HIGHER DIMENSIONAL METRICS OF COLLIDING GRAVITATIONAL PLANE WAVES

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We give a higher even dimensional extension of vacuum colliding gravitational plane waves with the combinations of collinear and non-collinear polarized four-dimensional metric. The singularity structure of space-time depends on the parameters of the solution.

I. INTRODUCTION

One of the main fields of interest in general relativity is the collision of gravitational plane waves. The structure of the field equations, physical and geometrical interpretations, and various solution-generating techniques have been described in [1]. Khan-Penrose [2] and Szekeres [3] have found exact solutions of the vacuum Einstein equations describing the collision of impulsive and shock plane waves with collinear polarizations. Nutku-Halil [4] generalized the Khan-Penrose metric to the case of non-collinear polarizations. Later several authors have studied exact solutions of the Einstein-vacuum and Einstein-Maxwell equations, describing the collision of the gravitational and electromagnetic plane waves. In general relativity, various techniques are known for generating different solutions of vacuum and electrovacuum Einstein field equations [5]. In this context recently various new solution-generating techniques have been given for vacuum and electrovacuum cases [6-8]. Also a formulation of the colliding gravitational plane waves in metric-affine theories is given in [9]. In connection with string theory colliding gravitational plane waves were studied in [10, 11].

Recently, motivated by the results obtained in [12, 13, 14, 15], we showed that starting from a Ricci flat metric of a four-dimensional geometry admitting two Killing vector fields it is possible to generate a whole class of two dimensional geometry with an arbitrary signature (0 or ±2), then the metric of the 2N = 2 + 2n dimensional geometry defined below

\[ ds^2 = e^{-M_i} \eta_{ab} dx^a dx^b + e^U (h_{0 i})_{ab} dy^a dy^b, \]

where \( \eta_{ab} \) is the metric of the flat 2-geometry with an arbitrary signature (0 or ±2), then the metric of the 2N = 2 + 2n dimensional geometry defined below

\[ ds^2 = e^{-M} \eta_{ab} dx^a dx^b + \sum_{i=1}^{n} \epsilon_i e^{u_i} (h_{0 i})_{ab} dy^a_i dy^b_i, \]

solves the vacuum equations, where \( \epsilon_i = \pm 1 \), \( M = \bar{M} + M_i \), \( M = \sum_{i=1}^{n} M_i \), \( M_i \) solves

\[ \frac{1}{2} (\nabla^2 M) \eta_{ab} + (n - 1) \mathcal{U}_{ab} - \frac{1}{2} \bar{M} \mathcal{U}_{ab} + \bar{M} \mathcal{U}_{ab} \]

\[ - \bar{M}_d \mathcal{U}^d \eta_{ab} \] - \( \frac{1}{2} \sum_{i=1}^{n} \partial_a u_i \partial_b u_i + \frac{1}{2} \eta_{ab} \mathcal{U} \partial_a \mathcal{U} = 0, \] (3)

and \( \mathcal{U} \) and \( u_i \) solve the following equations, respectively:

\[ \partial_a [\eta^{ab} e^U \partial_b U] = 0, \] (4)

\[ \nabla^2 u_i + \eta^{ab} \mathcal{U}_{ab} u_i = 0. \] (5)

Here the local coordinates of the \((2n+2)\) dimensional geometry are given by \( x^a = (x^a, y^a_1, y^a_2, \ldots, y^a_n) \). Given any
four-dimensional metric of colliding vacuum gravitational plane wave geometry we have their extensions to higher dimensions for arbitrary $2N$ without solving further differential equations. In particular, taking $u_i = m_i u$, where $m_i (i = 1, 2, ..., n)$ are real constants satisfying

$$\sum_{i=1}^{n} m_i = 1, \quad \sum_{i=1}^{n} (m_i)^2 = m^2, \quad (6)$$

and the signature of flat-space metric with null coordinates is

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x^1 = u, \quad x^2 = v,$$

then the solutions to the Eqs. (3) and (4) are found to be

$$e^{-M} = (f_a g_a)^{-n+1} (f + g) \frac{m^2 - n}{n} e^{-\sum_{i=1}^{n} M_i}, \quad (7)$$

$$e^{M} = f(u) + g(v), \quad (8)$$

where $f$ and $g$ are arbitrary functions of their arguments and Eq. (3) is automatically satisfied as a result of Eq. (8). Therefore, the above exact solutions describe the collision of gravitational waves for arbitrary $n > 1$.

In [16] we found a family of solutions when the four-dimensional metrics are the Szekeres metrics [3] (collinear four-dimensional metrics). In the next section we generalize this solution by adding non-collinear metrics to the Szekeres metrics. We also show that, in our formalism, there is no higher dimensional metric constructed by the non-collinear four dimensional metric alone.

### III. HIGHER DIMENSIONAL VACUUM SOLUTIONS

We take the following metric as the metric describing a plane wave geometry in $2N$ dimensions.

$$ds^2 = 2 e^{-M} dudv + \sum_{j=1}^{n_a} (f + g)^{m_j} \frac{[(1 - E_j) dx_j + i (1 + E_j) dy_j)]^2}{1 - E_j E_j} + \sum_{j=n_a+1}^{n} (f + g)^{m_j} (E_j dx_j^2 + e^{V_j} dy_j^2), \quad (9)$$

where the complex functions $E_i$ (non-collinear case) and the real functions $V_i$ (collinear case) satisfy the Ernst and Euler-Poisson-Darboux equations, respectively,

$$\begin{align*}
(1 - E_i \dot{E}_i)^{2} [2 (f + g) E_i f_g + E_i f + E_i g] &= -4 (f + g) \dot{E}_i E_i f E_i g, \quad i = 1, 2, ..., n_a, \quad (10) \\
2 (f + g) V_i f_g &= -V_i f - V_i g, \quad i = 1, 2, ..., n - n_a. \quad (11)
\end{align*}$$

with the following solutions:

$$E_i = e^{\alpha_i} \left( \frac{1}{2} - f \right)^{1/2} \left( \frac{1}{2} + g \right)^{1/2} e^{\beta_i} \left( \frac{1}{2} + f \right)^{1/2} \left( \frac{1}{2} - g \right)^{1/2}, \quad (12)$$

$$V_i = -2k_i \tanh^{-1} \left( \frac{\frac{1}{2} - f}{\frac{1}{2} + g} \right)^{1/2} - 2\ell_i \tanh^{-1} \left( \frac{\frac{1}{2} - g}{\frac{1}{2} + f} \right)^{1/2}, \quad (13)$$

where $\alpha_i, \beta_i, k_i, \ell_i$ are arbitrary constants and there is no sum on $i$ in Eq. (10). The metric function $M$ given in (3) is

$$e^{-M} = (f_a g_a)^{-n+1} (f + g) \frac{m^2 - n}{n} e^{-\sum_{i=1}^{n_a} M^{(1)}_i} e^{-\sum_{i=n_a+1}^{n} M^{(2)}_i}, \quad (14)$$

where the metric functions $M^{(1)}_i$ and $M^{(2)}_i$ are, respectively,

$$\begin{align*}
e^{-M^{(1)}_i} &= \frac{f_a g_a \left[ -\gamma_i^2 (f + g)^2 + 2 (\gamma_i^2 - 4) (1 + f g) g + \frac{g^2}{4} - 1 \right]}{(f + g) (1 + f g) + 2 \gamma_i (\frac{1}{4} - f^2)^{1/2} (\frac{1}{4} - g^2)^{1/2} (\frac{1}{4} - f^2)^{1/2} (\frac{1}{4} - g^2)^{1/2} e_i}, \quad (15) \\
e^{-M^{(2)}_i} &= \frac{c_i f_a g_a (f + g)^{2\gamma_i}}{(\frac{1}{4} - f)^{\gamma_i^2/2} (\frac{1}{4} - g)^{\gamma_i^2/2} (\frac{1}{4} + f)^{\gamma_i^2/2} (\frac{1}{4} + g)^{\gamma_i^2/2} (\frac{1}{4} - f)^{1/2} (\frac{1}{4} - g)^{1/2} + (\frac{1}{4} + f)^{1/2} (\frac{1}{4} + g)^{1/2} 2k_i e_i}, \quad (16)
\end{align*}$$

where $\gamma_i = 2 \cos (\alpha_i - \beta_i), \quad \tau_i = k_i^2 + \ell_i^2 + 2k_i \ell_i - 1$, and $e_i, c_i$ are constants.

The metric function $e^{-M}$ must be continuous across the null boundaries. To make it so we define

$$k^2 \equiv \sum_{i=1}^{n-n_a} k_i^2, \quad \ell^2 \equiv \sum_{i=1}^{n-n_a} \ell_i^2, \quad s \equiv \sum_{i=1}^{n-n_a} k_i \ell_i, \quad (17)$$

and we assume that the functions $f$ and $g$ take the form

$$f = \frac{1}{2} - (e_1 u)^{n_1}, \quad g = \frac{1}{2} - (e_2 v)^{n_2},$$

(18)

where $e_1, e_2, n_1 \geq 2, n_2 \geq 2$ are real constants. Then the metric function $e^{-M}$ is continuous across the boundaries if

$$k^2 + n_a = 2(1 - \frac{1}{n_1}), \quad \ell^2 + n_a = 2(1 - \frac{1}{n_2})$$

(19)

$$e^{-M} = \frac{(f + g)^{\frac{n}{2}}}{(\frac{1}{2} + f)(n_a + \ell^2)/2(\frac{1}{2} + g)(n_a + k^2)/2} \left[\left(\frac{1}{2} - f\right)^{1/2}(\frac{1}{2} - g)^{1/2} + \left(\frac{1}{2} + f\right)^{1/2}(\frac{1}{2} + g)^{1/2}\right]^{2s} e^{-\Gamma},$$

(21)

where

$$e^{-\Gamma} = \Pi_{i=1}^{a_2} \left(\frac{(f + g)^2}{2} - 2(\gamma^2 - 4)(1 + 4fg)fg - \frac{2}{\gamma^2} + 1\right) (1 + 4fg) + 2\gamma(1 - f^2)^{1/2} + g(1 - g^2)^{1/2}$$

and $\gamma = m^2 + n - n_a + k^2 + \ell^2 + 2s - 3$. We may set

$$-2e_1e_2n_1n_2\Pi_{i=1}^{a_2} = \Pi_{i=1}^{a_2} d_i.$$  

(22)

When we consider only the non-collinear case where $n_a = n$ and $k^2 = \ell^2 = s = 0$, conditions in Eq. (22) imply $n_1 = n_2$. Then Eqs. (14) and (20) reduce to

$$n = 2(1 - \frac{1}{n_1}),$$

(23)

where $n_1 \geq 2$ and $n$ is a positive integer. The only possible solution is $n = 1, n_1 = 2$ which corresponds to the Nutku-Halil solution. Hence if we take all $E_i$ as the Nutku-Halil metric functions it is not possible to find an appropriate metric function $M$ which is continuous across the null boundaries for $n > 1$. This is the reason why one has to take the higher dimensional metric as the combinations of collinear and non-collinear polarizations.

### A. Singularity structure

We now discuss the nature of the space-time singularity. That is, we study the behavior of the metric function $M$ as $f + g$ tends to zero. For this purpose, using the result obtained in [11] for the curvature invariant

$$I = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta},$$

(24)

as

$$I \sim e^{2M} \frac{(f_a g_e)^2}{(f + g)^3},$$

(25)

with

$$1 \leq k^2 < 2, \quad 1 \leq \ell^2 < 2.$$  

(20)

Therefore, the metric function $e^{-M}$ reads

$$I \sim (f_a g_e)\frac{2}{(f + g)^{-\mu}},$$

(26)

as $f + g \to 0$. Here $\mu = k^2 + \ell^2 + m^2 + 2s + 4n_a + 2$. For the four dimensional case ($n = 1$) $n_a = 0$ with $k = \ell = s = 1$ and $m^2 = 1$ this corresponds to the singularity structure of the Khan-Penrose solution. $n_a = 1, k^2 = \ell^2 = s = 0$ and $m^2 = 1$ corresponds to the Nutku-Halil solution. It is known that both solutions have the same singularity structures. $n_a = 0, k = k_1, \ell = \ell_1$, and $m^2 = 1$ corresponds to the singularity structure of the Szekeres solution. The singularity structure in the higher dimensional spacetimes can be made weaker or stronger than the four dimensional cases by choosing the constants $m_i, k_i$, and $\ell_i$ properly.

### IV. CONCLUSION

In this work we gave a higher even dimensional generalization of vacuum colliding gravitational plane waves with the combinations of collinear and non-collinear polarizations. We also discussed the singularity structure of the corresponding spacetimes, and showed that the strength of the singularity depends on arbitrary parameters $m_i$. We also showed that it is not possible to construct the higher dimensional metric by non-collinear four dimensional metric functions $E_i$ (11) alone. Einstein’s equations and continuity conditions force us to superpose the collinear and non-collinear metric functions.

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