real ϵ and Hermitian $\hat{G}$. $\hat{\Omega}$ acts on the operator $\hat{O}$ as

$$\hat{\Omega} : \hat{O} = \hat{\Omega}\hat{O}\,\hat{\Omega}^\dagger = \hat{O} + \mathrm{i}\,\epsilon\, [\hat{G}, \hat{O}] + \cdots + \frac{(\mathrm{i}\epsilon)^n}{n!} \underbrace{[\hat{G} \ldots [\hat{G}}_{n}, [\hat{O}, \hat{G}] \ldots] + \cdots \quad (15)$$

In analogy with the classical case, $\hat{G}$ above is the generator of the continuous quantum canonical maps in the Hilbert space. We describe the Weyl symbol of $\hat{\Omega}$ by $\Omega_{\mathcal{Z}_\star}(z)$ and its adjoint $\hat{\Omega}^\dagger$ by $\Omega^*_{\mathcal{Z}_\star}(z)$. Such a map is given by the commuting diagram

$$\begin{array}{ccc} \hat{z}_j & \overset{\hat{\Omega}}{\longleftrightarrow} & \hat{Z}_j = \hat{\Omega}\, \hat{z}_j\, \hat{\Omega}^\dagger \\ \mathcal{W} \updownarrow & & \updownarrow \mathcal{W} \\ z_j & \overset{\Omega_{\mathcal{Z}_\star}}{\longleftrightarrow} & Z_j = \Omega_{\mathcal{Z}_\star} \, : \, z_j \end{array} \quad (16)$$

It was shown [16,18] that $\Omega_{\mathcal{Z}_\star}$ acts on functions of z as

$$\Omega_{\mathcal{Z}_\star} : \tilde{o}(z) = \tilde{o}'(z) = \Omega_{\mathcal{Z}_\star}(z) \star_z \tilde{o}(z) \star_z \Omega^*_{\mathcal{Z}_\star}(z) \quad (17)$$

and it is the *quantum* counterpart of the classical unitary generator $\Omega_{\mathcal{Z}}$. We actually gained an advantage in the search for the methods to reach beyond the Hilbert space operator approaches by formulating the canonical maps in the phase space $\mathcal{Z}_\star$. The reason is that the quantum canonical maps can, in general, be handled in the phase space $\mathcal{Z}_\star$ in a similar way to the classical case in $\mathcal{Z}$ and this can be done in a totally independent way from the Hilbert space operator methods. Namely, the operator connections induced by the Weyl correspondence $\mathcal{W}$ can be totally and consistently ignored in (16). Examining (4), what remains is the Weyl map W between double (bilocal) Hilbert space and phase space. We now proceed to discuss the ⋆-covariance in Eq. (17).

If we use two such functions $\tilde{o}_1$ and $\tilde{o}_2$, Eq. (17) implies that

$$[\Omega_{\mathcal{Z}_\star} : \tilde{o}_1 \star_z \tilde{o}_2] = [\Omega_{\mathcal{Z}_\star} : \tilde{o}_1] \star_z [\Omega_{\mathcal{Z}_\star} : \tilde{o}_2]. \quad (18)$$

Equation (18) is to be regarded as the extended version of the classical covariance in (13). In the context of this work it will be referred to as the ⋆-covariance[16]. Also note that, if we denote the canonical map in $\mathcal{Z}_\star$ by $\Omega_{\mathcal{Z}_\star} : z_j \to Z_j$, Eq. (18) implies violation of the classical covariance, e.g. $[\Omega_{\mathcal{Z}_\star} : \tilde{o}](z) \neq \tilde{o}(Z)$. Comparing (13) and (18) we note that $\Omega_{\mathcal{Z}}$ in (13) preserves the commutativity of the standard product between the functions $\tilde{o}_1$ and $\tilde{o}_2$ whereas for $\Omega_{\mathcal{Z}_\star}$ in (18) the non commutativity by ⋆-product is invariant. The MB is therefore transformed by $\Omega_{\mathcal{Z}_\star}$ as

$$[\Omega_{\mathcal{Z}_\star} : \{\tilde{o}_1(z), \tilde{o}_2(z)\}^{(M)}] = \{[\Omega_{\mathcal{Z}_\star} : \tilde{o}_1], [\Omega_{\mathcal{Z}_\star} : \tilde{o}_2]\}^{(M)} \neq \{\tilde{o}_1(Z), \tilde{o}_2(Z)\}^{(M)}. \quad (19)$$

At the limit $\hbar \to 0$ the classical and star covariances coincide.

4.3. *$\hbar$-independent canonical maps*

A particular subgroup of canonical maps in types I and II is distinguished by its simultaneous unitary representations both in $\mathcal{Z}$ and $\mathcal{Z}_\star$. This is the subgroup of $\hbar$-independent canonical maps and may be crucial for a unified understanding of the classical and quantum phase spaces.

In what follows, we pay specific attention to this subgroup and derive some of its properties. We start with the canonically conjugated pair $Z_j = Z_j(z_1, z_2)$; $(j = 1, 2)$ with $\{z_j, z_k\}^{(M)} = \{Z_j, Z_k\}^{(M)} = \mathrm{i}\theta_{jk}$ where both the old (i.e. z_k) and the new (i.e. Z_k) canonical variables are assumed to be independent of $\hbar$. The canonical transformations generating the map $z_k \to Z_k$ are then in $\hbar$-independent canonical group.

Expanding the MB in powers of $\theta_{jk} = \hbar J_{jk}$ we have

$$\{Z_j, Z_k\}^{(M)}_z = \mathrm{i}\hbar\, \{Z_j, Z_k\}^{(P)}_z + \mathcal{O}(\theta^3) \quad (20)$$

where the first term is the Poisson Bracket (PB) $\{Z_j, Z_k\}^{(P)}_z \equiv J_{\ell m}\, (\partial_{z_\ell} Z_j)(\partial_{z_m} Z_k)$ generating a linear term in $\hbar$. The higher order terms are in odd powers of $\hbar$ and start with the cubic dependence. By assumption Z_j's have no dependence on $\hbar$. Considering that and by equating (20) to $\mathrm{i}\theta_{jk}$, we match the powers of $\hbar$ to deduce that the $\mathcal{O}(\theta^3)$ and higher order terms in (20) must all vanish. The non-vanishing part of (20) is therefore the linear term in $\hbar$

$$\{Z_j, Z_k\}^{(M)}_z = \mathrm{i}\hbar\, \{Z_j, Z_k\}^{(P)}_z = \mathrm{i}\hbar J_{jk} \quad (21)$$

which unambiguously implies that $\{Z_j, Z_k\}_z^{(P)} = J_{jk}$. On the other hand this is the condition for the pair Z_1, Z_2 to be classically canonical. We learn that any $\hbar$ independent quantum canonical map $\Omega_{\mathcal{Z}_\star}: z_j \to Z_j$ in (20) implies (21). Therefore such maps are also canonical in the (classical) Poisson sense and conversely all $\hbar$-independent Poisson maps are also quantum canonical. We will refer to them as type-I as well. A quick corollary of this result is that an $\hbar$ independent quantum canonical phase space transformation can also be obtained by an appropriate classical canonical map $\Omega_{\mathcal{Z}}$. Now, we consider a second $\hbar$-independent classical map $\Omega'_{\mathcal{Z}}$. We let it act classically on the canonical MB as

$$\begin{aligned}\Omega'_{\mathcal{Z}}: \frac{1}{\mathrm{i}\hbar}\{Z_j, Z_k\}^{(M)} &= \Omega'_{\mathcal{Z}}: \{Z_j, Z_k\}^{(P)} \\ &= \{[\Omega'_{\mathcal{Z}}: Z_j](z), [\Omega'_{\mathcal{Z}}: Z_k](z)\}^{(P)} \\ &= \{Z_j(z'), Z_k(z')\}^{(P)} \\ &= J_{jk}\end{aligned} \tag{22}$$

where $z' = \Omega'_{\mathcal{Z}}: z$. In (22) the second line is obtained by the classical covariance of the Poisson bracket in (14). The third line is an application of (13). The last line is the statement of the invariance of the canonical MB under $\hbar$-independent canonical maps which is a corollary of (21). Therefore the classically covariant [see (13)] canonical maps define an $\hbar$-independent automorphism on the *canonical* MB. It should be kept in mind that this result is correct only between the canonical pairs and not between any arbitrary functions of z.

In the literature, it almost goes without saying that all quantum canonical maps fall in type-II. As we have seen above, type-I maps also preserve the canonical MB although their action is truly different from that of type-II. Therefore the space of canonical maps in $\mathcal{Z}_\star$ is actually a union of types I and II which we now refer to as the extended picture in the quantum phase space.

4.4. *Non separability and the type-I maps*

In standard quantum mechanics, the canonical transformations of type-II have integral transforms. Denoting by $\phi(x)$ some local Hilbert space function, a typical map $\hat{\Omega}$ on $\phi(x)$ is expressed by

$$(\hat{\Omega}: \phi)(x') = \int \mathrm{d}x\, u(x, x')\, \phi(x). \tag{23}$$

In the rest of this work, we will be interested in examining similar integral transforms adopted for the bilocal representations and for the canonical maps of type-I. For separable cases these bilocal transforms reduce to the direct products of the integral transforms like in (23) of which an example is given below from the linear canonical group [see Eqs. (29) and (30)]. Denoting by $\Omega_{\mathcal{Z}}$ a generic type-I map, its action can be written as

$$\begin{aligned} o'(y, x) &= (\Omega_{\mathcal{Z}}: o)(y, x) \\ &= \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \mathcal{K}_z(y, x)\, \tilde{o}'(z) \end{aligned} \tag{24}$$

with $\tilde{o}'(z) = (\Omega_{\mathcal{Z}}: \tilde{o})(z) = \tilde{o}(Z)$ as dictated by (13). Using the inverse of (3) we convert (24) into an integral transform between the old and the new functions of BLC as

$$o'(y, x) = \int \mathrm{d}u \int \mathrm{d}v\, \mathcal{L}_{\Omega_{\mathcal{Z}}}(y, x; v, u)\, o(v, u) \tag{25}$$

with the integral kernel

$$\mathcal{L}_{\Omega_{\mathcal{Z}}}(y, x; u, v) = \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \mathcal{K}_z(y, x)\, (\Omega_{\mathcal{Z}}: \mathcal{K}_z)(v, u). \tag{26}$$

In Eq. (26) we used the classical covariance in (13) which implies that $\Omega_{\mathcal{Z}}: \mathcal{K}_z = \mathcal{K}_{\Omega_{\mathcal{Z}}:z} = \mathcal{K}_Z$. Equation (25) is the bilocal extension of (23). We now examine the separability of the general canonical map in (25) by three general examples.

4.4.1. *Linear canonical group.*

We describe the unitary generators of the linear canonical group by $\Omega_{\mathcal{Z}}^{(a)} = \Lambda_g \in Sp_2(\mathbb{R})$ which act on the phase space as

$$\Lambda_g: \tilde{o}(z) = \tilde{o}(g^{-1}: z), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{27}$$

where $\det g = 1$ and $g^{-1}: z = Z$ is the transformed coordinate. It is known that the linear canonical group is the only group of transformations for which the classical (type-I) and $\star$-covariances (type-II) coincide. Based on this fact we already expect (27) not to have an effect on the separability. Nevertheless, we carry on the explicit calculation and demonstrate this fact for illustration.

Using the inverse of the Weyl kernel in (3) we map the transformed and initial solutions by calculating the kernel in (26) for Λ_g as

$$\begin{aligned}\mathcal{L}_g &= \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \mathcal{K}_z(y, x)\, (\Lambda_g: \mathcal{K}_z)(v, u) \\ &= \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \mathcal{K}_z(y, x)\, \mathcal{K}_{g^{-1}:z}(v, u) \end{aligned} \tag{28}$$

$$= \exp\left\{\frac{\mathrm{i}}{2\hbar\, c}[d(u^2 - v^2) + a(x^2 - y^2) - 2(xu - yv)]\right\}. \tag{29}$$

Here $\mathcal{L}_g$ is the kernel of the map Λ_g. As expected it is manifestly separable, i.e. $\mathcal{L}_g(y, x; u, v) = \mathcal{L}_g(0, x; u, 0)\, \mathcal{L}_g(y, 0; 0, v)$. Therefore the transformed solution is separable when $o(y, x) = \psi_1(y)\,\psi_2(x)$ and non separable when $o(y, x)$ is non-separable. In the first case both dependences on the coordinates are transformed separately and identically as

$$\begin{aligned} Y'(y)\, X'(x) &= \left[\int \mathrm{d}v\, \mathcal{L}_g(y; v)\, Y(y)\right] \\ &\quad \times \left[\int \mathrm{d}u\, \mathcal{L}_g(x; u)\, X(x)\right]. \end{aligned} \tag{30}$$

where each square bracket is an integral transform of type-II as in (23).

4.4.2. *Non linear gauge transformation-type maps.*

We describe the unitary generators of these type of maps by $\Omega_{\mathcal{Z}}^{(b)}$ whose action on the phase space is defined by

$$\Omega_{\mathcal{Z}}^{(b)}: \tilde{o}(z) = \tilde{o}(z_1 + \tau_1 A_1(z_2), z_2 + \tau_2 A_2(z_1)). \tag{31}$$

It is crucial again that the characteristic functions A_j are $\hbar$ independent. The free and real parameter τ_j is either momentum, ($\tau_1 = \tau, \tau_2 = 0$) or coordinate-like ($\tau_1 = 0$, $\tau_2 = \tau$). Two or higher dimensional versions of (31) are normally referred to as gauge transformations (not considered here). We consider here a momentum-like map ($\tau_1 = \tau \neq 0, \tau_2 = 0$). Using (31) in (3) one finds

$$o'(y, x) = \mathrm{e}^{\mathrm{i}\frac{\tau}{\hbar} x_- A(x_+)}\, o(y, x) \tag{32}$$

where $x_- = (x - y)$ and $x_+ = (x + y)/2$. Due to the exponential factor, the transformation kernel in (32) is, for general cases, manifestly non-separable. An exception is only when $A(x_+) \propto \alpha + \beta x_+$, with α, β as arbitrary constants, in which case the exponential separates.

4.4.3. *Contact transformations.* We describe the unitary generators of the contact transformations by $\Omega^{(c)}_{\mathcal{Z}}$ whose action on the phase space is defined by

$$\Omega^{(c)}_{\mathcal{Z}} : \tilde{o}(z) = \tilde{o}(T(z_1), z_2/T'(z_1)) \tag{33}$$

or $1 \leftrightarrow 2$. Here, as usual, $T(z_1)$ describes an invertible and $\hbar$-independent function. Using (33) in (3) one finds

$$o'(y, x) = T'(x_+) \times o\Big(T(x_+) + \frac{x_-}{2} T'(x_+), T(x_+) - \frac{x_-}{2} T'(x_+)\Big). \tag{34}$$

This example again is also manifestly non separable for general cases even though the original $o(y, x)$ is separable. We thus have the conclusion that these two type of maps in (b) and (c) transform a local (separable) picture into a nonlocal (non-separable) one and visa versa.

Any non-separable canonical map can have various compositions of elementary maps of type-II but the basis requirement is that it must also include type-I maps such as $\Omega^{(b)}_{\mathcal{Z}}$ and/or $\Omega^{(c)}_{\mathcal{Z}}$ above.

Concerning the type-II case, these are standard unitary transformations of which we discussed as $Sp_2(\mathbb{R})$ [part (a) above] as an example. The type-II maps have Hilbert space operator representations and the results in Section 3 are valid for them. For the purpose of separability the standard type-II maps do not offer interesting results by themselves.

4.5. *Unitarity of the type-I maps*

Returning to the type-I maps, our discussion leading to Eq. (21) also classifies them as quantum canonical. An interesting question is then how to incorporate them into the standard unitary quantum mechanical picture. To the author's knowledge, the type-I maps have been first examined in a quantum mechanical context by the authors of Ref. [19]. Instead of the phase space, these authors searched for the standard Hilbert space representations. The equivalence generated by the type-I maps is known as isometry [19], viz. they map a Hilbert space to an equivalent Hilbert space with a generally different inner product. This point has been also advocated more recently by Anderson [20].

On the other hand, we now demonstrate that the phase space representations of the type-I and type-II maps can be done within the same phase space as opposed to their different (isometric) Hilbert space representations. Due to their classical origin, the action of the type-I maps is well-defined in $\mathcal{Z}$. Based on the discussions earlier in this section we have the clue that one should be able to incorporate them as unitary transformations also in $\mathcal{Z}_\star$. Therefore the function space $\mathcal{L}_2(\mathcal{Z})$ becomes the appropriate Hilbert space for such maps. That they conserve the norm of functions in $\mathcal{L}_2(\mathcal{Z})$, and hence the unitarity, can be shown by using (3) and its inverse. For two arbitrary bilocal functions o_1, o_2 and their Weyl maps $\tilde{o}_1, \tilde{o}_2$ we write

$$\begin{aligned}
\int \mathrm{d}y\, \mathrm{d}x\, o_1'(y, x)\, o_2(y, x) &\equiv \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \tilde{o}_1'^*(z) \star_z \tilde{o}_2'(z) \\
&= \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \tilde{o}_1^*(\Omega_{\mathcal{Z}} : z) \star_z \tilde{o}_2(\Omega_{\mathcal{Z}} : z) \\
&= \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \tilde{o}_1^*(Z) \star_z \tilde{o}_2(Z) \\
&= \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \tilde{o}_1^*(Z) \tilde{o}_2(Z) \\
&= \int \frac{\mathrm{d}^2 Z}{2\pi\hbar}\, \tilde{o}_1^*(Z) \tilde{o}_2(Z) \\
&= \int \frac{\mathrm{d}^2 Z}{2\pi\hbar}\, \tilde{o}_1^*(Z) \star_Z \tilde{o}_2(Z) \\
&\equiv \int \mathrm{d}y\, \mathrm{d}x\, o_1^*(y, x)\, o_2(y, x).
\end{aligned} \tag{35}$$

The preservation of the norm in $\mathcal{L}_2(\mathcal{Z})$ follows by comparing the top and the second line from the bottom in the case when $o_1 = o_2$. In the first line we used $\tilde{o}'(z) = (\Omega_{\mathcal{Z}} : \tilde{o})(z)$. The second and the third lines are the application of classical covariance in (13). The fourth line is the general property of the $\star$-product, i.e. $\int \mathrm{d}^2 z \tilde{o}_1 \star \tilde{o}_2 = \int \mathrm{d}^2 z\, \tilde{o}_1 \tilde{o}_2$. In the fifth line the invariance of the integral measure, i.e. $\mathrm{d}^2 z = \mathrm{d}^2 Z$ is employed. The sixth line is the general property of the $\star$-product again. The left hand side of the first line in Eq. (35) can be adopted as the definition of the inner product in the Hilbert and the right hand side as that in the phase space where the remaining lines demonstrate the preservation of the inner product under the type-I maps. Hence, this general inner product as adopted therein is invariant under all canonical transformations in the extended picture.

5. Non locality, entanglement and generalized Bell states

As also mentioned above, the key reason why a general type-I transformation is not representable locally is that such a map connects a local representation to a nonlocal one and, in this respect it can be viewed as an isometry between two Hilbert spaces which we write formally as $\hat{\Omega} : H \rightarrow H'$, where the Hilbert spaces H and H' are distinguished by their different inner products [20]. Consider two copies H_x and H_y of the same Hilbert space and an operator defined in the direct product space $H_x \times H_y$. In view of Section 4.4, a type-I map acting on this operator *entangles* the local coordinates x and y and, in turn, the transformed operator does not have a direct product representation. An important point which follows from the discussions in the previous sections is that the same map can be represented unitarily in an appropriate two-dimensional non commutative space. In the context of this work we considered the non- commutative space in question as the phase space $\mathcal{Z}_\star$.

Now we come to the point where one needs to incorporate the concept of entanglement within this extended picture. Since the canonical transformations can transform between the local versus non-local pictures, it is expected that they also play a role as entanglement changing transformations. How a canonical transformation can induce entanglement/dis-entanglement in terms of the bilocal coordinates can be demonstrated as follows. Consider for instance the operator $|n\rangle\langle m|$, where n, m denote the harmonic oscillator energy eigenstates, represented bilocally in $H_x \times H_y$. Below, we will use the fact that this operator is represented as a generalized Wigner function [14] $\tilde{W}_{n,m}(z)$ in the phase space $\mathcal{Z}_\star$.

A type-I map acting on this operator can be defined in the bilocal representation by using (25) and (26) as

$$(\hat{\Omega}: |n\rangle\langle m|)(y, x) = \int \mathrm{d}u \int \mathrm{d}v\, \mathcal{L}(y, x; v, u)\, \psi_n^*(y)\, \psi_m(x) \tag{36}$$

where $\psi_m(x) = \langle x|n\rangle$ is the m'th harmonic oscillator Hermite Gaussian and

$$\mathcal{L}(y, x; v, u) = \int \frac{\mathrm{d}z_1}{2\pi\hbar}\, \mathrm{e}^{-\mathrm{i}[Z_1(z_1, x_+)u_- - z_1 x_-]} \times \delta(Z_2(z_1, x_+) - u_+) \tag{37}$$

where $x_+ = (x+y)/2$, $x_- = (x-y)$, $u_+(u+v)/2$, $u_- = (u-v)$. Now consider the action of the classical transformation $\Omega_{\mathcal{Z}}$ on $\tilde{W}_{n,m}(z)$. Using the results in Section 4.1 we write this as $\Omega_{\mathcal{Z}}: \tilde{W}_{n,m}(z) = \tilde{W}_{n,m}(\Omega_{\mathcal{Z}}: z) = \tilde{W}_{n,m}(Z) \equiv \tilde{W}'_{n,m}(z)$ where Z standardly denotes the transformed phase space variables under the action of $\Omega_{\mathcal{Z}}$. Since the set $\{\tilde{W}_{n,m}(z); 0 \le n, m < \infty\}$ forms a basis (the Wigner basis) in the phase space via the orthogonality relation $\int \mathrm{d}^2 z/2\pi\hbar\, \tilde{W}^*_{n,m}(z)\, \tilde{W}_{n',m'}(z) = \delta_{n,n'}\, \delta_{m,m'}$ the transformed Wigner function $\tilde{W}'_{n,m}(z)$ can be uniquely expanded in this basis as

$$\Omega_{\mathcal{Z}}: W_{n,m}(z) = \sum_{0 \le n', m'}^{\infty} \omega^{(n,m)}_{n',m'}\, \tilde{W}_{n',m'}(z)$$

$$\omega^{(n,m)}_{n',m'} = \int \frac{\mathrm{d}^2 z}{2\pi\hbar}\, \tilde{W}^*_{n,m}(z)\, \tilde{W}_{n',m'}(Z). \tag{38}$$

Let us consider a simple case when the initial Wigner function is diagonal: $\tilde{W}_{n,m} = \delta_{n,m}\, \tilde{W}_n$. This Wigner function is given by the well known expression [21]

$$\tilde{W}_n(z) = 2\,(-1)^n\, \mathrm{e}^{-(z_1^2+z_2^2)}\, L_n(2(z_1^2 + z_2^2)) \tag{39}$$

which is rotationally invariant in $\mathcal{Z}_\star$. In Eq. (39) L_n is the n'th Laguerre polynomial. The abovementioned rotational invariance in (39) is implied by the zero eigenvalue of the *phase space angular momentum* operator $K_0 = \mathrm{i}\hbar\, z_j\, \theta_{j,k}\, \partial_{z_k}$ viz., $K_0 : \tilde{W}_n(z) = 0$ [a more general fact is $K_0 : \tilde{W}_{n,m} = (n-m)\hbar\, \tilde{W}_{n,m}$]. Let us further assume that the transformation $\Omega_{\mathcal{Z}}$ is also rotationally invariant; i.e. $[\Omega_{\mathcal{Z}}, K_0] = 0$. Therefore the transformed Wigner function $W_n(Z)$ is also rotationally invariant. It can therefore be expanded in a rotationally invariant Wigner sub basis (in terms of the diagonal elements only)

$$\tilde{W}_n(Z) = \sum_{n'} \omega^{(n)}_{n'}\, \tilde{W}_{n'}(z) \tag{40}$$

where the normalization on both sides and the invariance of the measure $\mathrm{d}^2 z = \mathrm{d}^2 Z$ requires $\sum_{n'} \omega^{(n)}_{n'} = 1$. We have already seen that the local symmetry algebra in the phase space is defined by the linear canonical transformations. The nonlocal symmetries are generated by the remaining type-I generators forming an infinite Lie sub algebra of the canonical algebra [20,22]. A general rotationally invariant transformation as in (40) can be generated in two steps where the first step is the transformation $T_1: (z_1, z_2) \to (J, \theta)$ with (J, θ) describing the generalized action-angle coordinates. For this specific example they are the harmonic oscillator action-angle variables: $J = (z_1^2 + z_2^2)$, $\theta = \tan^{-1} z_1/z_2$. The second step is the transformation within the action angle variables given by $T_2: (J, \theta) \to (g(J), \theta/g'(J))$ where we require T_2 to be of type-I hence $g(J)$ is $\hbar$ independent. The final transformation $T_2\, T_1$ produces (40) and it can be written [22] as a composition of the generators in the canonical Lie subgroup and $SP_2(\mathbb{R})$. Once the contact map $g(J)$ is known, the coefficients $\omega^{(n)}_{n'}$ are calculated by [from (39) and (40)]

$$\omega^{(n)}_{n'} = 2\,(-1)^{n+n'} \int_0^\infty \mathrm{d}J\, \mathrm{e}^{-(J+g(J))}\, L_{n'}(2J)\, L_n(2g(J)). \tag{41}$$

Equation (41) characterizes a general nonlocal, radial (rotationally invariant) transformation. These transformations generate bilocal Hilbert space analogs of the standard (entangled) Bell states as shown below.

In order to demonstrate this here, we continue with a specific example of what we call a generalized Bell state. The simplest one should have two nonzero coefficients in the expansion (40) which can be written as $\omega^{(n)}_{n'} = a_{n,n_1}\, \delta_{n',n_1} + a_{n,n_2}\, \delta_{n',n_2}$. Now let us find a canonical map that generates this Bell state. Consider the specific case when the initial Wigner function is given with $n = 0$ and the transformed one is with $n_1 = 0$ and $n_2 = 2$. For $n = 0$ one of the Laguerre polynomials in (41) drops out ($L_0 = 1$). We next choose $g(x) = x - \ln[a_{0,0} + a_{0,2}\, L_2(2x)]$ (here we consider $a_{0,2} < a_{0,0}$ so that the logarithm is real). Using this in (41) it can be seen that this transformation yields

$$|0\rangle\langle 0| \to a_{0,0}|0\rangle\langle 0| + a_{0,2}\,|2\rangle\langle 2|. \tag{42}$$

In principle, more general polynomial or infinite series Bell states can also be obtained. For instance, consider $n = 0$ again with the map $g(x) = x - \ln[\sum_{n'=0}^{M_{\max}} \omega^{(0)}_{n'}\, L_{n'}(2x)]$ where $M_{\max}$ can be finite or infinite. This produces the map

$$|0\rangle\langle 0| \to \sum_{n'}^{M_{\max}} a_{0,n'}|n'\rangle\langle n'|. \tag{43}$$

Even ordered Laguerre polynomials are bounded from below. This insures that it is always possible to choose a well defined (single valued and real) $g(x)$ by choosing the constant term $a_{0,0}$ appropriately and requiring the leading Laguerre function in the polynomial $M_{\max}$ to be of even order.

Since we are examining the entanglement properties of the generalized Bell states, it is illustrative as a side remark to recall the conventional measure of entanglement by the use of the Schmidt number [23]. In the simplest sense, the Schmidt number is the minimum number of Hilbert space dimensions onto which the reduced density matrix (for one degree of freedom) can be mapped. Such a reduced density matrix can be written in a diagonal Hilbert space representation as

$$\rho = \sum_i P_i \, |i\rangle\langle i| \tag{44}$$

where the Schmidt number is the number of nonzero partial probabilities P_i in the sum. Entanglement is said to be finite when the Schmidt number is larger than unity corresponding to a mixed state density matrix ρ. The absence of entanglement is indicated by a single nonzero term in (44) corresponding to a pure state in which case the Schmidt number is unity. The observation we make here is that, this conventional measure of entanglement can be directly applied to the density matrix in Eq. (43). Comparing Eq. (44) with (43) we infer that the nonlinear canonical map used to generate (43) induces a Schmidt number larger than one. Further implications of this conventional entanglement measure in the context of non-locality generating maps are currently under investigation. We now remark on the other implications of the generalized Bell states in a different context in the frame of non-commutative field theories.

5.1. *Generalized Bell states as non commutative solitons*

Recently radially symmetric nonlocal solutions similar to Eq. (40) have been observed (see for instance the first reference in Ref. [25] for a survey) in non-commutative field theories as vacuum soliton solutions. We will not enter the details of these solutions here. The reader is invited to examine complete reviews such as Harvey's in Ref. [24]. We rather confine ourselves to the remark that in the context of these works $|0\rangle\,\langle 0|$ is an example of a level-one non-commutative soliton and Eq. (43) is an example of a unitary map between two vacuum configurations of a level-one soliton. The type-I and type-II transformations are joined in the canonical group of area preserving diffeomorphisms in the phase space. Therefore, we expect that this canonical group is somewhat related to the $U(\infty)$ symmetry [24]. In the language of Ref. [24] we identify the type-II maps with the *local* and the type-I maps with the *nonlocal* sectors in $U(\infty)$. Whether the canonical group covers this $U(\infty)$ entirely is a subject of further investigation.

The type-I maps are fundamentally different from the non-unitary isometries presently discussed in the literature. For instance Harvey [25] studied the non-unitary *phase operator* $\hat{S} = \sum_{n=0}^{\infty} |n\rangle\,\langle n+1|$ in the context of non commutative field theories as a generating map of fixed-level index non-commutative soliton solutions. The fact that $\hat{S}$ is representable in a local Hilbert space implies that its action is defined within the local (nonlocal) sector; viz. $\mathcal{H}_x \times \mathcal{H}_y \to \mathcal{H}'_x \times \mathcal{H}'_y$. From the arguments above it is clear that such maps cannot create entanglement and therefore they cannot induce transformations as in (43). In this view one immediate use of type-I transformations in non-commutative field theory is in the generation of entangled vacuum soliton configurations. When the type-I maps can be used in composition with the $\hat{S}$ operator further soliton configurations can be obtained. As a typical composition of the two and considering for instance the action of $\hat{S}^\dagger$ on (43), one finds a transformation from $|1\rangle\,\langle 1|$ to the generalized Bell state $\sum_{n'}^{M_{\max}} a_{0,n'} |n'+1\rangle\,\langle n'+1|$ where the leading term $M_{\max}+1$ has an odd Laguerre polynomial.

Now we briefly consider the radially non symmetric configurations. We start by identifying the phase space *angular momentum* operator $K_0 = \mathrm{i}\hbar\, z_j\, \theta_{j,k}\, \partial_{z_k} = h_0 \star - \star\, h_0$ as the third generator of $sp_2(\mathbb{R})$. Here $h_0 = (z_1^2 + z_2^2)/2$ is the harmonic oscillator Hamiltonian. The left and the right $\star$ multiplication is equivalent to a doubling of the degrees of freedom whence, K_0 has an infinitely degenerate eigen basis $|n+k\rangle\langle n|$ with eigenvalue k for all $0 \le n$. The other two generators of $sp_2(\mathbb{R})$, namely $K_\pm$ raise and lower k by unity for all n and, they are used as generators of non-unitary symmetries. These type of generators break the rotational symmetry by introducing non-zero phase space *angular momentum k* and, in the context of Ref. [24], they generate soliton solutions with fixed phase space angular momentum. We will examine the relations between the nonlinear canonical group and the $U(\infty)$ in this context in a separate work [26].

6. Discussion

The general approach to the canonical maps in general, and the type-I maps in particular, require reaching beyond the standard Hilbert space formalism. Type-I canonical maps have also been examined in the context of generating Darboux transformations between two partner Hamiltonians [27]. In this context they play a role in mapping one integrable system to a large set of its integrable partners. In addition to these earlier results, the current work demonstrates that type-I maps can be unitarily incorporated into an extended quantum mechanical picture in the phase space. More specifically, type-I maps establish unitary isomorphism in the phase space and isometry in the Hilbert space and, they give rise to a nonlocal Hilbert space formulation of quantum mechanics. This result can be illustrated in Fig. 1.

In the view of this figure, the standard representations are specific cases of, and can be obtained from, the bilocal ones at one end and the phase space representations at the other. The bilocal and the phase space representations are connected by a W map which is not shown in the figure. The second case

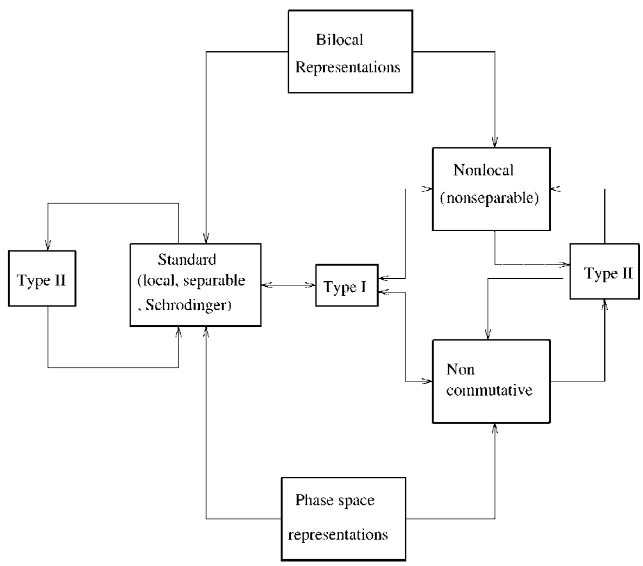

Fig. 1. Schematic of the bilocal and the phase space representations in quantum mechanics and their interconnections by the type I and II canonical maps.

that one can obtain from the bilocal ones is the nonlocal (non separable in BLC) representations. These are connected to the non-commutative picture of the phase space representations by the same Weyl map that connects the general bilocal and the phase space representations. Furthermore, each (local as well as nonlocal) representation is an independent automorphism created by the type II maps. The type I maps join these otherwise disjoint representations.

Note that, in the context of this work, the figure above resulted from a quantum mechanical analysis with one degree of freedom. One trivial extension is to carry out the analysis in N coordinate ($2N$ phase space) dimensions. We have indications that for $1 < N$ the linear canonical group has nonlocal realizations [26]. More interestingly, it has also implications for the field theories on non-commutative spaces. In particular, the field equations in such theories are reminiscent of the $\star$-Schrödinger equation in (6) with the nonlinear field interactions added. The representations of these theories in the non commutative space $\mathcal{Z}_\star$ as well as in the BLC can be fitted manifestly in the context of Fig. 1. We have also specifically shown how nonlocal maps generate generalized Bell states. Interesting explorations of such maps exist in the generation and characterization of entangled soliton configurations in the non-commutative theories and in nonlocal quantum mechanics.

Acknowledgements

The author is thankful to C. Zachos (HEP Theory Division, Argonne National Laboratories); D. Fairlie (University of Durham) and C. Deliduman (Feza Gürsey Institute) for helpful discussions and critical comments.

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