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Characterization of self-selective social choice functions on the tops-only domain

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Abstract. Self-selectivity is a new kind of consistency pertaining to social choice rules. It deals with the problem of whether a social choice rule selects itself from among other rival such rules when a society is also to choose the choice rule that it will employ in making its choice from a given set of alternatives. Koray [3] shows that a neutral and unanimous social choice function is universally self-selective if and only if it is dictatorial. In this paper, we confine the available social choice functions to the tops-only domain and examine whether such restriction allow us to escape the dictatoriality result. A neutral, unanimous, and tops-only social choice function, however, turns out to be self-selective relative to the tops-only domain if and only if it is top-monotonic, and thus again dictatorial.

1 Introduction

Self-selectivity is a new kind of consistency pertaining to social choice rules introduced by Koray [3]. Here we consider a society that will make a collective choice from a set of alternatives, which can be regarded as the ordinary choice level. Now, imagine that our society is also to choose the choice rule that will be used in making this ordinary-level choice. If we think of the process of choosing the choice rule as the "constitutional" level, a natural question that arises concerns the consistency between the ordinary and constitutional levels of choice. More specifically, the society's preference profile on the underlying set of alternatives induces a preference profile on any set of social choice functions, SCFs, in a natural fashion, where the SCFs are ranked according to the alternatives they choose at the ordinary level. The question now is whether an SCF which the society decides to use in choosing an alternative at the ordinary level also selects itself at the constitutional level from among other such functions that are available to our society. In the case where a particular SCF selects some other SCF rather than itself at the induced preference profile on the set of available SCFs, it is not unnatural to ascribe this phenomenon to a certain lack of consistency on the part of this SCF, for it is exactly according to its own rationale that it rejects itself.

Roughly speaking, we call an SCF self-selective at a particular preference profile if it selects itself from among any finite number of such rival functions at the induced profile. Moreover, an SCF is said to be universally self-selective if it is self-selective at each preference profile. The question now is which SCFs are universally self-selective. It is easy to see that dictatorial SCFs are universally self-selective. In fact, Koray [3] shows that a neutral and unanimous SCF is universally self-selective if and only if it is dictatorial. Can one escape this negative result by relaxing some conditions possibly necessitating it? There are two standard methods used in social choice theory to achieve similar aims. One is the restriction of the domain of the social choice rules considered, for example, to single-peaked preference profiles. Another is allowing the social choice rules, SCRs, under consideration to be set-valued rather than confining oneself to SCFs only. Both of these approaches turn out to work in the present context.

Before reporting the results that these two approaches lead, we wish to note that there is a third approach peculiar to the present context. One can restrict the set of SCFs against which self-selectivity is to be tested. Naturally, the smaller the set of test SCFs, the easier will it be for any SCF to pass the consistency test. It is quite possible, however, that the above kind of "monotonicity" is not strict in the sense that an SCF that fails the test of self-selectivity may continue to be non-self-selective even though the set of test SCFs is shrunk to a much smaller set than the initial one. In the present paper, we will confine ourselves to tops-only SCFs. Roughly speaking, an SCF is called "tops-only", if whenever each individual's best alternative is the same in any two given preference profiles, then outcomes of the SCF under these two preference profiles will also be the same. But this restriction does not change the results regarding self-selectivity: the only SCFs that are self-selective on this domain are dictatorial SCFs. The reason why we consider tops-only SCFs as our test functions is twofold. One is, of course, that most of the widely used electoral systems are actually tops-only. Secondly, tops-onliness conjoined with unanimity seems to single out the genuine rival SCFs to test self-selectivity. As we will see, self-selective, unanimous, neutral, tops-only functions turn out to choose from among top alternatives only. It is intuitively clear that the presence of SCFs that do not choose from among top alternatives as test functions is bound to go unnoticed regarding self-selectivity.

Turning back to the first two approaches to escape the negative dictatoriality result without giving up self-selectivity, both approaches seem to have lead to more "promising" results than restriction of test SCFs so far. Unel [6] provides a whole class of non dictatorial self-selective SCFs by restricting the domain to the single-peaked ones. Allowing the SCRs being multivalued, on the other hand, leads to a rediscovery of the Condorcet rule. Koray [2] characterizes the Condorcet rule as the maximal neutral top-majoritarian and universally self-selective SCR.

The rest of the paper is organized as follows. In the next section, we formalize the concept of self-selectivity and introduce other basic notions used in the paper. Section 3 reports a sequence of results about neutral unanimous tops-only self-selective SCFs, leading to a characterization of such voting rules as just dictatorialities. Section 4 concludes the paper with some closing remarks.

2 Basic notions

We let N stand for a finite nonempty society and keep it fixed throughout the paper. We will allow, however, the alternative set to change so long as it has a positive finite cardinality. As we will confine ourselves to neutral social choice functions here, only the size of the alternative set will matter. Thus, we write $I_m = \{1, \dots, m\}$ for each $m \in \mathbb{N}$ to represent an m-element set of alternatives, where $\mathbb{N}$ denotes the set of all positive integers as usual. Letting $\mathscr{L}(I_m)$ stand for the set of all linear orders[1] on I_m, we call a function

$$F : \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N \to \mathbb{N}$$

a *social choice function* (SCF) if and only if, for all $m \in \mathbb{N}$ and $\mathscr{R} \in \mathscr{L}(I_m)^N$, one has $F(\mathscr{R}) \in I_m$. Note that our definition of an SCF allows us to consider its action on preference profiles for alternative sets of different sizes. This is, of course, an appropriate approach in the context of voting rules, where the set of candidates is mostly unknown when the voting rule is decided upon. It is needed here for our analysis since we will be interested in what an SCF will choose from different sets of available SCFs even if the basic alternative set is kept fixed.

Given any $m \in \mathbb{N}$, $\mathscr{R} \in \mathscr{L}(I_m)^N$ and a permutation σ on I_m, we define the permuted linear order profile $\mathscr{R}^\sigma$ on I_m as follows: For any $i \in N$, $k, \ell \in I_m$, we say that $k \mathscr{R}_i^\sigma \ell$ if and only if $\sigma(k) \mathscr{R}_i \sigma(\ell)$. Now an SCF F is called *neutral* if and only if, for each $m \in \mathbb{N}$ and every permutation σ on I_m, one has

$$\sigma(F(\mathscr{R}^\sigma)) = F(\mathscr{R}).$$

We will denote the class of all neutral SCFs by $\mathscr{N}$.

We now wish to extend the domain of an SCF so as to cover linear order profiles on any nonempty finite set. The natural way of doing this seems to be by renaming the elements of the given set using an initial segment of natural numbers. As we wish the alternative chosen by our SCF to be independent of how we do this renaming, we will confine ourselves to neutral SCFs. Now let

[1] Formally, a linear order $\mathscr{R}$ on a set S is a binary relation, which is reflexive ($x \mathscr{R} x$, $\forall x \in S$), transitive (if $x \mathscr{R} y$ and $y \mathscr{R} z$, then $x \mathscr{R} z$, $\forall x, y, z \in S$), and total (for any $x, y \in S$ with $x \neq y$: $x \mathscr{R} y$ or $y \mathscr{R} x$, but not both.).

$F \in \mathcal{N}$, and take any finite set A with $|A| = m \in \mathbb{N}$, where $|A|$ stands for the cardinality of A. Let $\mu : I_m \to A$ be a bijection. Denoting $\mathcal{L}(A)$ for the set of all linear orders on A, take any linear order profile $L \in \mathcal{L}(A)^N$. Now L induces a linear order profile L^μ on I_m in a natural way as follows: For any $i \in N$ and any $k, \ell \in I_m$, we say that $kL_i^\mu \ell$ if and only if $\mu(k)L_i\mu(\ell)$. Finally, we simply define $F(L) = \mu(F(L^\mu))$. Note that $F(L) \in A$ and if $\mu : I_m \to A$ and $\mu' : I_m \to A$ are two bijections, then $\sigma = \mu^{-1} \circ \mu'$ is a permutation on I_m. Set $\mathcal{R} = L^\mu$, then $R^\sigma = L^{\mu'}$ and by the definition of neutrality $\sigma(F(\mathcal{R}^\sigma)) = F(\mathcal{R})$ which implies that $\mu(F(L^\mu)) = \mu'(F(L^{\mu'}))$. That is, $F(L)$ does not depend upon which bijection $\mu : I_m \to A$ is employed.

Let the underlying set of alternatives be represented by I_m, our society N be endowed with a preference profile $\mathcal{R} \in \mathcal{L}(I_m)^N$, and a nonempty set $\mathcal{A}$ of SCFs be available to this society to employ in making its choice from I_m. The agents in this society are naturally expected to rank the SCFs in $\mathcal{A}$ in accordance with what these choose from I_m at $\mathcal{R}$. This induces a preference profile on $\mathcal{A}$. Formally, we define these induced relations $\mathcal{R}_i^{\mathcal{A}}$ $(i \in N)$ on $\mathcal{A}$ as follows: For any $i \in N$ and $F, G \in \mathcal{A}$, we say that $F\mathcal{R}_i^{\mathcal{A}}G$ if and only if $F(\mathcal{R})\mathcal{R}_iG(\mathcal{R})$. Note that, although each agent preference ordering is linear order, $\mathcal{R}^{\mathcal{A}}$ is a complete preorder[2] profile on $\mathcal{A}$, and it will be called the preference profile on $\mathcal{A}$ induced by $\mathcal{R}$.

Now imagine that our society endowed with the preference profile $\mathcal{R}$ on I_m is also to choose an SCF from among those in $\mathcal{A}$ to employ in making its choice from $\mathcal{A}$. But then it also needs a choice rule to choose this SCF from $\mathcal{A}$ on which it already has an induced preference profile $\mathcal{R}^{\mathcal{A}}$. Now whatever $F \in \mathcal{A}$ is chosen to make the choice from I_m, it is only natural to ask whether this F would choose itself if it were also employed in making the choice from $\mathcal{A}$. If the induced profile $\mathcal{R}^{\mathcal{A}}$ is linear order profile, what we are asking here is nothing but whether $F(\mathcal{R}^{\mathcal{A}}) = F$. Since $\mathcal{R}^{\mathcal{A}}$ need not be a linear order profile in general, however, we relax our consistency test by asking whether there is a linear order profile L on $\mathcal{A}$ compatible with $\mathcal{R}^{\mathcal{A}}$ such that $F(L) = F$.

Formally, given a complete preorder ρ on a finite nonempty set A, we say that a linear order λ on A is compatible with ρ if and only if, for all $x, y \in A$, $x\lambda y$ implies $x\rho y$. For each $m \in \mathbb{N}$, $\mathcal{R} \in \mathcal{L}(I_m)^N$ and every nonempty finite subset $\mathcal{A}$ of $\mathcal{N}$, we set

$$\mathcal{L}(\mathcal{A}, \mathcal{R}) = \{L \in \mathcal{L}(\mathcal{A})^N \mid L_i \text{ is compatible with } \mathcal{R}_i^{\mathcal{A}}, \text{ for each } i \in N\},$$

and we refer to $\mathcal{L}(\mathcal{A}, \mathcal{R})$ as the set of all linear order profiles on $\mathcal{A}$ induced by $\mathcal{R}$.

This construction now turns our consistency test (in the sense of a certain self-selectivity) for SCFs into a well-posed question. Thus, we are ready to formally introduce the central notion of this paper.

Given $F \in \mathcal{N}$, $m \in \mathbb{N}$, $\mathcal{R} \in \mathcal{L}(I_m)^N$ and a finite subset $\mathcal{A}$ of $\mathcal{N}$ with $F \in \mathcal{A}$,

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[2] Formally, a complete preorder $\mathcal{R}$ on a set S is a binary relation, which is reflexive ($x\mathcal{R}x$, $\forall x \in S$), transitive (if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$, $\forall x, y, z \in S$), and complete (for any $x, y \in S$ $x\mathcal{R}y$ or $y\mathcal{R}x$, or both.).

we say that F is *self-selective at $\mathscr{R}$ relative to $\mathscr{A}$* if and only if there exists some $L \in \mathscr{L}(\mathscr{A}, \mathscr{R})$ with $F(L) = F$. Given a nonempty subclass $\mathscr{T}$ of $\mathscr{N}$, we say that $F \in \mathscr{T}$ is *$\mathscr{T}$-self-selective at $\mathscr{R}$* if and only if F is self-selective at $\mathscr{R}$ relative to any subset $\mathscr{A}$ of $\mathscr{T}$ with $F \in \mathscr{A}$. Moreover, F is said to be *$\mathscr{T}$-self-selective* if and only if F is $\mathscr{T}$-self-selective at each $\mathscr{R} \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$. We refer to $\mathscr{T}$-self-selectivity as *universal self-selectivity* when $\mathscr{T} = \mathscr{N}$. Given a nonempty finite set A, $a \in A$ and a linear order λ on A, we set $L(a, \lambda) = \{x \in A \mid a\lambda x\}$ and refer to $L(a, \lambda)$ as the lower contour set of λ at a. Moreover, we write $\tau(\lambda) = a$ if and only if $L(a, \lambda) = A$, and call $\tau(\lambda)$ the top alternative of λ. An SCF F is called *unanimous* if and only if, for all $m \in \mathbb{N}$, $\mathscr{R} \in \mathscr{L}(I_m)^N$ and $a \in I_m$, one has

$$(\forall i \in N : \tau(R_i) = a) \Rightarrow F(R) = a.$$

Moreover we say that $j \in N$ is a dictator for F if and only if, for all $m \in \mathbb{N}$ and $\mathscr{R} \in \mathscr{L}(I_m)^N$, one has $F(\mathscr{R}) = \tau(\mathscr{R}_j)$. We refer to F as a dictatorial SCF in case there is a dictator $j \in N$ for F.

Before proceeding further, it will be both illuminating and instructive to see the concept of self-selectivity in an example. The following example is taken from Koray [3] with some modifications.

Example. Consider a society $N = \{\alpha, \beta, \gamma, \delta\}$ consisting of four agents. Let F_1 be the plurality function where all ties are broken in favor of α. Given any $m \in \mathbb{N}$ and $\mathscr{R} \in \mathscr{L}(I_m)^N$, an outcome $a \in I_m$ is said to be a *Condorcet winner* at $\mathscr{R}$ if and only if, for all $b \in I_m \backslash \{a\}$, $|\{i \in N \mid a\mathscr{R}_i b\}| \geq |N|/2 = 2$. In case the set of Condorcet winners at $\mathscr{R}$ is nonempty, we define F_2 to be the Condorcet winner most preferred by α if m is odd, and the Condorcet winner most preferred by β if m is even; if there is no Condorcet winner at $\mathscr{R}$ at all, we set $F_2(\mathscr{R}) = \tau(\mathscr{R}_\alpha)$. We let F_3 stand for the Borda function where ties are broken in favor of γ and the scoring vector employed on I_m is the standard one, namely $(m, m-1, \ldots, 1)$, for each $m \in \mathbb{N}$. Finally, F_4 will denote the dictatorial SCF where δ is dictator, i.e. $F_4 = \tau(\mathscr{R}_\delta)$ at each $\mathscr{R} \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$. It is clear that F_1, F_2, F_3, and F_4 are all neutral and unanimous SCFs. Note that F_1 and F_4 are tops-only, and F_2, F_3 are not. Now let us consider the following linear order profile $\mathscr{R}$ on I_3:

$\mathscr{R}_\alpha$	$\mathscr{R}_\beta$	$\mathscr{R}_\gamma$	$\mathscr{R}_\delta$
2	1	3	1
1	3	2	2
3	2	1	3

First consider the case where the set $\mathscr{A}$ of available SCFs is $\{F_1, F_2, F_3\}$. We have $F_1(\mathscr{R}) = 1$, $F_2(\mathscr{R}) = 2$, and $F_3(\mathscr{R}) = 1$. The complete preorder $\mathscr{R}_\mathscr{A}$ on $\mathscr{A}$ induced by $\mathscr{R}$ is represented in the following table with a comma separating alternatives indicating an indifference class:

$\mathscr{R}_\alpha^\mathscr{A}$	$\mathscr{R}_\beta^\mathscr{A}$	$\mathscr{R}_\gamma^\mathscr{A}$	$\mathscr{R}_\delta^\mathscr{A}$
F_2	F_1, F_3	F_2	F_1, F_3
F_1, F_3	F_2	F_1, F_3	F_2

Now consider $\mathscr{L}(\mathscr{A}, \mathscr{R})$ that consists of 2^4 linear order profiles compatible with the above complete preorder profile in each component. The linear order profile L below is a member of $\mathscr{L}(\mathscr{A}, \mathscr{R})$:

L_α	L_β	L_γ	L_δ
F_2	F_3	F_2	F_3
F_3	F_1	F_3	F_1
F_1	F_2	F_1	F_2

Since $F_2(L) = F_2$ and $F_3(L) = F_3$, we conclude that both F_2 and F_3 are self-selective at $\mathscr{R}$ relative to $\mathscr{A}$. However, not only is it true that $F_1(L) = F_2 \neq F_1$, but we also have $F_1(\tilde{L}) \neq F_1$ for any $\tilde{L} \in \mathscr{L}(\mathscr{A}, \mathscr{R})$ since, at each such $\tilde{L}$, F_2 is top-ranked by two members of N including α to whose favor all ties broken under F_1.

Now consider the case where $\mathscr{A}' = \{F_2, F_3\}$. Here $\mathscr{L}(\mathscr{A}', \mathscr{R})$ consists of one member L' only, where

L'_α	L'_β	L'_γ	L'_δ
F_2	F_3	F_2	F_3
F_3	F_2	F_3	F_2

Now $F_2(L') = F_3 \neq F_2$ and $F_3(L') = F_2 \neq F_3$. Since $\mathscr{L}(\mathscr{A}', \mathscr{R}) = \{L'\}$, this means that neither F_2 nor F_3 is self-selective at $\mathscr{R}$ relative to $\mathscr{A}'$.

Finally, assume that our society's available set $\mathscr{A}''$ of SCFs is $\{F_3, F_4\}$. Note that $F_4(\mathscr{R}) = 1 = F_3(\mathscr{R})$. Now consider two profiles $L, L' \in \mathscr{L}(\mathscr{A}'', \mathscr{R})$ such that at L all agents in N top rank F_3, at L' all agents in N top rank F_4. Clearly, $F_3(L) = F_3$ and $F_4(L') = F_4$. Thus, both F_3 and F_4 are self-selective at $\mathscr{R}$ relative to $\mathscr{A}''$. Actually, it is trivially true that F_4 is universally self-selective. Moreover, we have seen that none of the F_1, F_2, F_3 is universally self-selective. □

We know from Koray [3] that a neutral and unanimous SCF is universally self-selective if and only if it is dictatorial. The question we deal with here is to find out what happens if we relax our consistency test by confining ourselves to "tops-only" SCFs. We call an SCF F *tops-only* if and only if, for any $m \in \mathbb{N}$, $\mathscr{R}, \mathscr{R}' \in \mathscr{L}(I_m)^N$, one has

$$(\forall i \in N : \tau(\mathscr{R}_i) = \tau(\mathscr{R}'_i)) \Rightarrow F(\mathscr{R}) = F(\mathscr{R}').$$

Denoting the class of neutral and tops-only SCFs by Θ, the question posed above can now be rephrased as characterizing Θ-self-selective SCFs. The next section deals with this problem.

3 Results

We will first find some conditions which are necessary for Θ-self-selectivity of unanimous SCFs. Note that our definition of a tops-only SCF does not guarantee the choice of an alternative which is top-ranked by at least one agent at a given linear order profile, but only requires the invariance of the chosen alternative so long as the list(N-tuple) of alternatives top-ranked by agents stays the same. It turns out, however, that only top alternatives will be chosen by a tops-only SCF if it is unanimous and Θ-self-selective as well. For any $m \in \mathbb{N}$, $\mathscr{R} \in \mathscr{L}(I_m)^N$, we let $T(\mathscr{R})$ stand for the collection of all top-ranked alternatives at $\mathscr{R}$, i.e. $T(\mathscr{R}) = \{\tau(\mathscr{R}_i) \mid i \in N\}$. Before stating and proving any results, we also note the following simple fact which will be used extensively throughout the paper: For any $m \in \mathbb{N}$, $\mathscr{R} \in \mathscr{L}(I_m)^N$ and $a \in T(\mathscr{R})$, there exists some $F \in \Theta$ with $F(\mathscr{R}) = a$. In what follows, m will always stand for an arbitrary positive integer.

Proposition 1. *If $F \in \Theta$ is unanimous and Θ-self-selective, then $F(\mathscr{R}) \in T(\mathscr{R})$ for each $\mathscr{R} \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$.*

Proof. Suppose that $F \in \Theta$ is unanimous and Θ-self-selective, but there is some $\mathscr{R} \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$ with $F(\mathscr{R}) \notin T(\mathscr{R})$. Set $F(\mathscr{R}) = a$. Now let $\tilde{\mathscr{R}}$ be the linear order profile for which $L(a, \tilde{\mathscr{R}}_i) = \{a\}$ and $L(x, \tilde{\mathscr{R}}_i) \backslash \{a\} = L(x, \mathscr{R}_i) \backslash \{a\}$ for each $i \in N$ and $x \in A \backslash \{a\}$. In other words, $\tilde{\mathscr{R}}$ is simply the linear order profile obtained from $\mathscr{R}$ by pushing a down to the bottom in each agent's preference ordering and leaving the relative positions of all the other alternatives fixed. Since we assumed $a \notin T(\mathscr{R})$, we have that $\tau(\mathscr{R}_i) = \tau(\tilde{\mathscr{R}}_i)$ for all $i \in N$, implying that $F(\tilde{\mathscr{R}}) = a$ since F is tops-only.

Now choose $b \in T(\tilde{\mathscr{R}})$. Now there is some $G \in \Theta$ with $G(\tilde{\mathscr{R}}) = G(\mathscr{R}) = b$. Set $\mathscr{A} = \{F, G\}$. Clearly, $\mathscr{L}(\mathscr{A}, \tilde{\mathscr{R}}) = \{\tilde{L}\}$, where $G\tilde{L}_iF$ for each $i \in N$ by construction of $\tilde{\mathscr{R}}$. Now by unanimity of F, one should have $F(\tilde{L}) = G$, while $F(\tilde{L}) = F$ is implied by Θ-self-selectivity of F. Since clearly $F \neq G$, this contradiction implies that $F(\mathscr{R}) \in T(\mathscr{R})$ for each $\mathscr{R} \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$. □

Now remember that we call an SCF F *Paretian* if and only if, for all $\mathscr{R} \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$, $F(\mathscr{R})$ is Pareto optimal with respect to $\mathscr{R}$. Since an alternative top-ranked by at least one agent is clearly Pareto optimal, we obtain the following corollary.

Corollary 1. *If $F \in \Theta$ is unanimous and Θ-self-selective, then F is Paretian.*

The following lemma specifies another simple necessary condition for Θ-self-selectivity of unanimous SCFs which turns out to play a crucial role in what follows.

Lemma 1. *Let $F \in \Theta$ be a unanimous and Θ-self-selective SCF and $\mathscr{R} \in \mathscr{L}(I_m)^N$ with $F(\mathscr{R}) = a$. If $B \subset I_m$ is such that $a \notin B$, then $F(\mathscr{R}|_{I_m \backslash B}) \notin T(\mathscr{R}) \backslash \{a\}$.*

Proof. Suppose that $B \subset I_m$, $a \notin B$, but $F(\mathscr{R}|_{I_m \backslash B}) = b \in T(\mathscr{R}) \backslash \{a\}$. Now there is some $G \in \Theta$ with $G(\mathscr{R}) = b$. Set $\mathscr{A}_1 = \{F, G\}$. Since $a \neq b$, we have $\mathscr{L}(\mathscr{A}_1, \mathscr{R}) = \{L_1\}$ for some $L_1 \in \mathscr{L}(\mathscr{A}_1)^N$. But then $F(L_1) = F$ by Θ-self-selectivity of F. Now set $\mathscr{R}' = \mathscr{R}|_{I_m \backslash B}$. We know by proposition 1 that $a \in T(\mathscr{R})$, so that $a \in T(\mathscr{R}')$ as well since $a \in I_m \backslash B$. Then, however, there is some $H \in \Theta$ with $H(\mathscr{R}') = a$. Set $\mathscr{A}_2 = \{F, H\}$. Now $\mathscr{L}(\mathscr{A}_2, \mathscr{R}') = \{L_2\}$ for some $L_2 \in \mathscr{L}(\mathscr{A}_2)^N$, since $F(\mathscr{R}') = b \neq a = H(\mathscr{R}')$. But then $F(L_2) = F$

again by Θ-self-selectivity of F. Define $\sigma : \mathscr{A}_1 \to \mathscr{A}_2$ by $\sigma(F) = H$, $\sigma(G) = F$. Since σ is a bijection, by neutrality of F, $F(L_2) = \sigma(F(L_2^\sigma))$. Note that $L_2^\sigma = L_1$, hence $\sigma(F(L_2^\sigma)) = \sigma(F(L_1)) = \sigma(F) = H$, in contradiction with $F(L_2) = F$. Thus, the proof is complete. □

The above lemma is clearly related with some version of independence of irrelevant alternatives. In our context, we will say that an SCF $F \in \mathscr{N}$ satisfies *Independence of Irrelevant Alternatives* (IIA) if and only if, for any $m \in \mathbb{N}$, $\mathscr{R} \in \mathscr{L}(I_m)^N$, one has

$$[B \subset I_m \text{ and } F(\mathscr{R}) \notin B] \Rightarrow F(\mathscr{R}) = F(\mathscr{R}|_{I_m \backslash B}).$$

We will show below that a neutral, unanimous and Θ-self-selective SCF actually satisfies IIA which is much stronger than the condition stated in Lemma 1. We first need a sequence of intermediate results, however. The following proposition tells that a unanimous Θ-self-selective, tops-only SCF does not distinguish between different sizes of alternative sets, so long as the list(N-tuple) of top-ranked alternatives are the same.

Proposition 2. *Let $F \in \Theta$ be a unanimous and Θ-self-selective SCF, and let $\mathscr{R}, \tilde{\mathscr{R}} \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$. If $\tau(\mathscr{R}_i) = \tau(\tilde{\mathscr{R}}_i)$ for each $i \in N$, then $F(\mathscr{R}) = F(\tilde{\mathscr{R}})$.*

Proof. Suppose that $\tau(\mathscr{R}_i) = \tau(\tilde{\mathscr{R}}_i)$ for each $i \in N$, but $F(\mathscr{R}) = a \neq b = F(\tilde{\mathscr{R}})$. By proposition 1, $a, b \in T(\mathscr{R}) = T(\tilde{\mathscr{R}})$. But then there exist $G, H \in \Theta$ such that $G(R) = b$ and $H(\tilde{\mathscr{R}}) = a$. Setting $\mathscr{A}_1 = \{F, G\}$ and $\mathscr{A}_2 = \{F, H\}$, we have that $\mathscr{L}(\mathscr{A}_1, \mathscr{R}) = \{L_1\}$ and $\mathscr{L}(\mathscr{A}_2, \tilde{\mathscr{R}}) = \{L_2\}$ for some $L_1 \in \mathscr{L}(\mathscr{A}_1)^N$ and $L_2 \in \mathscr{L}(\mathscr{A}_2)^N$. By Θ-self-selectivity of F, it follows that $F(L_1) = F = F(L_2)$. On the other hand, defining $\sigma : \mathscr{A}_1 \to \mathscr{A}_2$ by $\sigma(F) = H$, $\sigma(G) = F$, it now follows by neutrality of F that $F(L_2) = \sigma(F(L_2^\sigma)) = \sigma(F(L_1)) = \sigma(F) = H$, in contradiction with $F(L_2) = F$. □

We now let $S_a(\mathscr{R}) = \{i \in N \mid \tau(\mathscr{R}_i) = a\}$ for each $m \in \mathbb{N}$, $\mathscr{R} \in \mathscr{L}(I_m)^N$ and $a \in I_m$.

Lemma 2. *Let $F \in \Theta$ be unanimous and Θ-self-selective. Assume that $|N| \geq 2$, and let $m \in \mathbb{N}$. Let $\mathscr{R}, \tilde{\mathscr{R}} \in \mathscr{L}(I_m)^N$ be such that $T(\mathscr{R}) = \{a, b\}$ with $a \neq b$, $T(\tilde{\mathscr{R}}) \subset \{a, b\}$, $S_a(\mathscr{R}) \subset S_a(\tilde{\mathscr{R}})$, and $S_a(\tilde{\mathscr{R}}) \backslash S_a(\mathscr{R}) = \{k\}$ for some $k \in N$. Now if $F(\mathscr{R}) = a$, then $F(\tilde{\mathscr{R}}) = a$.*

Proof. Assume that $F(\mathscr{R}) = a$. First consider the case, where $m \geq 3$, and pick $c \in I_m \backslash \{a, b\}$. Now take $\bar{\mathscr{R}}, \hat{\mathscr{R}} \in \mathscr{L}(I_m)^N$ such that the following are satisfied for any $x \in I_m \backslash \{a, b, c\}$:

$$\forall i \in S_a(\mathscr{R}) : a\bar{\mathscr{R}}_i c\bar{\mathscr{R}}_i b\bar{\mathscr{R}}_i x;$$

$$c\bar{\mathscr{R}}_k b\bar{\mathscr{R}}_k a\bar{\mathscr{R}}_k x;$$

$$\forall i \in S_b(\tilde{\mathscr{R}}) : b\bar{\mathscr{R}}_i c\bar{\mathscr{R}}_i a\bar{\mathscr{R}}_i x;$$

$$\forall i \in N \backslash \{k\} : \hat{\mathscr{R}}_i = \bar{\mathscr{R}}_i;$$

$$c\hat{\mathscr{R}}_k a\hat{\mathscr{R}}_k b\hat{\mathscr{R}}_k x;$$

First note that $F(\mathscr{R}|_{\{a,b\}}) = a$ by lemma 1. We now wish to show that $F(\bar{\mathscr{R}}) = a$. We know that $F(\bar{\mathscr{R}}) \in T(\bar{\mathscr{R}}) = \{a, b, c\}$. First suppose that $F(\bar{\mathscr{R}}) = c$. Again by lemma 1, it follows that $F(\bar{\mathscr{R}}|_{\{a,c\}}) = c$. Considering the bijection $\sigma : \{a, c\} \to \{a, b\}$ with $\sigma(a) = a$, $\sigma(c) = b$, neutrality of F implies that $F(\mathscr{R}|_{\{a,b\}}) = b \neq a$, a contradiction. Now consider the case, where $F(\bar{\mathscr{R}}) = b$. Lemma 1 implies that $F(\bar{\mathscr{R}}|_{\{a,b\}}) = b$. Moreover, by construction of $\bar{\mathscr{R}}$, $\bar{\mathscr{R}}|_{\{a,b\}} = \mathscr{R}|_{\{a,b\}}$, so that $F(\mathscr{R}|_{\{a,b\}}) = b$, again a contradiction. Thus, $F(\bar{\mathscr{R}}) = a$.

Note that $\tau(\bar{\mathscr{R}}_i) = \tau(\hat{\mathscr{R}}_i)$ for each $i \in N$, so $F(\hat{\mathscr{R}}) = F(\bar{\mathscr{R}}) = a$ since F is tops-only. But then, again by lemma 1, $F(\hat{\mathscr{R}}|_{\{a,b\}}) = a$.

Finally, suppose that $F(\tilde{\mathscr{R}}) = b$, whence $F(\tilde{\mathscr{R}}|_{\{a,b\}}) = b$ by the same token since $T(\tilde{\mathscr{R}}) \subset \{a, b\}$ by hypothesis. But $\tilde{\mathscr{R}}|_{\{a,b\}} = \hat{\mathscr{R}}|_{\{a,b\}}$, i.e. $F(\tilde{\mathscr{R}}|_{\{a,b\}}) = a$, a contradiction. Hence, as $F(\tilde{\mathscr{R}}) \subset T(\tilde{\mathscr{R}})$, we conclude that $F(\tilde{\mathscr{R}}) = a$.

Now consider the case, where $m = 2$. Then $\{a, b\} = \{1, 2\}$. Define $\mathscr{R}', \tilde{\mathscr{R}}' \in \mathscr{L}(I_3)^N$ by letting, for all $i \in N$, $L(3, \mathscr{R}'_i) = L(3, \tilde{\mathscr{R}}'_i) = \{3\}$; $a\mathscr{R}'_i b$ iff $a\mathscr{R}_i b$; $a\tilde{\mathscr{R}}'_i b$ iff $a\tilde{\mathscr{R}}_i b$. In other words, we extend $\mathscr{R}$ and $\tilde{\mathscr{R}}$ to linear order profiles on I_3 by simply bottom ranking 3 everywhere. Now $F(\mathscr{R}') = F(\mathscr{R})$ by proposition 2, so that $\mathscr{R}'$ and $\tilde{\mathscr{R}}'$ satisfy all the hypotheses for the case with $m \geq 3$. Thus, $F(\tilde{\mathscr{R}}') = a$. But now $\tilde{\mathscr{R}}'|_{\{a,b\}} = \tilde{\mathscr{R}}$ and, by Lemma 1, it follows that $F(\tilde{\mathscr{R}}) = a$. □

Now utilizing Lemma 2, below we will show that if one agent, who was not top ranking the alternative chosen by a unanimous and Θ-self-selective SCF at some profile, changes his/her preferences so as to top rank it, while the remaining agents stick to their original preferences, then the same outcome continues to get chosen by the said SCF.

Proposition 3. *Let $F \in \Theta$ be unanimous and Θ-self-selective. Assume that $|N| \geq 2$, and let $\mathscr{R}, \tilde{\mathscr{R}} \in \mathscr{L}(I_m)^N$ for some $m \in \mathbb{N}$ with $F(\mathscr{R}) = a$. If there exists $j \in N$ such that $\tau(\mathscr{R}_j) \neq a$, $\tau(\tilde{\mathscr{R}}_j) = a$ and $\tau(\mathscr{R}_i) = \tau(\tilde{\mathscr{R}}_i)$ for all $i \in N \backslash \{j\}$, then $F(\tilde{\mathscr{R}}) = a$.*

Proof. Suppose that there is some $j \in N$ satisfying the given condition, but $F(\tilde{\mathscr{R}}) = b \neq a$. Now $b \in T(\tilde{\mathscr{R}})$ by Proposition 1. Since clearly $T(\tilde{\mathscr{R}}) \subset T(\mathscr{R})$, we have $b \in T(\mathscr{R})$. Thus, by Lemma 1, it follows that $F(\mathscr{R}|_{\{a,b\}}) = a$.

We define a linear order profile $\mathscr{R}'$ on $\{a, b\}$ through letting $\mathscr{R}'_i = \mathscr{R}_i|_{\{a,b\}}$ for all $i \in N \backslash \{j\}$ and $a\mathscr{R}'_j b$. Now either $\tau(\mathscr{R}_i|_{\{a,b\}}) = \tau(\mathscr{R}'_i)$ for all $i \in N$, so that $F(\mathscr{R}') = F(\mathscr{R}|_{\{a,b\}}) = a$, or $S_a(\mathscr{R}') \backslash S_a(\mathscr{R}|_{\{a,b\}}) = \{j\}$, in which case we again conclude that $F(\mathscr{R}') = a$ by Lemma 2.

On the other hand, $F(\tilde{\mathscr{R}}) = b$ and $a \in T(\tilde{\mathscr{R}})$ imply, by lemma 1, that $F(\tilde{\mathscr{R}}|_{\{a,b\}}) = b$. Note that $\mathscr{R}'$ is defined on $\{a, b\}$ and by the construction of $\tilde{\mathscr{R}}$, it is clear that $\mathscr{R}' = \tilde{\mathscr{R}}|_{\{a,b\}}$. But this means that $F(\mathscr{R}') = b$, yielding the desired contradiction. Therefore, $F(\tilde{\mathscr{R}}) = a$. □

Now we wish to pursue the *monotonicity* notion inherent to the preceding result further. Since we deal with tops-only SCFs here, any change in a profile which leaves the top-ranked alternatives unchanged will not have any impact upon the alternative chosen. Thus, an improvement of the relative position of an alternative in a profile should be deemed as possibly new agents joining the set of those who top rank that alternative. This line of reasoning leads us to the following definition of top-monotonicity. We say that an SCF is *top-monotonic* if and only if, for any $m \in \mathbb{N}$ and $\mathscr{R}, \mathscr{R}' \in \mathscr{L}(I_m)^N$ one has $F(\mathscr{R}') = a$ whenever $F(\mathscr{R}) = a$ and $S_a(\mathscr{R}) \subset S_a(\mathscr{R}')$. It will now be shown that a unanimous Θ-self-selective neutral tops-only SCF is top-monotonic.

Proposition 4. *If $F \in \Theta$ is unanimous and Θ-self-selective, then it is top-monotonic.*

Proof. Assume that $F \in \Theta$ is a unanimous Θ-self-selective SCF. Pick $\mathscr{R}, \tilde{\mathscr{R}} \in \mathscr{L}(I_m)^N$, where $m \in \mathbb{N}$, and assume that $F(\mathscr{R}) = a$ and $S_a(\mathscr{R}) \subset S_a(\tilde{\mathscr{R}})$. Let $\bar{\mathscr{R}} = (\mathscr{R}_{S_a(\tilde{\mathscr{R}})}, \tilde{\mathscr{R}}_{N \setminus S_a(\tilde{\mathscr{R}})})$, i.e., $\bar{\mathscr{R}}$ is the preference profile where the agents in $S_a(\tilde{\mathscr{R}})$ are endowed with their preference orderings in $\mathscr{R}$, while the remaining agents in $N \setminus S_a(\tilde{\mathscr{R}})$ are assigned their preference orderings in $\tilde{\mathscr{R}}$. But then $S_a(\bar{\mathscr{R}}) = S_a(\mathscr{R})$. We will first show that $F(\bar{\mathscr{R}}) = a$.

Set $V = \{i \in N \mid \tau(\mathscr{R}_i) \neq \tau(\bar{\mathscr{R}}_i)\}$. If $V = \emptyset$, then clearly $F(\bar{\mathscr{R}}) = a$, since F is tops-only. Now consider the case where $V \neq \emptyset$, and take any $j \in V$ with $\tau(\mathscr{R}_j) = b \neq c = \tau(\bar{\mathscr{R}}_j)$, and $a \neq b$, $a \neq c$. For all $k \in N \setminus S_a(\mathscr{R})$ with $c\mathscr{R}_k a \mathscr{R}_k b$, let $\hat{\mathscr{R}}_k = \mathscr{R}_k^{\sigma_1}$ where $\sigma_1(a) = b$, $\sigma_1(b) = a$ and $\sigma_1(x) = x$ for all $x \in I_m \setminus \{a, b\}$. For any $k \in N \setminus S_a(\mathscr{R})$ with $b\mathscr{R}_k a \mathscr{R}_k c$, on the other hand, let $\sigma_2(a) = c$, $\sigma_2(c) = a$, and $\sigma_2(x) = x$ for all $x \in I_m \setminus \{a, c\}$, and set $\hat{\mathscr{R}}_k = \mathscr{R}_k^{\sigma_2}$. Finally, let $\hat{\mathscr{R}}_k = \mathscr{R}_k$ for all the remaining agents in N. In other words, whenever b and c are separated by a in $\mathscr{R}_k$, we rearrange agent k's preference in such a way that now both b and c are preferred to a, while the restriction of the relevant linear orderings to $\{b, c\}$ remains same. Note that the tops do not change where one passes from $\mathscr{R}$ to $\hat{\mathscr{R}}$, i.e., $\tau(\mathscr{R}_i) = \tau(\hat{\mathscr{R}}_i)$ for all $i \in N$. Thus, $F(\hat{\mathscr{R}}) = a = F(\mathscr{R})$. Also, remembering that $\tau(\hat{\mathscr{R}}_j) = \tau(\mathscr{R}_j) = b$ and so $b \in T(\hat{\mathscr{R}})$, we conclude that $F(\hat{\mathscr{R}}|_{\{a,b\}}) = a$ in view of lemma 1.

We now define $\hat{\mathscr{R}}'$ by letting $\hat{\mathscr{R}}'_i = \hat{\mathscr{R}}_i$ for all $i \in N \setminus \{j\}$, whereas, for any $x, y \in I_m \setminus \{c\}$, we have $c\hat{\mathscr{R}}'_j x$, and $x\hat{\mathscr{R}}'_j y$ iff $x\hat{\mathscr{R}}_j y$. In other words, we obtain $\hat{\mathscr{R}}'$ from $\hat{\mathscr{R}}$ by making c the top-ranked alternative of agent j and leaving everything else unchanged. Note that $\hat{\mathscr{R}}|^{\sigma}_{\{a,b\}} = \hat{\mathscr{R}}'|_{\{a,c\}}$ is a direct consequence of the definitions of $\hat{\mathscr{R}}$ and $\hat{\mathscr{R}}'$, where $\sigma : \{a, b\} \to \{a, c\}$ is defined by $\sigma(a) = a$, and $\sigma(b) = c$. Now since F is neutral and $F(\hat{\mathscr{R}}|_{\{a,b\}}) = a$, we conclude that $F(\hat{\mathscr{R}}'|_{\{a,c\}}) = a$. Now this means that $F(\hat{\mathscr{R}}') \neq c$ according to Lemma 1. But $F(\hat{\mathscr{R}}') \neq b$ either, for otherwise we would have $F(\hat{\mathscr{R}}) = b \neq a$ by proposition 3. Suppose that $F(\hat{\mathscr{R}}') = d \in T(\hat{\mathscr{R}}') \setminus \{a\}$. Now clearly there is some $\hat{\mathscr{R}}'' \in \mathscr{L}(I_m)^N$ such that $\hat{\mathscr{R}}''|_{\{a,d\}} = \hat{\mathscr{R}}|_{\{a,d\}}$ and $\tau(\hat{\mathscr{R}}''_i) = \tau(\hat{\mathscr{R}}'_i)$ for all $i \in N$. But then $F(\hat{\mathscr{R}}'') = F(\hat{\mathscr{R}}') = d$, and thus $F(\hat{\mathscr{R}}''|_{\{a,d\}}) = d$ by Lemma 1. By the same lemma, however, we also have $F(\hat{\mathscr{R}}|_{\{a,d\}}) = a$, a contradiction. Therefore, $F(\hat{\mathscr{R}}') = a$. In the above procedure, we started from $\mathscr{R}$ with $F(\mathscr{R}) = a$ and obtained $\hat{\mathscr{R}}'$ such that $F(\hat{\mathscr{R}}') = a$ and $\tau(\mathscr{R}_i) = \tau(\hat{\mathscr{R}}'_i)$ for all $i \in N \subset \{j\}$.

Now if $k \in V \setminus \{j\}$, then we can find some $\hat{\hat{\mathscr{R}}} \in \mathscr{L}(I_m)^N$ by assigning the role of $\mathscr{R}$ in the above procedure to $\hat{\mathscr{R}}'$ such that $F(\hat{\hat{\mathscr{R}}}) = a$ and $\tau(\hat{\mathscr{R}}'_i) = \tau(\hat{\hat{\mathscr{R}}}_i)$

for all $i \in N \backslash \{k\}$ and $\tau(\hat{\hat{\mathscr{R}}}_k) = \tau(\bar{\mathscr{R}}_k)$, i.e., $\tau(\mathscr{R}_i) = \tau(\hat{\hat{\mathscr{R}}}_i)$ for all $i \in N \backslash \{j, k\}$, but $\tau(\hat{\hat{\mathscr{R}}}_j) = \tau(\bar{\mathscr{R}}_j)$ and $\tau(\hat{\hat{\mathscr{R}}}_k) = \tau(\bar{\mathscr{R}}_k)$. Applying this procedure successively to each agent in V, we end up with some $\mathscr{R}'$ such that $F(\mathscr{R}') = a$ and $\tau(\mathscr{R}'_i) = \tau(\bar{\mathscr{R}}_i)$ for all $i \in N$, implying that $F(\bar{\mathscr{R}}) = a$.

Finally, to show that $F(\tilde{\mathscr{R}}) = a$, take $i \in S_a(\tilde{\mathscr{R}}) \backslash S_a(\mathscr{R})$, if any, and let $\bar{\mathscr{R}}'_t = \bar{\mathscr{R}}_t$ for all $t \in N \backslash \{i\}$ and $\bar{\mathscr{R}}'_i$ be such that $\tau(\bar{\mathscr{R}}'_i) = a$. But then $F(\bar{\mathscr{R}}') = a$ by proposition 3. Applying this procedure successively to every agent in $S_a(\tilde{\mathscr{R}}) \backslash S_a(\mathscr{R})$, we end up with some $\tilde{\tilde{\mathscr{R}}} \in \mathscr{L}(I_m)^N$ such that $F(\tilde{\tilde{\mathscr{R}}}) = a$ and $\tau(\tilde{\tilde{\mathscr{R}}}_t) = \tau(\tilde{\mathscr{R}}_t)$ for all $t \in N$. Therefore, $F(\tilde{\mathscr{R}}) = a$, and hence F is top-monotonic. □

We are now ready to prove that neutral, unanimous and Θ-self-selective SCFs satisfy IIA.

Proposition 5. *If $F \in \Theta$ is unanimous and Θ-self-selective, then F satisfies IIA.*

Proof. Assume that $F \in \Theta$ is unanimous and Θ-self-selective. Let $m \in \mathbb{N}$, $\mathscr{R} \in \mathscr{L}(I_m)^N$, and set $F(\mathscr{R}) = a$. Take $B \subset I_m$ with $a \notin B$, and set $\mathscr{R}' = \mathscr{R}|_{I_m \backslash B}$. Suppose that $F(\mathscr{R}') = b \neq a$. Note that $b \notin T(\mathscr{R})$ in view of Lemma 1 and $a \in T(\mathscr{R}')$. Clearly, $\mathscr{R}|_{\{a,b\}} = \mathscr{R}'|_{\{a,b\}}$, which we will simply denote by $\mathscr{R}''$. Since $F(\mathscr{R}') = b$, we have $F(\mathscr{R}'') \neq a$ again by lemma 1, so that $F(R'') = b$. Now set $K = \{i \in N \backslash S_a(\mathscr{R}) \mid a \mathscr{R}_i b\}$ and $K' = \{i \in N \backslash S_a(\mathscr{R}) \mid b \mathscr{R}_i a\}$. For each $i \in K$, $j \in K'$ let $\hat{\mathscr{R}}_i, \hat{\mathscr{R}}_j \in \mathscr{L}(I_m)$ be such that $L(a, \hat{\mathscr{R}}_i) = L(b, \hat{\mathscr{R}}_j) = I_m$, and set $\hat{\mathscr{R}}_k = \mathscr{R}_k$ for all $k \in S_a(\mathscr{R})$. Clearly, $S_a(\mathscr{R}) \subset S_a(\hat{\mathscr{R}})$, implying that $F(\hat{\mathscr{R}}) = a$ since F is top-monotonic by proposition 4 and $F(\mathscr{R}) = a$. Moreover, the construction of $\hat{\mathscr{R}}$ is such that $\hat{\mathscr{R}}|_{\{a,b\}} = \mathscr{R}''$ and $\tau(\mathscr{R}''_i) = \tau(\hat{\mathscr{R}}_i)$ for all $i \in N$. But then $F(\mathscr{R}'') = F(\hat{\mathscr{R}}) = a$ by Proposition 2 and neutrality of F, contradicting that $F(\mathscr{R}'') = b$. Hence, $F(\mathscr{R}') = a$, i.e., F satisfies IIA. □

There are three different ways two obtain the main result. The first one is a corollary to Proposition 4 via Müller-Satterthwaite Theorem [5]. The second one is a corollary to Proposition 5 through Koray [3] which itself utilizes Arrow's Theorem [1]. Unel [6] contains the third proof which does not utilize any kind of impossibility theorems, and based on proposition 4 and 5. For the sake of brevity, we will consider the first two ways. In both ways, as the said theorems only apply to I_m with $m \geq 3$, the case $m = 2$ is treated separately.

Corollary 2. *Let $F \in \Theta$ be a unanimous SCF. F is Θ-self-selective if and only if F is dictatorial.*

Proof. The "if" part is obvious. Conversely, if F is Θ-self-selective, then F is top-monotonic by proposition 4. Now, for each $m \in \mathbb{N}$, let F_m stand for the restriction of F to $\mathscr{L}(I_m)^N$. Note that F_m is monotonic for each $m \in \mathbb{N}$ and thus dictatorial by Müller-Satterthwaite Theorem [5] whenever $m \geq 3$. That is, for each $m \in \mathbb{N}$, there is some $i_m \in N$ who is a dictator for $F_m : \mathscr{L}(I_m)^N \to I_m$. We will now show that all the dictators i_m for $m \geq 3$ coincide and, moreover, this common agent is also a dictator when $m \in \{1, 2\}$.

First consider $k, \ell \in \mathbb{N}$ with $k > \ell \geq 3$. Let $\mathscr{R} \in \mathscr{L}(I_k)^N$ be such that, for any $t \in I_{k-1}$, $t\mathscr{R}_{i_k}(t+1)$ and $(t+1)\mathscr{R}_j t$ for all $j \in N\backslash\{i_k\}$. Now $F(\mathscr{R}) = F_k(\mathscr{R}) = \tau(\mathscr{R}_{i_k}) = 1$, and, moreover, $F(\mathscr{R}|_{I_\ell}) = 1$ since F satisfies IIA by proposition 5. But since $\tau((\mathscr{R}|_{I_\ell})_j) = \ell \neq 1$ for each $j \in N\backslash\{i_k\}$, this implies that $i_k = i_\ell$.

To show that the same agent is also a dictator for F_2, take any $\mathscr{R} \in \mathscr{L}(I_2)^N$. Define $\tilde{\mathscr{R}} \in \mathscr{L}(I_3)^N$ by letting, for any $i \in N$ and any $x, y \in I_2$, $x\tilde{\mathscr{R}}_i y \Leftrightarrow x\mathscr{R}_i y$, and $x\tilde{\mathscr{R}}_i 3$. But then $\tau(\mathscr{R}_i) = \tau(\tilde{\mathscr{R}}_i)$ for any $i \in N$, so that $F(\tilde{\mathscr{R}}) = F(\mathscr{R})$ by Proposition 2. This, however, simply means that i_3 is also a dictator for F_2. As the same agent is trivially a dictator for F_1 as well, we conclude that F is dictatorial. □

Another way of proving the above corollary is, as mentioned before, by utilizing Theorem 1 in Koray [3] which states that a neutral unanimous SCF is universally self-selective if and only if it is Paretian and satisfies IIA. The conjunction of Corollary 1 and Proposition 5 here implies that a unanimous Θ-self-selective SCF $F \in \Theta$ is Paretian and satisfies IIA, and therefore F is universally self-selective. It is known from Koray [3] again that neutral, unanimous, and universally self-selective SCFs are dictatorial, yielding yet another proof for the above corollary.

4 Concluding remarks

In this paper, we dealt with the question of characterizing self-selective SCFs on the tops-only domain. We showed that, for a neutral, unanimous, and tops-only SCF F, the restriction of rival SCFs used for testing the consistency of F in the sense of self-selectivity to the class Θ of neutral and tops-only SCFs goes unnoticed. In other words, deleting neutral, unanimous SCFs which are not tops-only from the set of potential rivals does not make the self-selectivity test any easier for an SCF in Θ, for again only dictatorial SCFs turn out to pass this consistency test.

It might be illuminating to give an estimation about the relative sizes of the class of unanimous and neutral SCFs and the class of unanimous, neutral and tops-only SCFs. For the case where there are n agents and m alternatives, the number of neutral and unanimous SCFs is computed as $m^{(m^{n-1}-1)((m-1)!)^{n-1}}$ in [3]. For $n = 3$ and $m = 4$, this number is equal to 16^{270}. On the other hand, the number of tops-only and unanimous SCFs can be seen to equal $m^{(m^n-m)}$, which is equal to 16^{30}, for $n = 3$ and $m = 4$. The number of tops-only, unanimous and neutral SCFs is surely much smaller than this number. Note that, the proportion of the number of unanimous, neutral and tops-only SCFs to the number of unanimous and neutral SCFs is smaller than $\left(\frac{1}{16}\right)^{240}$. Thus, even for small values of m, the shrinkage in the size of the set of test functions obtained by confining these to tops-only ones is huge. The deletion of an immense class of SCFs from the test set goes unnoticed regarding self-selectivity.

One natural extension of this paper is to search for other sets of SCFs with which one can escape the dictatoriality result. In a companion paper, Koray and Slinko [4] pursue the research in this line by considering various classes of potential rival SCFs which play the same role as the particular class Θ of neutral and tops-only SCFs here. Starting with any neutral hereditary social choice correspondence π, $\mathscr{F}$ is taken to be a π-complete collection of SCFs in the sense that, for every linear order profile $\mathscr{R}$ and every $a \in \pi(\mathscr{R})$, $\mathscr{F}$ owns an SCF F with $F(\mathscr{R}) = a$. Koray and Slinko [4] show that, in the presence of at least three alternatives, an SCF F (not necessarily in $\mathscr{F}$) is $\mathscr{F}$-self-selective if and only if F is π-dictatorial or π-antidictatorial. π-dictatoriality (π-antidictatoriality) means that there exists some π-dictator (π-antidictator) i in the sense that, for any preference profile $\mathscr{R}$, one has $F(\mathscr{R}) = \arg\max_{\pi(\mathscr{R})} \mathscr{R}_i$ ($F(\mathscr{R}) = \arg\min_{\pi(\mathscr{R})} \mathscr{R}_i$).

By restricting the domain of SCFs to single-peaked preference profiles and modifying the relevant notions appropriately, Unel [6] finds a class of nondictatorial self-selective SCFs. The characterization of the set of the nondictatorial SCFs on the single-peaked preference profiles is yet to be done.

As already mentioned in the introduction, if we allow the social choice rules dealt with to be set-valued, the picture is expected to change quite radically. Although it is not known, thus yet to be found, what universally self-selective SCRs exactly are, Koray [2] answers this question for voting rules which are defined as nonempty-valued neutral top-majoritarian social choice correspondences (SCC), where an SCC is called *top-majoritarian* if and only if, at all linear order profiles with a strict majority of agents top-ranking one and the same alternative, it chooses the singleton consisting of that alternative only. In this context, Koray [2] rediscovers the Condorcet rule as the maximal neutral top-majoritarian and self-selective social choice rule.

The notion of self-selectivity as a novel consistency criterion seems to belong to a class of concepts which are appealing but difficult to achieve, and thus lead to impossibility theorems. On the other hand, the question of whether there exist any further well-established social-choice-theoretic concepts other than the Condorcet rule which possess some version of this kind of consistency as one of their major characteristics sounds interesting, and is yet to be explored.

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