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BERGMAN PROJECTIONS ON BESOV SPACES ON BALLS

H. TURGAY KAPTANOĞLU

ABSTRACT. Extended Bergman projections from Lebesgue classes onto all Besov spaces on the unit ball are defined and characterized. Right inverses and adjoints of the projections share the property that they are imbeddings of Besov spaces into Lebesgue classes via certain combinations of radial derivatives. Applications to the Gleason problem at arbitrary points in the ball, duality, and complex interpolation in Besov spaces are obtained. The results apply, in particular, to the Hardy space H^2, the Arveson space, the Dirichlet space, and the Bloch space.

1. Introduction

The inner product and the norm in $\mathbb{C}^N$ are $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$ and $|z| = \sqrt{\langle z, z\rangle}$, where $\overline{(\ )}$ denotes the complex conjugate (or the closure of a set if the context requires it). We let ν be the Lebesgue (volume) measure on the unit ball $\mathbb{B}$ of $\mathbb{C}^N$ normalized with $\nu(\mathbb{B}) = 1$, which is the area measure on the unit disc $\mathbb{D}$ when $N = 1$. We define on $\mathbb{B}$ also the measures

$$d\nu_c(z) = (1 - |z|^2)^c \, d\nu(z) \qquad (c \in \mathbb{R}),$$

which are finite only when $c > -1$. Unless otherwise specified or restricted, our main parameters are the following:

$$q \in \mathbb{R}, \quad 0 < p \leq \infty, \quad s \in \mathbb{C}, \quad \sigma = \operatorname{Re} s, \quad t \in \mathbb{C}, \quad \tau = \operatorname{Re} t.$$

Let $H(\mathbb{B})$ denote the space of holomorphic functions on $\mathbb{B}$. For $q > -1$, a function $f \in H(\mathbb{B})$ belongs to the (*weighted*) *Bergman space* A^p_q whenever f lies in the Lebesgue class $L^p(\nu_q)$. The norm $\|f\|_{A^p_q}$ is simply the $L^p(\nu_q)$ norm of f, where we use the term norm even when $0 < p < 1$. So the inclusion map $i : A^p_q \to L^p(\nu_q)$ is an isometric imbedding.

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Bergman projections are the linear operators P_s defined for $\sigma > -(N+1)$ by

$$P_s f(z) = \int_{\mathbb{B}} \frac{(1-|w|^2)^s}{(1-\langle z,w\rangle)^{N+1+s}}\, f(w)\, d\nu(w) \qquad (z \in \mathbb{B})$$

for suitable f. It is clear that $P_s f$ is a member of $H(\mathbb{B})$. Complex powers are always understood to be principal branches.

The following result is classical; see [FR], [C], and [HKZ, §1.2], for example.

Theorem 1.1. *For $1 \le p < \infty$, P_s is a bounded operator from $L^p(\nu_q)$ onto A^p_q if and only if*

$$q + 1 < p\,(\sigma + 1). \tag{1}$$

For such a value s,

$$(P_s \circ i) f = \frac{N!}{(1+s)_N}\, f \qquad (f \in A^p_q). \tag{2}$$

The inequality (1) implies $\sigma > -1$ since $q > -1$ for Bergman spaces. The expression $(a)_b$ in (2) is the *Pochhammer symbol* given by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)} \tag{3}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ.

Besov spaces extend weighted Bergman spaces to all q. To define them, we first take a radial differential operator D^t_s of order t and consider the linear transformation I^t_s defined for $f \in H(\mathbb{B})$ by

$$I^t_s f(z) = (1-|z|^2)^t\, D^t_s f(z).$$

We say a function $f \in H(\mathbb{B})$ belongs to the *Besov space* B^p_q whenever $I^t_s f$ lies in $L^p(\nu_q)$ for some s, t satisfying

$$\begin{cases} q + p\,\tau > -1 & (0 < p < \infty), \\ \tau > 0 & (p = \infty). \end{cases} \tag{4}$$

The $L^p(\nu_q)$ norm of any one of the functions $I^t_s f$ can be used as an equivalent norm for $\|f\|_{B^p_q}$. It turns out that $B^p_q = A^p_q$ for $q > -1$.

We also need an extended notion of Bergman projections in order to be able to handle all q. Consider the kernel

$$H_s(\lambda) = \begin{cases} \dfrac{1}{(1-\lambda)^{N+1+s}} = \displaystyle\sum_{k=0}^{\infty} \frac{(N+1+s)_k}{k!}\,\lambda^k, & \text{if } \sigma > -(N+1), \\[2ex] \dfrac{{}_2F_1(1,1;1-N-s;\lambda)}{-N-s} = \displaystyle\sum_{k=0}^{\infty} \frac{k!\,\lambda^k}{(-N-s)_{k+1}}, & \text{if } \sigma \le -(N+1), \end{cases} \tag{5}$$

where ${}_2F_1$ is the hypergeometric function; see [BB, p. 13]. With no restriction on s, we define the (*extended*) *Bergman projections*, also denoted by P_s, as

$$P_s f(z) = \int_{\mathbb{B}} H_s(\langle z, w\rangle)\,(1-|w|^2)^s\, f(w)\, d\nu(w) \qquad (z \in \mathbb{B}).$$

Our main result is the following generalization of Theorem 1.1.

THEOREM 1.2. *For $1 \le p \le \infty$, P_s is a bounded operator from $L^p(\nu_q)$ onto B^p_q if and only if*

$$\begin{cases} q+1 < p\,(\sigma+1) & (1 \le p < \infty), \\ \sigma > -1 & (p = \infty). \end{cases} \tag{6}$$

Given a number s satisfying (6), *if t satisfies* (4), *then*

$$(P_s \circ I^t_s) f = \frac{N!}{(1+s+t)_N}\, f = C_{st}\, f \qquad (f \in B^p_q). \tag{7}$$

Note that (6) no longer implies $\sigma > -1$. On the other hand, (6) and (4) together imply $\sigma + \tau > -1$ so that $(1+s+t)_N$ never hits a pole of Γ.

The best partial result in this direction is [P, Theorem 3.11], in which s is restricted to $s > -1$; then $s > -(N+1)$ trivially and only the binomial part of the kernel (5) is used. Consequently the only-if part is also missing. The same restriction on s applies also to the right inverses given for P_s. A very special case of (7) is [Z2, Lemma 4.2.8]. Although H_s appears in [BB] in its full generality, this source considers only projections from differentiable classes onto Besov spaces.

We prove Theorem 1.2 in Section 5. Our proof is entirely different from that of [P]. We emphasize that Theorem 1.2 generalizes Theorem 1.1 also in the sense that i in (2) can be replaced by the more general I^t_s even when $q > -1$. We also find the adjoint of P_s in this section.

Section 4 is devoted to properties of Besov spaces that place the extended Bergman projections in context. This is required partly because the q we use in the definition of B^p_q is not standard. It is shown that $H_q(\langle z, w\rangle)$ is the reproducing kernel of B^2_q. The independence of B^p_q of the parameters s, t under (4) and their relation to Bergman and other spaces are established. Equation (2) in [Kap], where some of the results in this paper are announced, graphically shows the classification of B^p_q with our q.

The operators D^t_s are introduced in Section 3. They are defined by using coefficient multipliers on the homogeneous expansions of functions in $H(\mathbb{B})$.

Section 2 introduces the notation and some preliminary formulas.

Sections 6, 7, and 8 give applications of Theorem 1.2. First we solve the Gleason problem at an arbitrary point $a \in \mathbb{B}$ in Besov spaces. Then we study the duality of B^p_q spaces under some new pairings. Finally we investigate complex interpolation in these spaces and identify some linear operators that leave them invariant.

Proofs of a few technical results are deferred to the Appendix. Particular cases of our results refer to the Hardy space $H^2 = B^2_{-1}$, the Arveson space B^2_{-N}, and the Dirichlet space $B^2_{-(N+1)}$.

2. Notation and preliminaries

Constants appearing in formulas are all denoted by C, although they may have different values. The constants may depend on various parameters, but never on the functions in the formula in which they appear.

For fixed a, b, Stirling's formula and (3) give

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \quad (c)_a \sim c^a, \quad \frac{(a)_c}{(b)_c} \sim c^{a-b} \quad (\operatorname{Re} c \to \infty), \tag{8}$$

where $x \sim y$ means that $|x/y|$ is bounded above and below by two positive constants that are independent of any parameter present (c here).

For $1 \leq p < \infty$, the symbol p' denotes the exponent conjugate to p; that is, $1/p + 1/p' = 1$. The *dual space* X^* of a Banach space X is the space of all bounded linear functionals on X.

We use multi-index notation in which $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ is an N-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$, and $0^0 = 1$. Then

$$H_s(\langle z, w\rangle) = \begin{cases} \displaystyle\sum_\alpha \frac{(N+1+s)_{|\alpha|}}{\alpha!}\, z^\alpha \overline{w}^\alpha, & \text{if } \sigma > -(N+1), \\ \displaystyle\sum_\alpha \frac{(|\alpha|!)^2}{\alpha!\,(-N-s)_{|\alpha|+1}}\, z^\alpha \overline{w}^\alpha, & \text{if } \sigma \leq -(N+1). \end{cases} \tag{9}$$

When $s = -(N+1)$, (9) sums to

$$H_{-(N+1)}(\langle z, w\rangle) = \sum_{k=0}^\infty \frac{1}{k+1}\, \langle z, w\rangle^k = \frac{1}{\langle z, w\rangle}\, \log \frac{1}{1 - \langle z, w\rangle}. \tag{10}$$

By (8), the coefficient of λ^k in (5) is $\sim k^{N+\sigma}$ for large k. Thus (5) converges, in particular, when $\lambda = \langle z, w\rangle$ with $z, w \in \mathbb{B}$.

Let Σ be the Lebesgue (surface) measure on the boundary $\partial\mathbb{B}$ of $\mathbb{B}$ normalized so that $\Sigma(\partial\mathbb{B}) = 1$. The following result extends [R, Proposition 1.4.9] to $p \neq 2$ and $q \neq 0$. Its proof follows similar lines and is omitted.

Proposition 2.1. *For a multi-index α, $0 < p < \infty$, and $\sigma > -1$, we have*

$$\int_{\partial\mathbb{B}} |\zeta^\alpha|^p \, d\Sigma(\zeta) = \frac{(N-1)!\, \prod_{j=1}^N \Gamma(1 + \alpha_j p/2)}{\Gamma(N + |\alpha| p/2)}$$

and

$$\int_{\mathbb{B}} |z^\alpha|^p \, (1-|z|^2)^s \, d\nu(z) = \frac{N! \, \Gamma(1+s) \, \prod_{j=1}^N \Gamma(1+\alpha_j p/2)}{\Gamma(N+1+s+|\alpha|p/2)}.$$

Remark 2.2. The case $p=2$ of the second integral is [AK1, Lemma 1]. A similar orthogonality result, $\int_{\mathbb{B}} z^\alpha \overline{z}^\beta \, (1-|z|^2)^s \, d\nu(z) = 0$ if $\alpha \neq \beta$, appears in [FR, Proposition 2.4].

Proposition 2.3. *For any $c \in \mathbb{R}$, $L^\infty(\nu_c) = L^\infty(\nu)$.*

Proof. It suffices to show that the null sets of the measures ν_c and ν are the same. Note that ν_c is σ-finite. We have $d\nu(z) = (1-|z|^2)^{-c} \, d\nu_c(z)$ with $z \mapsto (1-|z|^2)^{-c}$ integrable with respect to ν_c, and $d\nu_c(z) = (1-|z|^2)^c \, d\nu(z)$ with $z \mapsto (1-|z|^2)^c$ locally integrable with respect to ν. Since neither measure has atoms, it follows that either measure is absolutely continuous with respect to the other. □

Now for $a, b \in \mathbb{C}$ and suitable g, consider the operator

$$V_b^a g(z) = (1-|z|^2)^a \int_{\mathbb{B}} \frac{(1-|w|^2)^b}{(1-\langle z, w\rangle)^{N+1+a+b}} \, g(w) \, d\nu(w).$$

Theorem 2.4.

(a) *For $1 \leq p \leq \infty$, V_b^a is bounded on $L^p(\nu_c)$ if and only if*

$$-\operatorname{Re} a < \frac{c+1}{p} < \operatorname{Re} b + 1.$$

(b) *For $0 < p \leq 1$, if*

$$0 < \operatorname{Re} b + 1 - N\left(\frac{1}{p} - 1\right) \quad \text{and} \quad -\operatorname{Re} a < \frac{c+1}{p} < \operatorname{Re} b + 1 - N\left(\frac{1}{p} - 1\right),$$

then V_b^a is a continuous map from $L^p(\nu_c) \cap H(\mathbb{B})$ to $L^p(\nu_c)$.

Proof. (a) This is essentially [HKZ, Theorem 1.9] for $1 \leq p < \infty$ and contained in [Z3, Theorem 9] for $p = \infty$.

(b) See the Appendix. □

For $\sigma > -(N+1)$, clearly P_s is the operator V_s^0. Then the first part of Theorem 1.1 follows immediately from Theorem 2.4 (a).

Remark 2.5. Theorem 2.4 is true also for *operators of type V_b^a* that have the form

$$Mg(z) = \int_{\mathbb{B}} k(z,w) \, |g(w)| \, d\nu(w),$$

where $k(z,w)$ is a measurable kernel satisfying

$$|k(z,w)| \leq C \, \frac{(1-|z|^2)^{\operatorname{Re} a} \, (1-|w|^2)^{\operatorname{Re} b}}{|1-\langle z,w\rangle|^{N+1+\operatorname{Re} a+\operatorname{Re} b}}.$$

3. Radial differential operators

Let $f \in H(\mathbb{B})$ be given by its *homogeneous expansion* $f(z) = \sum_{k=0}^{\infty} f_k(z)$, where f_k is a homogeneous polynomial of degree k. The *radial derivative* at z of f is

$$Rf(z) = \sum_{m=1}^{N} z_m \frac{\partial f}{\partial z_m}(z) = \sum_{k=1}^{\infty} k\, f_k(z).$$

In particular, $R(z^\alpha) = |\alpha| z^\alpha$ and $R(\langle z, w\rangle) = \langle z, w\rangle$, where R acts on the holomorphic variable z. What is nice about Rf is that it is also holomorphic and dominates the derivatives of f in tangential directions; see [R, §6.4]. By imitating the passage across $\sigma = -(N+1)$ in (5) and following [AU, §3], we extend R to arbitrary orders.

Definition 3.1. Let $f \in H(\mathbb{B})$. We define $D_s^t f = \sum_{k=0}^{\infty} {}_s^t d_k\, f_k$, where

$${}_s^t d_k = \begin{cases} \dfrac{(N+1+s+t)_k}{(N+1+s)_k}, & \text{if } \sigma > -(N+1),\ \sigma+\tau > -(N+1), \\ \dfrac{(N+1+s+t)_k\,(-(N+s))_{k+1}}{(k!)^2}, & \text{if } \sigma \le -(N+1),\ \sigma+\tau > -(N+1), \\ \dfrac{(k!)^2}{(N+1+s)_k\,(-(N+s+t))_{k+1}}, & \text{if } \sigma > -(N+1),\ \sigma+\tau \le -(N+1), \\ \dfrac{(-(N+s))_{k+1}}{(-(N+s+t))_{k+1}}, & \text{if } \sigma \le -(N+1),\ \sigma+\tau \le -(N+1). \end{cases}$$

Clearly, $D_s^t f \in H(\mathbb{B})$, $D_s^0 = I$, $D_s^t(1) = 1$,

$$D_{-N}^1 = R + I, \qquad {}_s^t d_k \neq 0, \qquad \text{and} \qquad D_s^t(z^\alpha) = {}_s^t d_{|\alpha|} z^\alpha. \tag{11}$$

Moreover, by (8),

$${}_s^t d_k \sim k^\tau \qquad (k \to \infty). \tag{12}$$

Theorem 3.2. *Any D_s^t is a continuous operator on $H(\mathbb{B})$.*

Proof. This is one direction of [Ara, Theorem 5], using the estimate (12). □

Hence identities for D_s^t can be proved by checking their action on z^α since $\{z^\alpha\}$ generates $H(\mathbb{B})$. So by (5) we have the important identity

$$D_s^t H_s(\langle z, w\rangle) = H_{s+t}(\langle z, w\rangle), \tag{13}$$

where D_s^t acts on the holomorphic variable z. The properties (11), (12), and (13) allow us to state the following result.

Proposition 3.3. *Any D_s^t is a radial differential operator of order t.*

The parameter s does not affect the order and is there for convenience in the proofs. These operators are truly differential when t is a positive integer, and integral when t is negative. By (11), any D_s^t is a bijection on $H(\mathbb{B})$ and thus invertible. A case by case checking as in Definition 3.1 yields that

$$D_{s+t}^r D_s^t = D_s^{t+r}. \tag{14}$$

This formula for $s > -1$, $s+t > -1$, and $r+s+t > -(N+1)$ appears in [P, Remark 3.6 (b)]. The two-sided inverse of D_s^t is obtained by taking $D_s^0 = I$ on the right in (14). Therefore

$$(D_s^t)^{-1} = D_{s+t}^{-t}. \tag{15}$$

Consequently, if $f \in H(\mathbb{B})$ and $D_s^t f \equiv 0$ or $I_s^t \equiv 0$ for some s, t, then $f \equiv 0$.

Let us note in passing that the operators $R_a = (R - aI)^{-1} = -a^{-1} D_{-(N+a)}^{-1}$ satisfy the resolvent equation $R_a - R_b = (a-b) R_a R_b$.

THEOREM 3.4. *Suppose b satisfies $\operatorname{Re} b > -1$ and $\operatorname{Re}(u-t+b) > -(N+1)$. Let $f \in H(\mathbb{B})$. Then $(1-|z|^2)^{u-t} D_r^u(f) = M(D_s^t(f))$ for an operator M of type V_b^{u-t}. In particular, $D_r^t(f) = M(D_s^t(f))$ for an operator M of type V_b^0.*

Proof. See the Appendix. □

The parameter b can be chosen at will as long as it satisfies the two inequalities stated. This provides great flexibility as we show next.

4. Besov spaces

We first make sure that the B_q^p spaces are well-defined.

THEOREM 4.1. *Suppose $f \in H(\mathbb{B})$, $q \in \mathbb{R}$, and $r, s, t, u \in \mathbb{C}$.*

(a) *Let $0 < p < \infty$. For $q + p \operatorname{Re} t > -1$ the function $I_s^t f$ belongs to $L^p(\nu_q)$ if and only if for some r and u satisfying $q + p \operatorname{Re} u > -1$ the function $I_r^u f$ belongs to $L^p(\nu_q)$, and the $L^p(\nu_q)$ norms of these two functions are equivalent.*

(b) *Let $p = \infty$. For $\operatorname{Re} t > 0$ the function $I_s^t f$ is bounded on $\mathbb{B}$ if and only if for some r and $\operatorname{Re} u > 0$ the function $I_r^u f$ is bounded on $\mathbb{B}$, and the supremums of these two functions on $\mathbb{B}$ are equivalent norms for f.*

Note that there is absolutely no restriction on the lower parameters r, s of the differential operators.

Proof. (a) The relations $I_s^t f \in L^p(\nu_q)$ and $I_r^u f \in L^p(\nu_q)$ can be restated in the forms $D_s^t f \in L^p(\nu_{q+p\tau})$ and $(1-|z|^2)^{u-t} D_r^u f(z) \in L^p(\nu_{q+p\tau})$, respectively. So we apply Theorem 2.4 (a) or (b) with $c = q + p\tau$ and $a = u - t$. These values satisfy the first inequalities there since $q + p \operatorname{Re} u > -1$. We take a sufficiently large real b in Theorem 3.4 which also satisfies the second inequality in Theorem 2.4 (a) or (b). Theorem 2.4 then says that if $I_s^t f \in L^p(\nu_q)$, then $I_r^u f \in L^p(\nu_q)$.

In the opposite direction, we interchange the roles of the pairs s, t and r, u, and use the condition $q + p \operatorname{Re} t > -1$.

(b) By choosing a large enough b in Theorem 3.4 and then replacing b by $b - t$, we can write $I_r^u f = (M_1 \circ I_s^t) f$ with an operator M_1 of type V_{b-t}^u. Interchanging the pair s, t with r, u, we get an operator M_2 that is of type V_{b-u}^t. We take first $c = 0$, $a = u$, and b large, and then $c = 0$, $a = t$, and b large in Theorem 2.4(a). The inequalities on a and c are satisfied by the hypotheses $\operatorname{Re} u > 0$ and $\operatorname{Re} t > 0$, respectively. Then $I_r^u f$ is uniformly bounded on $\mathbb{B}$ if and only if $I_s^t f$ is uniformly bounded on $\mathbb{B}$. □

Corollary 4.2. *The space B_q^p is independent of the particular choice of s, t as long as* (4) *holds. The $L^p(\nu_q)$ norms of $I_{s_1}^{t_1} f$ and $I_{s_2}^{t_2} f$ are equivalent as long as* (4) *is satisfied by t_1 and t_2.*

Hence, if (4) holds, the map $I_s^t : B_q^p \to L^p(\nu_q)$ is an isometric imbedding modulo the equivalence of norms much like the map i is for Bergman spaces.

The reference [BB] uses the differential operators $(R + sI)^t$ instead of D_s^t, and as remarked there (p. 41), the corresponding spaces are the same. Thus our B_q^p spaces are the *holomorphic Sobolev spaces* $A_{q+pt+1,t}^p$ of [BB], which imposes the restriction $q + pt + 1 > 0$ as in (4). In fact, our Corollary 4.2 is contained in [BB, Theorem 5.12 (i)], and we could have referred to it instead of proving Theorems 3.4 and 4.1.

However, we have worked out the details because our relatively restricted approach makes the exposition simpler and our definition of Besov spaces uses the same parameters as those of weighted Bergman spaces, but with respect to $I_s^t f$ rather than f, thus making the roles of various functions and parameters clearer. Also Corollary 4.2 precedes Theorem 1.2, in contrast to many treatments of the subject; see [HKZ, Proposition 1.11] for comparison.

When $q > -1$ and $0 < p < \infty$, (4) is satisfied by $t = 0$ independently of p; then the spaces B_q^p and the weighted Bergman spaces A_q^p coincide. On the other hand, for such q and p, (4) is satisfied for certain t with $\tau < 0$ too. Then Corollary 4.2 gives a new characterization of weighted Bergman spaces using integrals of the functions contained in them rather than their derivatives.

In contrast, when $q \leq -1$ and $1 \leq p < \infty$, $t = -q$ always satisfies (4) independently of p. Then we see that the holomorphic Besov spaces $B_p(\mathbb{B})$ of [Z3] are our $B_{-(N+1)}^p$ spaces for such p. This value of t is also used in the pairings of Theorems 7.1 and 7.2 when identifying the dual of B_q^p.

When $p = \infty$, Corollary 4.2 combined with Proposition 2.3 says that the spaces B_q^∞ are the same for all q. Using $t = 1$, by [AFJP, Theorem 2] we see that this space is the *Bloch space* $\mathcal{B}$. The subspace $\mathcal{B}_0$ of $\mathcal{B}$ consisting of those functions f for which $I_s^t f$ restricts to 0 on $\partial\mathbb{B}$ for some t with $\tau > 0$ is called the *little Bloch space*. These results are stated in [Z4, Theorem 5] for $t > N$.

Let us denote by BSV_p^v the *diagonal Besov spaces* as defined in [AFJP, Remark 5.2], [P, Definition 1.1], or [AC], where the expression *diagonal* refers to the equality of two parameters out of three in the full Besov-space family. A straightforward checking of parameters yields that B_q^p and $\mathrm{BSV}_p^{-(q+1)/p}$, and conversely BSV_p^v and $B_{-(vp+1)}^p$, coincide.

Each B_q^2 space, being a Hilbert space (see [BB]), is equipped with several equivalent inner products

$$ {}_q[f,g]_s^t = \int_{\mathbb{B}} I_s^t f \, \overline{I_s^t g} \, d\nu_q, \tag{16} $$

one for each s, t satisfying $q + 2\tau > -1$. Using ${}_q[\cdot,\cdot]_0^0$ is standard for Bergman spaces ($q > -1$). The monomials $\{z^\alpha\}$ form an orthogonal set with respect to each of these inner products by Remark 2.2.

The following result clearly explains our choice of kernel in defining the extended Bergman projections. Hypergeometric kernels are not rare; see [Kar].

THEOREM 4.3. *Each B_q^2 space is a reproducing kernel Hilbert space. The reproducing kernel of B_q^2 is $K_q(z,w) = H_q(\langle z,w\rangle)$.*

Proof. See [BB, pp. 13–14]. □

The spaces B_q^2 are known as *Dirichlet-type spaces*, with $B_{-(N+1)}^2$ being *the Dirichlet space* $\mathcal{D}$ by (10), B_{-1}^2 the Hardy space H^2, and B_{-N}^2 the *Arveson space* $\mathcal{A}$. The space $\mathcal{A}$ is important in operator theory (see, for example, [Arv], [AM], [AK2]) due to a universal property of its kernel in Nevanlinna-Pick interpolation. A slightly different description of B_q^2 spaces for $q \geq -(N+1)$ that does not involve any derivatives on the functions is given in [AK3, Proposition 2.1]. For similar results on bounded symmetric domains, essentially for $q > -(N+1)$, see [Y].

5. Bergman projections

We start by deriving an integral formula for D_s^t.

LEMMA 5.1. *If $\sigma > -1$ and $f \in H(\mathbb{B})$, then for any t,*

$$ D_s^t f(z) = \frac{(s+1)_N}{N!} \lim_{r\to 1-} \int_{\mathbb{B}} H_{s+t}(\langle z,w\rangle)\,(1-|w|^2)^s\, f(rw)\, d\nu(w). $$

Proof. This is a direct computation using $f(z) = z^\alpha$, (9), and Proposition 2.1. To finish the proof, we invoke Theorem 3.2. □

Hence I_s^t is a constant multiple of V_s^t on suitable f for $\sigma > -1$. The more precise relationship (17) below complements this. The restriction on s can be weakened using the method of [AK1, §5].

As a matter of fact, our differential operators are defined in other sources using such integrals with binomial kernel and hence for limited t. Up to constant multiples, $\mathcal{R}_s^\mu$ of [P] is our D_s^μ for $s > -1$ and $\mu + s > -(N+1)$; D^s and D_s of [Z3] are our D_0^s and D_s^{-s} for $s > -1$; $D^{\alpha,\beta}$ and $D_{\alpha,\beta}$ of [Z4] are our D_β^α and $D_{\alpha+\beta}^{-\alpha}$ for $\alpha > -1$ and $\beta > -1$.

Proof of Theorem 1.2. Let $\varphi \in L^p(\nu_q)$; it is clear that $P_s\varphi \in H(\mathbb{B})$. We pick a t satisfying (4) and $\sigma + \tau > -(N+1)$, and apply I_s^t to $P_s\varphi$. By differentiating under the integral sign and employing (13), we obtain

$$I_s^t(P_s\varphi)(z) = (1-|z|^2)^t \int_{\mathbb{B}} \frac{(1-|w|^2)^s}{(1-\langle z,w\rangle)^{N+1+s+t}}\, \varphi(w)\, d\nu(w) = V_s^t\varphi(z).$$

By Theorem 2.4 (a) and Corollary 4.2, $P_s\varphi$ lies in B_q^p if and only if (6) holds. Now (6) and (4) together give $\sigma + \tau > -1$, so that the extra assumption $\sigma + \tau > -(N+1)$ above is not necessary.

Further, $\|P_s\varphi\|_{B_q^p} = \|V_s^t\varphi\|_{L^p(\nu_q)} \le \|V_s^t\|\,\|\varphi\|_{L^p(\nu_q)}$. So $\|P_s\| \le \|V_s^t\|$ whenever P_s is bounded.

Now let s be as above, and pick a possibly different t satisfying (4). Then

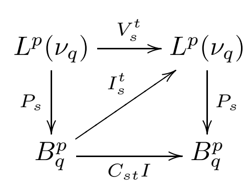
$$\begin{aligned} P_s(I_s^t f)(z) &= \int_{\mathbb{B}} \frac{(1-|w|^2)^{s+t}}{(1-\langle z,w\rangle)^{N+1+s}}\, D_s^t f(w)\, d\nu(w) \\ &= \frac{N!}{(1+s+t)_N}\, D_{s+t}^{-t} D_s^t f(z) = \frac{N!}{(1+s+t)_N}\, f(z) \end{aligned}$$

by Lemma 5.1 and (15). Lemma 5.1 applies, because (6) and (4) together imply $\sigma + \tau > -1$, which also ensures that $(1+s+t)_N$ is always defined. □

Remark 5.2. The proof of Theorem 1.2 reveals the following interesting fact. When $1 \le p \le \infty$, if V_s^t is bounded on $L^p(\nu_q)$, then it factors through B_q^p as

$$V_s^t = I_s^t \circ P_s. \tag{17}$$

Theorem 1.2 and (17) can be summarized in a commutative diagram:



Right inverses similar to I_s^t appear in limited cases also in [C] and [BB, Corollary 6.5], the latter for a different kind of projection.

The case $p = \infty$ is covered by Theorem 1.2, but deserves separate mention. It is more general than [C, Theorem 2], because it provides the only-if part and a whole family of right inverses.

Corollary 5.3. *The Bergman projection P_s maps $L^\infty(\nu)$ boundedly onto $\mathcal{B}$ if and only if $\sigma > -1$. If also $\tau > 0$, then* (7) *holds for all $f \in \mathcal{B}$.*

Let us isolate one other important case when $N = 1$. The operator P_s maps $L^2(\nu_{-1})$ boundedly onto $H^2 = \mathcal{A} = B^2_{-1}$ if and only if $\sigma > -1$, and if $\tau > 0$, then $(P_s \circ I^t_s) f = Cf$ for $f \in H^2$. Letting $s = 0$ and $t = 1$, for $f \in H^2$ and $f(0) = 0$, (7) amounts to the representation

$$f(z) = C \int_{\mathbb{D}} \frac{1 - |w|^2}{(1 - \langle z, w\rangle)^2} \, f'(w) \, d\nu(w).$$

Our purpose now is to compute the adjoint of P_s. First, for $1 \le p < \infty$ we have $(L^p(\nu_q))^* = L^{p'}(\nu_q)$ under the pairing ${}_q[\cdot,\cdot]^0_0$, and it is shown in Theorem 7.1 below that $(B^p_q)^* = B^{p'}_q$ under the general pairings given in (20). Given s satisfying (6) and $1 \le p < \infty$, by an *adjoint* of $P_s : L^p(\nu_q) \to B^p_q$ we mean a linear operator $P^*_s : B^{p'}_q \to L^{p'}(\nu_q)$ such that ${}_q[P_s f, g]^{t,-q+s}_{s,q+t} = {}_q[f, P^*_s g]^0_0$ for some t satisfying (4) and for all $f \in L^p(\nu_q)$ and $g \in B^{p'}_q$.

Theorem 5.4. *The adjoint of P_s as defined above is $P^*_s = C_{st} \, I^{-q+s}_{q+t}$.*

Proof. Expanding the definition of P^*_s in integrals and using (13) and Fubini's theorem, we obtain

$$P^*_s g(z) = (1 - |z|^2)^{-q+s} \int_{\mathbb{B}} H_{s+t}(\langle z, w\rangle) \, (1 - |w|^2)^{s+t} \, D^{-q+s}_{q+t} g(w) \, d\nu(w),$$

which is bounded if and only if (6) and (4) hold because of P_s. But for such s, t, the kernel H_{s+t} is binomial. Thus, when P^*_s is bounded, we have

$$\begin{aligned} P^*_s g(z) &= (1 - |z|^2)^{-q+s} \int_{\mathbb{B}} \frac{(1 - |w|^2)^{q+t}}{(1 - \langle z, w\rangle)^{N+1+s+t}} \, I^{-q+s}_{q+t} g(w) \, d\nu(w) \\ &= V^{-q+s}_{q+t} (I^{-q+s}_{q+t} g)(z). \end{aligned}$$

This form of P^*_s is entirely similar to and generalizes even for $q > -1$ the one given in [C, Corollary 7]. The only notable difference is the presence of I^{-q+s}_{q+t}, which is expected by our definition of Besov spaces and which can be checked to imbed $B^{p'}_q$ into $L^{p'}(\nu_q)$ under (6) and (4) with p' in place of p. The boundedness condition of P^*_s can now be read off also directly from Theorem 2.4 (a). By factoring V^{-q+s}_{q+t} as in (17) and using Theorem 1.2, we obtain the desired result. □

Note that no matter what value of t is used in the pairing of (20) to define the adjoint, P^*_s turns out to be essentially the same since the lower parameter of radial derivatives is a mere technicality. We can take $t = -q + s$ for symmetry.

Bounded projections from Lebesgue classes keep on playing important roles in the theory of Bergman-type spaces; see [CKY], for example.

6. The Gleason problem

Let X be a space of functions defined, say, on $\mathbb{B}$. Given $a \in \mathbb{B}$ and $f \in X$, the *Gleason problem* is to determine whether $f_1, \dots, f_N \in X$ exist such that

$$f(z) - f(a) = \sum_{m=1}^{N} (z_m - a_m)\, f_m(z) \qquad (z \in \mathbb{B}).$$

The point here is that $f_1, \dots, f_N$ must be in the same space as f. Explicit solutions are given in [Z1] and [C] in Bergman spaces A^p_q for $1 \le p < \infty$ at $a = 0$. In [AK3, §3], it is proved that solutions exist in Dirichlet-type spaces B^2_q at arbitrary $a \in \mathbb{B}$. For further recent results on the Gleason problem and their applications to interpolation see also [AK2], [CKY], and [AD].

In this section, we give explicit solutions to the Gleason problem in B^p_q for all q and $1 \le p \le \infty$, including $p = \infty$, at an arbitrary point $a \in \mathbb{B}$. Our solutions take the modification in [AK3] of the Ahern-Schneider solution (see [R, §6.6.2] and [AS]) one step further by employing Theorems 1.2 and 2.4.

We need integer values of s that satisfy (6). If $q > -1$, then $s = \lceil q+1 \rceil$, the least integer greater than or equal to $q+1$, works for all $1 \le p < \infty$. If $q \le -1$, then $s = 0$ works for all $1 \le p \le \infty$, including $p = \infty$. In any case, $s \ge 0$ and H_s is binomial.

THEOREM 6.1. *Given q, $1 \le p \le \infty$, and $a \in \mathbb{B}$, there exist bounded linear operators ${}_aG_1, \dots, {}_aG_N : B^p_q \to B^p_q$ satisfying*

$$f(z) - f(a) = \sum_{m=1}^{N} (z_m - a_m)\, {}_aG_m f(z) \qquad (f \in B^p_q,\ z \in \mathbb{B}). \tag{18}$$

Proof. Let $s > -(N+1)$ be an integer satisfying (6), let t satisfy (4), and define

$${}_aG_m f(z) = \frac{1}{C_{st}} \int_{\mathbb{B}} \frac{H_s(\langle z, w\rangle) - H_s(\langle a, w\rangle)}{\langle z - a, w\rangle}\, \overline{w}_m\, (1 - |w|^2)^s\, I^t_s f(w)\, d\nu(w)$$

for $m = 1, \dots, N$ and $f \in B^p_q$. The crucial difference to the Ahern-Schneider solution is the presence of the imbedding I^t_s. Then the right side of (18) is

$$\begin{aligned} &\frac{1}{C_{st}} \int_{\mathbb{B}} \big(H_s(\langle z, w\rangle) - H_s(\langle a, w\rangle) \big)\, (1 - |w|^2)^s\, I^t_s f(w)\, d\nu(w) \\ &\qquad = \frac{1}{C_{st}} \big(P_s(I^t_s f)(z) - P_s(I^t_s f)(a) \big) = f(z) - f(a) \end{aligned}$$

by Theorem 1.2. Hence ${}_aG_1f,\dots,{}_aG_Nf$ satisfy (18). It remains to show that ${}_aG_m$ is bounded.

Using that s is an integer and the finite binomial expansion, we can write

$$ {}_aG_mf(z)=\frac{1}{C_{st}}\sum_{j=0}^{N+s}\int_{\mathbb{B}}\frac{\overline{w}_m\,(1-|w|^2)^s}{(1-\langle z,w\rangle)^{N+1+s-j}\,(1-\langle a,w\rangle)^{1+j}}\,I_s^tf(w)\,d\nu(w). $$

Take a u satisfying (4) with $\operatorname{Re}u$ in place of τ, and apply I_{s-j}^u to the jth term in the sum, which we denote by $T_jf(z)$, $j=0,\dots,N+s\geq 1$. By (13), the result is

$$ \begin{aligned} &\frac{1}{C_{st}}(1-|z|^2)^u\int_{\mathbb{B}}\frac{\overline{w}_m\,(1-|w|^2)^s}{(1-\langle z,w\rangle)^{N+1+s+u-j}\,(1-\langle a,w\rangle)^{1+j}}\,I_s^tf(w)\,d\nu(w)\\ &\qquad=\frac{1}{C_{st}}(1-|z|^2)^u\int_{\mathbb{B}}\frac{(1-|w|^2)^s}{(1-\langle z,w\rangle)^{N+1+s+u}}\,\frac{\overline{w}_m\,(1-\langle z,w\rangle)^j}{(1-\langle a,w\rangle)^{1+j}}\,I_s^tf(w)\,d\nu(w) \end{aligned} $$

The second fraction is bounded for all $z,w\in\mathbb{B}$ for fixed $a\in\mathbb{B}$. Hence

$$ \begin{aligned} |I_{s-j}^u(T_jf)(z)|&\leq C(1-|z|^2)^u\int_{\mathbb{B}}\frac{(1-|w|^2)^s}{|1-\langle z,w\rangle|^{N+1+s+u}}\,|I_s^tf(w)|\,d\nu(w)\\ &=T(I_s^tf)(z), \end{aligned} $$

where T is an operator of type V_s^u and is bounded on $L^p(\nu_q)$, which contains I_s^tf, if and only if (4) and (6) hold with $\operatorname{Re}u$ in place of τ by Theorem 2.4 (a). Thus T_j is bounded on B_q^p. Therefore ${}_aG_m$ is a bounded operator on B_q^p since it is a finite sum of the T_j. □

7. Duality

It is well-known that $(\mathrm{BSV}_p^v)^*=\mathrm{BSV}_{p'}^v$, which is equivalent to $(B_q^p)^*=B_q^{p'}$. Here we derive this relationship from Theorem 1.2 and give a whole family of pairings that realize it. Some of these pairings have already been used in Section 5 in finding P_s^*. Our results give some new pairings also for the classical duality $(A_q^p)^*=A_q^{p'}$ of weighted Bergman spaces.

THEOREM 7.1. *Let $q\leq -1$ and $1\leq p<\infty$. The dual space $(B_q^p)^*$ can be identified with $B_q^{p'}$ under the pairing ${}_q[\cdot,\cdot]_0^{-q}$. In particular, the Bloch space $\mathcal{B}$ is the dual space of all B_q^1. Explicitly, every $g\in B_q^{p'}$ induces a bounded linear functional M_g on B_q^p via*

$$ M_g(f)=\int_{\mathbb{B}}I_0^{-q}f\,\overline{I_0^{-q}g}\,d\nu_q, \tag{19} $$

and every bounded linear functional M on B_q^p is of the form M_g for a unique $g\in B_q^{p'}$.

Proof. First let $g \in B_q^{p'}$. Apply Hölder's inequality to the right side of (19) with p and p' to obtain $|M_g(f)| \le \|f\|_{B_q^p} \|g\|_{B_q^{p'}}$ for all $f \in B_q^p$. This gives that $B_q^{p'} \subset (B_q^p)^*$ and $\|M_g\| \le \|g\|_{B_q^{p'}}$.

Next let M be a bounded linear functional on B_q^p. By (4), I_0^{-q} imbeds B_q^p in $L^p(\nu_q)$. Let Q be the restriction of P_0 to $I_0^{-q}(B_q^p)$. Then $M \circ Q$ is a bounded linear functional on $I_0^{-q}(B_q^p)$ by Theorem 1.2. By the Hahn-Banach theorem, $M \circ Q$ extends to a bounded linear functional L on $L^p(\nu_q)$ with $\|L\| = \|M \circ Q\|$. By the Riesz representation theorem, there exists a unique φ in $L^{p'}(\nu_q)$ such that $L(h) = \int_{\mathbb{B}} h\, \overline{\varphi}\, d\nu_q$ for all $h \in L^p(\nu_q)$ and $\|L\| = \|\varphi\|_{B_q^{p'}}$. Taking $h = I_0^{-q} f = F$ for $f \in B_q^p$ gives $M(f) = \int_{\mathbb{B}} I_0^{-q} f\, \overline{\varphi}\, d\nu_q$. We can replace f by $CP_0(I_0^{-q} f) = CP_0 F$ by Theorem 1.2. Put $g = CP_0\varphi$. Then g is unique and clearly $g \in B_q^{p'}$. Now we have

$$\begin{aligned} M(f) &= C \int_{\mathbb{B}} I_0^{-q}(P_0 F)\, \overline{\varphi}\, d\nu_q = C \int_{\mathbb{B}} \overline{\varphi(z)} \int_{\mathbb{B}} \frac{F(w)}{(1 - \langle z, w\rangle)^{N+1-q}}\, d\nu(w)\, d\nu(z) \\ &= C \int_{\mathbb{B}} F(w)\, (1 - |w|^2)^{-q} \int_{\mathbb{B}} \frac{\overline{\varphi(z)}}{(1 - \langle z, w\rangle)^{N+1-q}}\, d\nu(z)\, d\nu_q(w) \\ &= C \int_{\mathbb{B}} F\, \overline{V_0^{-q}\varphi}\, d\nu_q = C \int_{\mathbb{B}} F\, \overline{I_0^{-q}(P_0\varphi)}\, d\nu_q = \int_{\mathbb{B}} I_0^{-q} f\, \overline{I_0^{-q} g}\, d\nu_q \end{aligned}$$

by (13), (17), and the Fubini theorem. The norms satisfy

$$\begin{aligned} \|g\|_{B_q^{p'}} &\le C\|P_0\|\, \|\varphi\|_{L^{p'}(\nu_q)} = C\|P_0\|\, \|L\| \\ &\le C\|P_0\|\, \|M\|\, \|Q\| \le C\|P_0\|^2\, \|M\|. \end{aligned}$$

So the norms of g and M need not be equal; in other words, the identification of dual spaces may not be isometric. □

We similarly have the following duality whose proof is omitted.

THEOREM 7.2. *Let $q \le -1$. The dual space ${\mathcal{B}_0}^*$ of the little Bloch space can be identified with each of B_q^1 under the pairing ${}_q[\cdot, \cdot]_0^{-q}$.*

The cases $q = -(N+1)$ of Theorems 7.1 and 7.2 are with respect to the invariant measure and given in [Z3, Theorems 17 and 18]. The corresponding identifications for $q > -1$ concern the weighted Bergman spaces and can be found essentially in [HKZ, Theorem 1.16, Theorem 1.21, p. 23].

REMARK 7.3. Retracing the proofs of Theorems 7.1 and 7.2, we see that the stated dualities are realized under each of the pairings

$$ {}_q[f, g]_{s, q+t}^{t, -q+s} = \int_{\mathbb{B}} I_s^t f\, \overline{I_{q+t}^{-q+s} g}\, d\nu_q, \tag{20}$$

now for all q, where s, t satisfy (6) and (4).

8. Complex interpolation

Another application of Theorem 1.2 is that we can apply *complex interpolation* between $B_q^{p_0}$ and $B_q^{p_1}$. These are *compatible* spaces, because they are contained in A_q^1 if $q > -1$ and in A^1 if $q \le -1$, as seen in [Kap, (2)]. So for $q \le -1$, for example, we have $B_q^{p_0} \cap B_q^{p_1} \subset B_q^p \subset B_q^{p_0} \cup B_q^{p_1} \subset B_q^{p_0} + B_q^{p_1} \subset A^1$, and the inclusions are dense since polynomials are dense in all B_q^p, see [BB, Lemma 5.2]. For relevant definitions and notation such as $[\cdot,\cdot]_\theta$, $\|\cdot\|_\theta$, $\mathcal{F}$, or $\|\cdot\|_{\mathcal{F}}$; see [Z2, §2.2]. The notation $[X,Y]_\theta$ here denotes the *complex interpolation space* between the Banach spaces X and Y, and must not be confused with the inner products in (16).

Theorem 8.1. *Suppose* $1 \le p_0 < p < p_1 \le \infty$ *with* $1/p = (1-\theta)/p_0 + \theta/p_1$ *for some* $\theta \in (0,1)$. *Then* $[B_q^{p_0}, B_q^{p_1}]_\theta = B_q^p$.

Proof. Given $f \in B_q^p$, we pick positive s,t satisfying (6) and (4) with p_0 (the smallest), and set $\varphi = I_s^t f \in L^p(\nu_q)$. We know by Theorem 1.2 that $P_s\varphi = Cf$ and $\|\varphi\|_{L^p(\nu_q)} = C\,\|f\|_{B_q^p}$. For ζ in the *strip* $\overline{S} = \{\,\zeta \in \mathbb{C} : 0 \le \operatorname{Re}\zeta \le 1\,\}$ and $z \in \mathbb{B}$ we define

$$\Phi_\zeta(z) = \frac{\varphi(z)}{|\varphi(z)|}\,\big|\varphi(z)\big|^{p\left(\frac{1-\zeta}{p_0} + \frac{\zeta}{p_1}\right)}$$

and $F_\zeta = P_s\Phi_\zeta$ as in the proof of [Z2, Theorem 5.3.8], which takes care of the case $q = -(N+1)$. Both Φ and F are continuous and bounded for $\zeta \in \overline{S}$, holomorphic for $\zeta \in S$, $\Phi_\theta = \varphi$, and $F_\theta = f$. On the left boundary of S, $|\Phi_{iy}(z)| = |\varphi(z)|^{p/p_0}$, $\|\Phi_{iy}\|^{p_0}_{L^{p_0}(\nu_q)} = \|\varphi\|^p_{L^p(\nu_q)}$, and $I_s^t F_{iy}(z) = M\Phi_{iy}(z)$, where M is an operator of type V_s^t by (17) and bounded on $L^{p_0}(\nu_q)$. Thus $\|F_{iy}\|_{B_q^{p_0}} \le \|M\|\,\|\varphi\|^{p/p_0}_{L^p(\nu_q)}$ for all $y \in \mathbb{R}$. Similarly, on the right boundary of S, $\|F_{1+iy}\|_{B_q^{p_1}} \le \|M\|\,\|\varphi\|^{p/p_1}_{L^p(\nu_q)}$ for all $y \in \mathbb{R}$. Thus $\|f\|_\theta \le \|F\|_{\mathcal{F}} \le C\,\|f\|_{B_q^p}$ and $f \in [B_q^{p_0}, B_q^{p_1}]_\theta$.

Conversely let $f \in [B_q^{p_0}, B_q^{p_1}]_\theta$. There is a function F_ζ such that $F_{iy} \in B_q^{p_0}$, $F_{1+iy} \in B_q^{p_1}$, and $F_\theta = f$. Put $\Phi_\zeta = I_s^t F_\zeta$. But then $\Phi_{iy} \in L^{p_0}(\nu_q)$ and $\Phi_{1+iy} \in L^{p_1}(\nu_q)$. Applying [Z2, Theorem 2.2.6] yields $\Phi_\theta \in L^p(\nu_q)$. Finally Theorem 1.2 gives $P_s\Phi_\theta = P_s(I_s^t F_\theta) = CF_\theta = Cf \in B_q^p$. □

Note that the interpolating space between $B_q^{p_0}$ and $B_q^{p_1}$ is not the same as the interpolating space between $\mathrm{BSV}^v_{p_0}$ and $\mathrm{BSV}^v_{p_1}$.

Let $\operatorname{Aut}(\mathbb{B})$ be the group of all automorphisms of $\mathbb{B}$, that is, one-to-one holomorphic maps of $\mathbb{B}$ onto $\mathbb{B}$. We recall that $\operatorname{Aut}(\mathbb{B})$ acts transitively on $\mathbb{B}$, and for each $\psi \in \operatorname{Aut}(\mathbb{B})$, there is a unique unitary transformation U of $\mathbb{C}^N$ such that $\psi = U \circ \phi_a$, where $a = \psi^{-1}(0)$ and ϕ_a is an involutive *Möbius transformation*, as explained in detail in [R, §2.2].

The measures ν_q have certain invariance properties. For $\psi \in \mathrm{Aut}(\mathbb{B})$, define the operators

$$(Z^q_\psi f)(z) = f(\psi(z))\,|J\psi(z)|^{2\left(1+\frac{q}{N+1}\right)},$$

where $J\psi$ is the *complex Jacobian* of ψ. Then

$$\int_{\mathbb{B}} (Z^q_\psi f)\,d\nu_q = \int_{\mathbb{B}} f\,d\nu_q \tag{21}$$

for $f \in L^1(\nu_q)$. This is stated in [BB, (3.5)] for $q > -1$, but it holds for all q. It reduces to the well-known invariance under compositions with members of $\mathrm{Aut}(\mathbb{B})$ (*Möbius-invariance*) of $\nu_{-(N+1)}$. Further, it is shown in [BB, Theorem 3.3] using (21) that the Bergman spaces A^p_q for $0 < p \le \infty$ and $q > -1$ are invariant under each of the isometries

$$U^{p,q}_\psi f(z) = f(\psi(z))\,\big(J\psi(z)\big)^{\frac{2}{p}\left(1+\frac{q}{N+1}\right)} \qquad (\psi \in \mathrm{Aut}(\mathbb{B})).$$

In our final theorem we apply interpolation methods to extend this result to certain other Besov spaces.

Theorem 8.2. *Suppose* $2 \le p \le \infty$, $-(N+1) \le q \le -1$, *and* $\psi \in \mathrm{Aut}(\mathbb{B})$. *Then* $U^{p,q}_\psi$ *is a bounded linear transformation on* B^p_q.

Proof. We know that $U^{2,q}_\psi$ is a unitary transformation for $q > -(N+1)$ on B^2_q by [BB, Theorem 1.10]. This holds also for $q = -(N+1)$ since it is equivalent to the Möbius invariance of the Dirichlet space $\mathcal{D}$. We also know that $U^{\infty,q}_\psi$ maps $\mathcal{B}$ onto itself isometrically, which is actually the Möbius invariance. To interpolate between these two ends, for $\zeta \in \overline{S}$, we let

$$T_\zeta f(z) = f(\psi(z))\,\big(J\psi(z)\big)^{\left(1+\frac{q}{N+1}\right)(1-\zeta)}$$

and $1/p = (1-\theta)/2$. Then $T_\theta f = U^{p,q}_\psi f$. Let $\omega(z) = \big(\arg(J\psi(z))\big)^{1+\frac{q}{N+1}}$. The Jacobian $J\psi(z)$, being the determinant of a linear map on $\mathbb{C}^N$, the Jacobian matrix, has bounded argument as z varies in $\mathbb{B}$. So $|\omega(z)| \le C$ for all $z \in \mathbb{B}$. Then $\|T_{iy} f\|_{B^2_q} \le e^{Cy}\,\|f\|_{B^2_q}$ and $\|T_{1+iy} f\|_{\mathcal{B}} \le e^{Cy}\,\|f\|_{\mathcal{B}}$ for all real y.

Now we proceed as in the proof of [Z2, Theorem 2.2.4]. Given $f \in B^p_q$, there is a function $F \in \mathcal{F}$ such that $F_\theta = f$ and $\|F\|_{\mathcal{F}} \le \|f\|_{B^p_q} + \varepsilon$ by Theorem 8.1. Put $G(\zeta) = e^{iC\zeta}\,T_\zeta F_\zeta$ so that $|G(\theta)| = |T_\theta f|$. On the boundary of S, we have $\|G(iy)\|_{B^2_q} = e^{-Cy}\,\|T_{iy}F_{iy}\|_{B^2_q} \le \|F_{iy}\|_{B^2_q}$ and $\|G(1+iy)\|_{\mathcal{B}} \le \|F_{1+iy}\|_{\mathcal{B}}$ for all $y \in \mathbb{R}$ by the above remark. This shows that $\|G\|_{\mathcal{F}} = \|F\|_{\mathcal{F}} = \|f\|_{B^p_q} + \varepsilon$. Since ε is arbitrary, $\|U^{p,q}_\psi f\|_{B^p_q} = \|T_\theta f\|_{B^p_q} = \|G(\theta)\|_\theta \le \|G\|_{\mathcal{F}} = \|f\|_{B^p_q}$ by Theorem 8.1. It follows that $\|U^{p,q}_\psi\| \le 1$. □

So, in particular, if $N = 1$ and $\psi \in \mathrm{Aut}(\mathbb{B})$, then the map $f \mapsto (f \circ \psi)\sqrt{J\psi}$ is unitary on $H^2 = \mathcal{A}$.

Appendix

Proof of Theorem 2.4 (b). For notational simplicity, without loss of generality, we take $a, b \in \mathbb{R}$. We apply [BB, Corollary 3.8 (iii)] with $p_2 = 1$, $q_2 = b + 1$, $p_1 = p$, and $q_1 = p(N + 1 + b) - N$ to the holomorphic function

$$w \mapsto \frac{f(w)}{(1 - \langle w, z\rangle)^{N+1+a+b}}.$$

The first condition on b is a result of the requirement $q_1 > 0$. The requirement $q_2 > 0$, that is, $b > -1$, is implied by this. (Actually, [BB] considers the case $q = 0$ too, but with a different kind of measure. Also note that the variable q of [BB] corresponds to $q - 1$ in our notation.) This lemma is valid for $0 < p_1 \le p_2$, which is equivalent to $0 < p \le 1$ as we assumed. We obtain

$$|V_b^a f(z)|^p \le C\,(1 - |z|^2)^{pa} \int_{\mathbb{B}} \frac{(1 - |w|^2)^{p(N+1+b)-(N+1)}}{|1 - \langle z, w\rangle|^{p(N+1+a+b)}}\, |f(w)|^p \, d\nu(w).$$

Then $\|V_b^a f(z)\|^p_{L^p(\nu_c)}$ is

$$\begin{aligned}
&= \int_{\mathbb{B}} |V_b^a f(z)|^p \,(1 - |z|^2)^c \, d\nu(z)\\
&\le C \int_{\mathbb{B}} (1 - |z|^2)^{c+pa} \int_{\mathbb{B}} \frac{(1 - |w|^2)^{(p-1)(N+1)+pb}}{|1 - \langle z, w\rangle|^{p(N+1+a+b)}}\, |f(w)|^p \, d\nu(w)\, d\nu(z)\\
&= C \int_{\mathbb{B}} |f(w)|^p \,(1 - |w|^2)^{(p-1)(N+1)+pb} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{c+pa}\, d\nu(z)}{|1 - \langle z, w\rangle|^{p(N+1+a+b)}}\, d\nu(w),
\end{aligned}$$

where we used Fubini's theorem. The condition on a amounts to $c + pa > -1$, and the second condition on b to $(p-1)(N+1) + pb - c > 0$, and when these conditions hold, [R, Proposition 1.4.10] gives us

$$(1 - |w|^2)^{(p-1)(N+1)+pb} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{c+pa}}{|1 - \langle z, w\rangle|^{p(N+1+a+b)}}\, d\nu(z) \le C\,(1 - |w|^2)^c,$$

which in turn yields that $\|V_b^a f(z)\|^p_{L^p(\nu_c)} \le C\,\|f(z)\|^p_{L^p(\nu_c)}$. □

Proof of Theorem 3.4. We show the details only for the case when s, $s+t$, r, and $r+u$ are real and exceed $-(N+1)$; the other cases are similar. Following [BB, p. 41], for $b > -1$, we consider

$$h(\lambda) = \sum_{k=0}^{\infty} \frac{(N+1+r+u)_k}{(N+1+r)_k}\, \frac{(N+1+s)_k}{(N+1+s+t)_k}\, \frac{(b+1)_{N+k}}{N!\,k!}\, \lambda^k,$$

which belongs to $H(\mathbb{D})$. Computing with $f(z) = z^\alpha$ and Proposition 2.1, we see that

$$D_r^u f(z) = \int_{\mathbb{B}} D_s^t f(w)\, h(\langle z, w\rangle)\,(1 - |w|^2)^b \, d\nu(w)$$

for $f \in H(\mathbb{B})$ by virtue of Theorem 3.2. By (8),

$$h(\lambda) \sim \sum_{k=0}^{\infty} \frac{(N+1+b)_k}{k!\,(k+1)^{t-u}}\, \lambda^k = g_{N+1+b,t-u}(\lambda),$$

where $g_{x,y}$ is defined in [BB, §0.5]. Then by the assumption on b and [BB, Corollary 2.4 (a)],

$$h(\lambda) = \frac{h_1(\lambda)}{(1-\lambda)^{N+1+u-t+b}}$$

for a holomorphic h_1 in the Lipschitz class $\Lambda_{N+1+u-t+b}$ on $\overline{\mathbb{D}}$. This finally implies, since $\operatorname{Arg}(1-\lambda)$ is bounded, that

$$|h(\lambda)| \leq \frac{C}{|1-\lambda|^{N+1+u-t+b}}$$

as in the proof of [BB, Lemma 5.6]. The proof is now complete. □

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Mathematics Department, Middle East Technical University, Ankara 06531, Turkey

Current address: Mathematics Department, Bilkent University, Ankara 06800, Turkey
E-mail address: kaptan@fen.bilkent.edu.tr
URL: http://www.fen.bilkent.edu.tr/~kaptan/