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Some Further Properties of the Accelerated Kerr-Schild Metrics

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We extend the previously found accelerated Kerr-Schild metrics for Einstein-Maxwell-null dust and Einstein-Born-Infeld-null dust equations to the cases including the cosmological constant. This way we obtain the generalization of the charged de Sitter metrics in static space-times. We also give a generalization of the zero acceleration limit of our previous Einstein-Maxwell and Einstein-Born-Infeld solutions.

KEY WORDS: Classical general relativity; exact solutions; differential geometry.

1. INTRODUCTION

Using a curve C in D-dimensional Minkowski space-time M_D, we have recently studied the Einstein-Maxwell-null dust [1] and Einstein-Born-Infeld-null dust field equations [2], Yang-Mills equations [3], and Liénard-Wiechert potentials in even dimensions [4]. In the first three works we found some new solutions generalizing the Tangherlini [5], Plebański [6], and Trautman [7] solutions, respectively. The last one proves that the accelerated scalar or vector charged particles in even dimensions lose energy. All of the solutions contain a function c which is assumed to depend on the retarded time τ_0 and all accelerations a_k, $(k = 0, 1, 2, \cdots)$, see [1–4]. We also assumed that when the motion is uniform or the curve C is a straight line in M_D, this function reduces to a function depending only on the retarded time τ_0. In this work we first relax this assumption and give the most

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general form of the function c when the curve C is a straight line. In addition, we also generalize our previous accelerated Kerr-Schild metrics by including the cosmological constant in arbitrary D-dimensions for the Einstein-Maxwell and in four dimensions for the Einstein-Born-Infeld theories. The solutions presented here can be interpreted as the solutions of the Einstein-Maxwell-null perfect fluid field equations with a constant pressure Λ or Einstein-Maxwell-null dust field equations with a cosmological constant Λ. In our treatment we adopt the second interpretation.

Our conventions are similar to the conventions of our earlier works [1, 2, 4]. In a D-dimensional Minkowski space-time M_D, we use a parameterized curve $C = \{x^\mu \in M_D : x^\mu = z^\mu(\tau)\,,\, \mu = 0, 1, 2, \cdots, D-1\,,\, \tau \in I\}$ such that τ is a parameter of the curve and I is an interval on the real line $\mathbb{R}$. We define the world function Ω as

$$\Omega = \eta_{\mu\nu}\,(x^\mu - z^\mu(\tau))\,(x^\nu - z^\nu(\tau)), \tag{1}$$

where x^μ is a point not on the curve C. There exists a point $z^\mu(\tau_0)$ on the non-spacelike curve C which is also on the light cone with the vertex located at the point x^μ, so that $\Omega(\tau_0) = 0$. Here τ_0 is the retarded time. By using this property we find that

$$\lambda_\mu \equiv \partial_\mu \tau_0 = \frac{x^\mu - z^\mu(\tau_0)}{R} \tag{2}$$

where $R \equiv \dot{z}^\mu(\tau_0)\,(x_\mu - z_\mu(\tau_0))$ is the retarded distance. Here a dot over a letter denotes differentiation with respect to τ_0. It is easy to show that λ_μ is null and satisfies

$$\lambda_{\mu,\nu} = \frac{1}{R}\,[\eta_{\mu\nu} - \dot{z}_\mu\,\lambda_\nu - \dot{z}_\nu\,\lambda_\mu - (A-\epsilon)\,\lambda_\mu\,\lambda_\nu] \tag{3}$$

where $A \equiv \ddot{z}^\mu\,(x_\mu - z_\mu(\tau_0))$ and $\dot{z}^\mu\,\dot{z}_\mu = \epsilon = -1, 0$. Here $\epsilon = -1$ and $\epsilon = 0$ correspond to the time-like and null velocity vectors, respectively. One can also show explicitly that $\lambda^\mu\,\dot{z}_\mu = 1$ and $\lambda^\mu R_{,\,\mu} = 1$. Define $a \equiv \frac{A}{R} = \lambda^\mu \ddot{z}_\mu$, then

$$\lambda^\mu\, a_{,\,\mu} = 0. \tag{4}$$

Furthermore defining (letting $a_0 = a$)

$$a_k \equiv \lambda_\mu\,\frac{d^{k+2}\,z^\mu(\tau_0)}{d\tau_0^{k+2}}, \qquad k = 0, 1, 2, \cdots \tag{5}$$

one can show that

$$\lambda^\mu\, a_{k,\,\mu} = 0, \qquad \forall k = 0, 1, 2, \cdots . \tag{6}$$

Hence any function c satisfying

$$\lambda^\mu\, c_{,\,\mu} = 0, \tag{7}$$

has arbitrary dependence on all a_k's and τ_0. Using the above curve kinematics, we showed that Einstein-Maxwell-perfect fluid equations with the Kerr-Schild metric give us the following result

Proposition 1. Let the space-time metric and the electromagnetic vector potential be respectively given by $g_{\mu\nu} = \eta_{\mu\nu} - 2V\lambda_\mu\lambda_\nu,\ A_\mu = H\,\lambda_\mu$, where V and H are some differentiable functions in M_D. Let V and H depend on R, τ_0 and functions c_i ($i = 1, 2, \cdots$) that satisfy (7), then the Einstein equations reduce to the following set of equations [see [1] for details]

$$\kappa p + \Lambda = \frac{1}{2}V'' + \frac{3D-8}{2R}V' + \frac{(D-3)^2}{R^2}V, \tag{8}$$

$$\kappa (H')^2 = V'' + \frac{D-4}{R}V' - \frac{2V}{R^2}(D-3), \tag{9}$$

$$\begin{aligned}\kappa(p+\rho) &= q - \kappa\eta^{\alpha\beta}\, H_{,\alpha} H_{,\beta} \\ &\quad + 2\left[\frac{2(A-\epsilon)(D-3)V}{R^2} - \sum_{i=1}\left(w_i\, c_{i,\alpha}\, \dot{z}^\alpha\right)\right],\end{aligned} \tag{10}$$

$$\sum_{i=1} w_i\, c_{i,\alpha} = \left[\sum_{i=1}\left(w_i\, c_{i,\beta}\, \dot{z}^\beta\right)\right]\lambda_\alpha, \tag{11}$$

where

$$w_i = V'_{,c_i} + \frac{D-4}{R}\,V_{,c_i} - \kappa H' H_{,c_i}, \tag{12}$$

and prime over a letter denotes partial differentiation with respect to R. Here κ is the gravitational constant, p and ρ are, respectively, the pressure and energy density of the fluid, Λ is the cosmological constant and the function q is defined by

$$q = \eta^{\alpha\beta}\, V_{,\alpha\beta} - \frac{4}{R}\dot{z}^\alpha\, V_{,\alpha} + 2(\epsilon - A)\,\frac{\lambda^\alpha V_{,\alpha}}{R} + [2\epsilon(-D+3) + 2A(D-2)]\,\frac{V}{R^2}.$$

Please refer to [1] for this Proposition.

For the case of the Einstein-Born-Infeld field equations with similar assumptions, we have the following proposition (please see [2] for the details of the Proposition)

Proposition 2. Let V and H depend on R, τ_0 and functions c_i ($i = 1, 2, \cdots$) that satisfy (7), then the Einstein equations reduce to the following set of equations

$$\kappa p + \Lambda = V'' + \frac{2}{R}V' - \kappa b^2\,[1 - \Gamma_0], \tag{13}$$

$$\kappa\,\frac{(H')^2}{\Gamma_0} = V'' - \frac{2V}{R^2}, \tag{14}$$

$$\kappa(p+\rho) = \sum_{i=1} \left[V_{,c_i} \left(c_{i,\alpha}{}^{,\alpha} \right) - \frac{4}{R} V_{,c_i} \left(c_{i,\alpha} \dot{z}^{\alpha} \right) \right.$$
$$\left. - \frac{\kappa}{\Gamma_0} (H_{,c_i})^2 \left(c_{i,\alpha}\, c_i{}^{,\alpha} \right) \right] - \frac{2A}{R} \left(V' - \frac{2V}{R} \right), \qquad (15)$$

$$\sum_{i=1} w_i\, c_{i,\alpha} = \left[\sum_{i=1} \left(w_i\, c_{i,\beta}\, \dot{z}^{\beta} \right) \right] \lambda_{\alpha}, \qquad (16)$$

where

$$w_i = V'_{,c_i} - \frac{\kappa H'}{\Gamma_0} H_{,c_i}, \qquad \Gamma_0 = \sqrt{1 - \frac{(H')^2}{b^2}}, \qquad (17)$$

and prime over a letter denotes partial differentiation with respect to R. Here $\kappa (= 8\pi)$ is the gravitational constant, p and ρ are, respectively the pressure and the energy density of the fluid, b is the Born-Infeld parameter (as $b \to \infty$ one arrives at the Maxwell limit), and Λ is the cosmological constant.

2. NULL-DUST EINSTEIN-MAXWELL SOLUTIONS IN *D*-DIMENSIONS WITH A COSMOLOGICAL CONSTANT

In this section we assume zero pressure. Due to the existence of the cosmological constant Λ, one may consider this as if there is a constant pressure as the source of the field equations. We shall not adopt this interpretation. Instead, we think of this as if there is a null dust, a Maxwell field and a cosmological constant as the source of the Einstein field equations. Hence assuming that the null fluid has no pressure in Proposition 1, we have the following result:

Proposition 3. Let $p = 0$. Then

$$V = \begin{cases} \frac{\kappa e^2 (D-3)}{2(D-2)} R^{-2D+6} + m R^{-D+3} + \frac{\Lambda}{(D-2)(D-1)} R^2, & (D \geq 4) \\ -\frac{\kappa}{2} e^2 \ln R + m + \frac{\Lambda}{2} R^2, & (D = 3) \end{cases} \qquad (18)$$

$$H = \begin{cases} c + \epsilon e R^{-D+3}, & (D \geq 4) \\ c + \epsilon e \ln R. & (D = 3) \end{cases} \qquad (19)$$

The explicit expressions of the energy density ρ and the current vector J_{μ} do not contain the cosmological constant Λ and are identical with the ones given in [1], so we don't rewrite those long formulas here.

Here $M = m + \epsilon\, \kappa (3-D) ec$ for $D \geq 4$ and $M = m + \frac{\kappa}{2} e^2 + \epsilon\, \kappa ec$ for $D = 3$. In all cases e is assumed to be a function of τ_0 only but the functions m and c which are related through the arbitrary function $M(\tau_0)$ (depends on τ_0 only) do depend on the scalars a_k $(k \geq 0)$. Of course Λ is any real number.

In Proposition 3 we have chosen the integration constants (R independent functions) as the functions c_i $(i = 1, 2, 3)$ so that $c_1 = m$, $c_2 = e$, $c_3 = c$ and

$$c = c(\tau_0, a_k), \quad e = e(\tau_0), \tag{20}$$

$$m = \begin{cases} M(\tau_0) + \epsilon\,\kappa(D-3)ec, & (D \geq 4) \\ M(\tau_0) - \frac{\kappa}{2}\,e^2 - \epsilon\,\kappa ec, & (D = 3) \end{cases} \tag{21}$$

where a_k's are defined in (5).

Remark 1. We can have pure null dust solutions when $e = 0$. The function c in this case can be gauged away, that is we can take $c = 0$. The Ricci tensor takes its simplest form $R_{\mu\nu} = \rho\lambda_\mu\,\lambda_\nu + \Lambda\,g_{\mu\nu}$ then. In this case we have

$$V = mR^{3-D} + \frac{\Lambda}{(D-2)(D-1)}\,R^2,$$
$$\rho = \frac{2-D}{\kappa}\,[a(1-D)m + \dot{m}]\,R^{2-D} \tag{22}$$

for $D \geq 4$ and

$$V = m + \frac{\Lambda}{2}\,R^2,$$
$$\rho = \frac{2ma - \dot{m}}{\kappa R} \tag{23}$$

for $D = 3$. Such solutions are usually called as the *Photon Rocket* solutions [8, 9]. We give here the D dimensional generalizations of this type of metrics with a cosmological constant.

Remark 2. If $e = m = 0$ we obtain a metric

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{2\Lambda\,R^2}{(D-1)(D-2)}\,\lambda_\mu\,\lambda_\nu, \quad (D \geq 3)$$

which clearly corresponds to the D-dimensional de Sitter space.

Remark 3. The static limit $a_0 \equiv a = 0$ of our solutions with a constant c are the charged Tangherlini solutions with a cosmological constant. If the function c is not chosen to be a constant, we obtain their generalizations (see Section 4).

3. NULL-DUST EINSTEIN-BORN-INFELD SOLUTIONS IN 4-DIMENSIONS WITH A COSMOLOGICAL CONSTANT

Using Proposition 2, and assuming zero pressure, we find the complete solution of the field equations.

Proposition 4. Let $p = 0$. Then

$$V = \frac{m}{R} - 4\pi e^2 \frac{F(R)}{R} + \frac{\Lambda}{6} R^2, \tag{24}$$

$$H = c - \epsilon\, e \int^R \frac{dR}{\sqrt{R^4 + e^2/b^2}}, \tag{25}$$

where

$$m = M(\tau_0) + 8\pi \epsilon e c, \tag{26}$$

$$F(R) = \int^R \frac{dR}{R^2 + \sqrt{R^4 + e^2/b^2}}. \tag{27}$$

Here e is assumed to be a function of τ_0 only but the functions m and c which are related through the arbitrary function $M(\tau_0)$ do depend on the scalars a_k, $(k \geq 0)$.

We have chosen the integration constants (R independent functions) as the functions c_i $(i = 1, 2, 3)$ so that $c_1 = m$, $c_2 = e$, $c_3 = c$ and

$$c = c(\tau_0, a_k), \quad e = e(\tau_0), \quad m = M(\tau_0) + 8\pi \epsilon e c.$$

Remark 4. In the static limit with a constant c, we obtain the Plebański solution with a cosmological constant [10]. If the function c is not a constant, we can also give a class of solutions of the Einstein-Born-Infeld-null dust equations with a cosmological constant (see Section 4).

4. STRAIGHT LINE LIMITS

When the accelerations a_k $(k \geq 0)$ vanish, the curve C is a straight line in M_D. In this limit we have the following: $\tau_0 = t - r$, $z^\mu = n^\mu\, \tau_0$, $n^\mu \equiv (1, 0, 0, 0)$, $\dot{z}^\mu = n^\mu$ and $R = -r$. Moreover,

$$x^\mu = (t, \vec{x}), \quad \lambda_\mu = \left(1, -\frac{\vec{x}}{r}\right), \quad r^2 = \vec{x} \cdot \vec{x}. \tag{28}$$

In this case the function c arising in the metrics introduced in the previous sections can be assumed to depend on some other functions $\xi_{(I)}$ so that $\lambda^\mu\, \xi_{(I),\mu} = 0$ $(I = 1, 2, \cdots, D-2)$ [11]. As an example let $\xi_{(I)} = \vec{l}_{(I)} \cdot \frac{\vec{x}}{r}$, where $\vec{l}_{(I)}$ are constant vectors. It is easy to show that $\lambda^\mu\, \xi_{(I),\,\mu} = 0$. Hence in this (straight line) limit we assume that $c = c(\tau_0, \xi_{(I)})$. From this simple example we may define more general functions satisfying our constraint equation $c_{,\mu}\, \lambda^\mu = 0$. Let X_μ be a vector satisfying

$$X_{\mu,\,\nu} = b_0\, \eta_{\mu\nu} + b_1(k_\mu\, \lambda_\nu + k_\nu\, \lambda_\mu) + b_2\, \lambda_\mu\, \lambda_\nu + Q_{\mu\nu}, \tag{29}$$

where b_0, b_1, b_2 are some arbitrary functions, k_μ is any vector and $Q_{\mu\nu}$ is any antisymmetric tensor in M_D. Then any vector X_μ satisfying (29) defines a scalar $\xi = \lambda^\mu X_\mu$ so that $\lambda^\mu \xi_{,\,\mu} = 0$.

The simple example given at the beginning of this section corresponds to a constant vector, $X_\mu = l_\mu = (l_0, \vec{l})$. Hence, ξ becomes a function of the spherical angles. For instance, in four dimensions, $\xi = l_0 + l_1 \cos\phi \sin\theta + l_2 \sin\phi \sin\theta + l_3 \cos\theta$ where l_0, l_1, l_2, and l_3 are the constant components of the vector l_μ. In the straight line limit, the metric can be transformed easily to the form

$$ds^2 = -(1+2V)\,dT^2 + \frac{1}{1+2V}\,dr^2 + r^2\,d\Omega^2_{D-2}, \tag{30}$$

where

$$dT = dt - \frac{2V\,dr}{1+2V},$$

and $d\Omega^2_{D-2}$ is the metric on the $D-2$-dimensional unit sphere. The above form of the metric is valid both for the Einstein-Maxwell and for the Einstein-Born-Infeld theories. For the case of the Einstein-Maxwell-null dust with a cosmological constant we have

$$V = \begin{cases} \frac{\kappa e^2\,(D-3)}{2(D-2)}\,r^{-2D+6} + m(-1)^{D+1}\,r^{-D+3} + \frac{\Lambda}{(D-2)(D-1)}\,r^2, & (D \geq 4) \\ m - \frac{\kappa}{2} e^2 \ln r + \frac{\Lambda}{2}\,r^2, & (D = 3) \end{cases} \tag{31}$$

and the function H defining the electromagnetic vector potential is given by

$$H = \begin{cases} c + \epsilon\, e\,(-1)^{D+1}\,r^{-D+3}, & (D \geq 4) \\ c + \epsilon e \ln r. & (D = 3) \end{cases} \tag{32}$$

This solution is a generalization of the Tangherlini solution [5]. The relationship between c and m are given in (21), but in this case the function c is a function of the scalars $\xi_{(I)}$ as discussed in the first part of this section. For the case of the Einstein-Born-Infeld-null dust with a cosmological constant, we have

$$V = -\frac{m}{r} + 4\pi e^2 \frac{F(r)}{r} + \frac{\Lambda}{6}\,r^2, \tag{33}$$

$$H = c + \epsilon\, e \int^r \frac{dr}{\sqrt{r^4 + e^2/b^2}} \tag{34}$$

where

$$m = M(\tau_0) + 8\pi\epsilon e c, \tag{35}$$

$$F(r) = -\int^r \frac{dr}{r^2 + \sqrt{r^4 + e^2/b^2}}. \tag{36}$$

This solution is a generalization of the Plebański et al. [6] static solution of the Einstein-Born-Infeld theory. Our generalization is with the function c depending arbitrarily on the scalars $\xi_{(I)}$.

Remark 5. When the function c is not a constant, the mass m defined through the relations (21) or (35) is not a constant anymore, it depends on the angular coordinates.

5. CONCLUSION

We have reexamined the accelerated Kerr-Schild geometries for two purposes. One of them is to generalize our earlier solutions of Einstein-Maxwell-null dust [1] and Einstein-Born-Infeld-null dust field equations [2] by including a cosmological constant. The other one is to generalize the static limit (straight line limit) of the above mentioned solutions. Previously in the static limit, we were assuming the function c to be a constant. As long as this function satisfies the condition $\lambda^{\mu} c_{,\mu} = 0$, as we have seen in Section 4 (although the acceleration scalars a_k $(k \geq 0)$ are all zero) we can still obtain the generalization of the charged Tangherlini [5] and Plebański [6] solutions.

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