ALMOST $p$-STRUCTURES ON VECTOR-BUNDLES

I. DIBAG

Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey
e-mail: dibag@fen.bilkent.edu.tr

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Abstract. For $p \geq 2$ we introduce the notion of an almost $p$-structure on vector-bundles which generalizes the notion of an almost-complex structure and investigate the existence of almost $p$-structures on spheres and complex projective spaces.

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0. Introduction. In this note we generalize the notion of an almost-complex structure on a real vector-bundle; i.e. a fibrewise linear map $J$ on a vector-bundle $\xi$ such that $J^2 = -1$. For $p \geq 2$ we consider a fibrewise linear map $J$ on $\xi$ such that $J^p = (-1)^{p-1}$. For $p = 2$ this gives an almost-complex structure, but for $p > 2$ this does not suffice. Let $a_p = R[x]/(x^p - (-1)^{p-1})$. This turns the fibre $\xi_x$ into an $a_p$-module.

We insert one more condition which guarantees this. We call such maps $J$ almost $p$-structures. We then study the structure of $a_p$ as an algebra and prove that

$$a_p = \begin{cases} \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C} \quad (p \text{ factors } \mathbb{C}) & \text{if } p \text{ is even} \\ \mathbb{R} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C} \quad (p-1 \text{ factors } \mathbb{C}) & \text{if } p \text{ is odd} \end{cases}$$

It follows from this that a vector-bundle of dimension $n$ admits an almost $p$-structure iff $n = kp$ for some $k \in \mathbb{Z}^+$ and splits into a direct-sum of $\frac{p}{2}$ complex vector-bundles of dimension $k$ if $p$ is even and into a direct-sum of a real vector-bundle and $(\frac{p-1}{2})$-complex vector bundles of dimension $k$ if $p$ is odd. Using this criterion we solve completely the existence problem of almost $p$-structures on spheres and complex projective spaces. The only non-trivial almost $p$-structures on spheres (i.e. on non-parallelisable ones) is an almost 3-structure on $S^{15}$ in addition to the almost-complex structures on $S^2$ and $S^6$. The only almost $p$-structures that exist on complex projective spaces is an almost 3-structure on $\mathbb{C}P^3$ in addition to the almost-complex structures that exist on all complex projective spaces. For this we rely heavily on [1].

1. Almost $p$-structures. For $p \geq 2$ let $J$ be a fibrewise linear map on a vector-bundle $\xi$ over a topological space $X$ such that $J^p = (-1)^{p-1}$.

**Definition 1.1.** Let $a_p = R[x]/(x^p - (-1)^{p-1})$. Then $a_p = \{1, x, \ldots, x^{p-1}/x^p = (-1)^{p-1}\}$. The fibre $\xi_x$ is an $a_p$-module, the module structure is given by $x^i v = J^i(v)$, $v \in \xi_x(0 \leq i \leq p - 1)$.

**Definition 1.2.** For $v \in \xi_x$ define $E(v)$ to be the subspace generated by $v, J(v), \ldots, J^{p-1}(v)$. 

DEFINITION 1.3. We call \( v \in \xi \) a cyclic vector iff \( \dim E(v) = p \), i.e. iff \( v, J(v), \ldots, J^{p-1}(v) \) are linearly-independent. For \( v \in \xi \) a cyclic-vector, \( E(v) = a_p \). For \( p = 2 \) every non-zero vector is a cyclic vector.

DEFINITION 1.4. A fibrewise linear map \( J \) on a vector-bundle \( \xi \) is called an almost \( p \)-structure on \( \xi \) iff

(i) \( J^p = (-1)^{p-1} \) and (ii) For every \( J \)-invariant proper subspace \( U \) of \( \xi \) there exists a cyclic vector \( v \not\in U \).

We deduce from (ii) that there exist cyclic vectors \( v_1, \ldots, v_k \) such that \( \xi = E(v_1) \oplus E(v_2) \oplus \cdots \oplus E(v_k) n = kp \) i.e. \( n \equiv 0 \) (mod \( p \)) and \( \xi \equiv a_p \). For \( p = 2 \) condition (ii) is vacuous and condition (i) suffices to define an almost \( 2 \)- (i.e. almost-complex) structure.

2. Algebraic structure of \( a_p \). For \( p \) even let \( \theta_k = \frac{(2k-1)\pi}{p} \) and \( \cos(\theta_k) (x^m - x^{p-m})(1 \leq k \leq \frac{p}{2}) \). Then \( x_k^2 = x_k, x_k x_\ell = 0 (k \neq \ell) \) and \( \sum_{k=1}^{\frac{p}{2}} x_k = 1 \). Thus \( a_p = \bigoplus_{k=1}^{\frac{p}{2}} I_k \) where \( I_k \) is the ideal generated by \( x_k \). The homomorphism \( R[x] \to I_k \) has kernel \( (x - e^{i\theta_k})(x - e^{-i\theta_k}) \) and this gives an isomorphism of algebras \( C = R[x]/(x^2 - 2x \cos \theta_k + 1) \cong I_k \). Thus \( a_p = C \oplus C \oplus \cdots \oplus C( \frac{p}{2} \text{-factors}) \).

For \( p \) odd let \( \psi_k = \frac{2k\pi}{p} (0 \leq k \leq \frac{1}{2}(p-1)) \). Then \( x^2 = x_k, x_k x_\ell = 0 (k \neq \ell) \) and \( \sum_{k=0}^{\frac{p}{2}-1} x_k = 1 \). Thus \( a_p = \bigoplus_{k=0}^{\frac{p}{2}-1} I_k \) where \( I_k \) is the ideal generated by \( x_k \). The homomorphism \( R[x] \to I_k \) has kernel \( (1 - x^2) \) for \( k = 0 \) and \( (x - e^{i\theta_k})(x - e^{-i\theta_k}) = x^2 - 2x \cos \psi_k + 1 (1 \leq k \leq \frac{1}{2}(p-1)) \). We obtain algebra isomorphisms \( R = R[x]/(1 - x^2) \cong I_0 \) and \( C = R[x]/(x^2 - 2x \cos \psi_k + 1) \cong I_k (1 \leq k \leq \frac{1}{2}(p-1)) \). Hence \( a_p = R \oplus C \oplus \cdots \oplus C( \frac{p}{2} \text{-factors}) \).

3. Almost \( p \)-structures on real vector-bundles. Let \( \xi \) be a real vector-bundle of dimension \( n \) over a topological space \( X \) with an almost \( p \)-structure \( J \). We know from Section 1 that \( n \equiv 0 \) (mod \( p \)). Let \( n = kp \). For \( x \in X \), the fibre \( \xi_x \) is an \( a_p \)-module. Let \( x_i \in a_p \) be the elements defined in Section 2 such that \( a_p \) is the direct-sum of the ideals generated by \( x_i \). Define \( \xi_{\{x\}}(x) = \{x_i \cdot v \mid v \in \xi \} \). Then \( \xi_{\{x\}} = \oplus \xi_{\{x\}}(x) \) and if we define \( \xi_{\{x\}}(x) = \bigcup_{x \in X} \xi_{\{x\}}(x) \), \( \xi \) decomposes into \( \xi = \oplus \xi_{\{x\}}(x) \). If \( p \) is even \( E_i \) is a complex vector-bundle of dimension \( k \) for \( 1 \leq i \leq \frac{p}{2} \). If \( p \) is odd \( E_0 \) is a real vector-bundle and \( E_i \) is a complex vector-bundle of dimension \( k \) for \( 1 \leq i \leq \frac{p-1}{2} \). The argument is reversible. Suppose \( p \) is even and \( \xi = \bigoplus_{i=1}^{\frac{p}{2}} \xi_i \) for complex vector-bundles \( \xi_i \). Let \( J_i \) be the almost-complex structure on \( \xi_i \). Define \( x_i \cdot v = J_i(v) \) for \( v \in \xi \). Then the \( i^{th} \)-factor \( C \) in the direct-sum decomposition of \( a_p \) acts on \( \xi_i \) and this defines an action of \( a_p \) on \( \xi \). An analogous argument holds in the case \( p \) odd. This leads to

\[ \text{THEOREM 3.1. A vector-bundle } \xi \text{ of dimension } n \text{ over a topological space } X \text{ admits an almost \( p \)-structure iff } n \equiv 0 \text{ (mod } p \text{)} \text{ i.e. } n = kp \text{ and } \]

(i) \( p \) is even \( \xi = \bigoplus_{i=1}^{\frac{p}{2}} \xi_i \) where \( \xi_i \) is a complex vector-bundle of dimension \( k \).

(ii) \( p \) is odd \( \xi = E_0 \oplus \bigoplus_{i=1}^{\frac{p-1}{2}} \xi_i \) where \( E_0 \) is a real vector-bundle and \( \xi_i \) is a complex vector-bundle of dimension \( k \). (1 \leq i \leq \frac{1}{2}(p-1)).
4. Almost \( p \) structures on spheres. It is well known that the even spheres which admit almost-complex structures are \( S^2 \) and \( S^6 \). We search for almost \( p \)-structures on spheres for \( p > 2 \). The only non-trivial almost \( p \)-structure that we can find is an almost 3-structure on \( S^{15} \). We rely heavily on [1] for machinery and details. Let \( L_k = 2v_2(M_k) \) be the 2-primary component of the Atiyah–Todd number i.e. \( v_2(M_k) = \sum_{1 \leq r \leq k-1} (r + v_2(r)) \).

We note that almost \( p \)-structures on \( S^k \) exist for all \( p/k \) when \( S^k \) is parallelisable i.e. if \( k = 1, 3, 7 \) and call such almost \( p \)-structures trivial. We call an almost \( p \)-structure non-trivial if the sphere in question is not parallelisable.

**Proposition 4.1.** Let \( p \) and \( k \) be odd. The only non-trivial almost \( p \)-structure on \( S^{pk} \) is an almost 3-structure on \( S^{15} \).

**Proof.** By Theorem 3.1 (ii), \( S^{pk} \) admits an almost \( p \)-structure iff the fibration

\[
SO(pk + 1)/SO(k) \times U(k) \times \cdots \times U(k) \xrightarrow{SO(pk)/SO(k) \times U(k) \times \cdots \times U(k)} S^{pk}
\]

admits a cross-section. Let’s fix one \( U(k) \). Since \( SO(k) \) and all the other \( U(k)’s \) can be imbedded in this fixed \( U(k) \), by using the idea of proof of [2, Theorem 27.16] we deduce that fibration 1 admits a cross-section iff the fibration

\[
SO(pk + 1)/U(k) \xrightarrow{SO(pk)/U(k)} S^{pk};
\]

admits a cross-section. If \( \frac{pk + 1}{2} = \frac{k}{2} \) is odd the existence of a cross-section to fibration 2 implies the existence of a cross-section to the Stiefel fibration

\[
V_{pk, (p-2)k+1} = SO(pk + 1)/SO(2k) \xrightarrow{V_{pk, (p-2)k+1} = SO(pk)/SO(2k)} S^{pk} \quad \text{i.e.} \quad a(p - 2)k\text{-frame on } S^{pk}.
\]

Since \( pk + 1 = 2 \pmod{4} \), \( S^{pk} \) admits at most a 1-frame and thus \( (p - 2)k = 1 \) or \( p = 3, k = 1 \). Since \( S^3 \) is parallelisable this is the only case when fibration 2 admits a cross-section when \( \frac{pk + 1}{2} = \frac{k}{2} \) is odd.

For \( \frac{pk + 1}{2} = \frac{k}{2} \leq 4 \) is even, \( \frac{pk + 1}{2} = 2, 4 \), \( S^{pk} \) is parallelisable and fibration 2 admits a cross-section. For \( \frac{pk + 1}{2} > 4 \) and is even we deduce from [1, Proposition 4.3] and the discussion following it that fibration 2 admits a cross-section iff \( L_{\frac{k}{2}}(p-2k+1)/(pk+1) \).

We observe that \( L_n > 4n \) for \( n > 4 \). To see this, note that \( L_5 = 2^6 > 4.5 \) and for \( k \geq 6, L_k \geq 2^{k-1} > 4k \).

For \( \frac{p-2k+1}{2} > 4 \), \( L_{\frac{p-2k+1}{2}} - \left( \frac{pk+1}{2} \right) > 4(\frac{p-2k+1}{2} - \left( \frac{pk+1}{2} \right)) \geq \frac{1}{2}(k(3p - 8) + 3) > 0 \) i.e. \( L_{\frac{p-2k+1}{2}} > \left( \frac{pk+1}{2} \right) \) and thus fibration 2 does not admit a cross-section. For \( \frac{p-2k+1}{2} \leq 4 \), we disregard the cases \( \frac{p-2k+1}{2} = 2, 4 \) since \( \frac{pk + 1}{2} \) is odd in either case. Let \( \frac{k(p-2)+1}{2} = 1, k = 1, p = 3, S^{pk} = S^5 \) is parallelisable. \( \frac{k(p-2)+1}{2} = 3, \frac{k(p-2)+1}{2} = 5 \). Either \( k = 1 \) and \( p = 7 \) and \( S^{pk} = S^7 \) is parallelisable or \( p = 3, k = 5, \frac{pk+1}{2} = 8 \) and \( L_3 = 8/8 \) and we obtain an almost 3-structure on \( S^{15} \).

**Lemma 4.2.** Let \( p/q \). Then the existence of an almost \( q \)-structure on a vector-bundle implies the existence of an almost \( p \)-structure.

**Corollary 4.3.** The only almost \( p \)-structures on spheres for \( p \) even are the almost-complex structures on \( S^2 \) and \( S^6 \).

**Proof.** By Lemma 4.2 if a sphere admits an almost \( p \)-structure for \( p \) even then it admits an almost-complex structure and hence the sphere in question is \( S^2 \) or \( S^6 \). Apart from the almost-complex structures on these spheres, \( S^6 \) may admit an almost 6-structure. It follows from the proof of Proposition 4.1 it is equivalent to the
cross-sectioning of the fibration $V_{7,5} = SO(7)/U(1)$ \(\xrightarrow{V_{6,4}} SO(6)/U(1)} \to S^6$; i.e. the existence of a 4-frame on $S^6$ which is impossible.

**Lemma 4.4.** An almost $p$-structure does not exist on $S^{pk}$ for $p$ odd and $k$ even.

**Proof.** The existence of an almost $p$-structure implies the existence of a frame on the even dimensional sphere $S^{pk}$ which is impossible.

We gather Proposition 4.1, Corollary 4.3 and Lemma 4.4 in a single Theorem.

**Theorem 4.5.** The only non-trivial almost $p$-structures that exist on spheres are the almost $2$-i.e. almost-complex) structures on $S^2$ and $S^6$ and the almost $3$-structure on $S^{15}$.

5. Almost $p$-structures on complex projective spaces.

**Proposition 5.1.** For $p > 2$ the only almost $p$-structure on complex projective spaces is an almost $3$-structure on $P_3(\mathbb{C})$.

**Proof.** Suppose $P_{n-1}(\mathbb{C})$ admits an almost $p$-structure for $p > 2$. Then $2(n - 1) = kp$. Let $\pi: S^{2n-1} \to P_{n-1}(\mathbb{C})$ be the projection. Since $T(S^{2n-1}) = \pi^*(T(P_{n-1}(\mathbb{C}))) \oplus 1$ the fibration

$$SO(2n)/U(k) \times \cdots \times U(k) \to S^{2n-1}$$

or the fibration

$$SO(2n)/SO(k) \times U(k) \times \cdots \times U(k) \to S^{2n-1}$$

admits a cross-section depending on whether $p$ is even or odd. By the proof of [2, Theorem 27.16], in either case the fibreation $SO(2n)/U(k) \to S^{2n-1}$ admits a cross-section and $L_{n-k}/n$ by [1, Proposition 4.3] and discussion following it. As in the proof of Proposition 4.1, $L_{n-k} = 4(n - k) > n$ for $n > k + 4$ and $n > 4$. Hence $L_{n-k} \nmid n$ for $n = kp + 1 > k + 4$ i.e. for 1. $(\frac{1}{2}p - 1)k > 3$. This is always satisfied for $p > 8$. For $p = 8$, $(\frac{1}{2}p - 1)k > 3$ unless $k = 1$ in which case $n = 5$, $n - k = 4$ and $L_4 \nmid 5$.

For $p = 7$, 1 is satisfied unless $k = 1$. $kp = 7$ is a contradiction since $kp$ is even. For $p = 6$, 1 is satisfied unless $k = 1$ in which case $n = 4$. The existence of an almost $6$-structure on $P_3(\mathbb{C})$ means that $(T(P_3(\mathbb{C})))$ is the direct-sum of three $U(1)$-bundles $\xi_i$, $(i = 1, 2, 3)$. $T(P_3(\mathbb{C})) \oplus 1 = 4\eta_3$ where $\eta_3$ is the complex Hopf bundle over $P_3(\mathbb{C})$. Taking Pontryagin classes, $p(P_3(\mathbb{C})) = (1 + y^2)^4$ where $y \in H^2(P_3; \mathbb{Z})$ is the generator. Suppose $\xi_i$ has Pontryagin class $1 + m_i^2 y^2$, $m_i \in \mathbb{Z}$. Equating $(1 + y^2)^4 = \prod_{i=1}^{4}(1 + m_i^2 y^2)$. Hence $m_1^2 + m_2^2 + m_3^2 = 4$ which has solution $m_1 = 2$ and $m_2 = m_3 = 0$. i.e. $\xi_2$ and $\xi_3$ are trivial. This implies the existence of a frame on $P_3(\mathbb{C})$ which is impossible.

For $p = 5$ again we consider $k = 1$ (otherwise 1 is satisfied). We disregard this case since $kp$ should be even.

For $p = 4$ and $k = 1, 2$. Let $k = 2$, $n = 5$, $L_3 = 8 \nmid 5$. Let $k = 1$, $n = 3$, $L_2 = 2 \nmid 3$. For $p = 3$ since $pk$ is even $k = 2, 4$. Let $k = 4$, $n = 7$, $L_3 \nmid 7 k = 2, n = 4$. Let $\tau: P_3(\mathbb{C}) \to P_1(Q)$ be the projection onto the one dimensional quaternionic projective space. Let $J$ be the quaternionic structure on $\mathbb{C}^4$ which anti-commutes with the complex structure. The assignment $x \mapsto J(x)(x \in S^7)$ defines a unit vector-field on $\pi^*(T(P_3(\mathbb{C})))$ and passes
to the quotient and generates a line sub-bundle $\xi$ of $T(P_3(\mathbb{C}))$ whose orthogonal complement is $\tau^1(T(P_1(\mathbb{Q})))$. Hence $\tau^1(T(P_1(\mathbb{Q})))$ admits an almost-complex structure and $T(P_3(\mathbb{C})) = \xi \oplus \tau^1(T(P_1(\mathbb{Q})))$ an almost 3-structure. □

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