Entanglement of Qutrits

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Abstract—For two types of qutrits specified by the dynamical symmetry SU(3) and SU(2), we consider the difference in entanglement caused by the lack of quantum observables in the latter case. In particular, we show that the SU(2) qutrits can have specific separable entanglement caused by quantum correlations of intrinsic degrees of freedom in a single party without interparty correlations.

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The main objective of this note is to accentuate the principle difference between the two possible realizations of qutrits with different dynamic symmetry properties.

Since the pioneering paper by late Professor David Klyshko and his coauthors [1], the problem of qutrits is widely discussed in the context of quantum ternary logic (see [2–6] and references therein). Qutrit is often defined as a quantum three-level or ternary system, that is, a system with states in the three-dimensional Hilbert space $\mathcal{H}_3$. We denote the orthonormal basis of $\mathcal{H}_3$ as follows:

$$|+1\rangle, |0\rangle, |-1\rangle. \tag{1}$$

In fact, the above definition of qutrits is incomplete. The point is that the general symmetry of the three-level system is given by the group $G = \text{SU}(3)$. This group is induced by the Lie algebra $\mathcal{L} = \text{su}(3)$

$$G = \exp(i\mathcal{L}),$$

whose orthogonal basis consists of eight independent Hermitian operators (Gell-Mann matrices) [7]

$$\lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2}$$

This means that a state in $\mathcal{H}_3$ is specified by measurement of these eight observables.

At the same time, a system with reduced symmetry $G' = \text{SU}(2) \subset \text{SU}(3)$ can also be considered in $\mathcal{H}_3$. This symmetry is associated with the “spin” algebra $\mathcal{L}' = \text{su}(2)$ with an orthogonal basis, consisting of only three spin-projection operators

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

This example clearly illustrates a principle of quantum mechanics that is often overlooked. Namely, the definition of a quantum system assumes specification of both the Hilbert space of states and the set of basic observables that can be accessed (e.g., see [8]). This fact has special importance for understanding of quantum entanglement [9, 10].

It is clear that measurement of observables (2) and (3) gives us qualitatively different information about the system. For example, the observable $S_z$ in (3) is represented by a linear combination of the Gell-Mann matrices in (2):

$$S_x = \frac{1}{\sqrt{2}} (\lambda_1 + \lambda_6),$$
so that the measurement of $S_x$ assumes that the noncommuting observables $\lambda_1$ and $\lambda_6$ are measured simultaneously. At the same time, quantum uncertainty of the measurement of $S_x$ is higher than the common uncertainty of measurements of $\lambda_1$ and $\lambda_6$: 

$$V(S_x) = \frac{1}{2} (V(\lambda_1) + V(\lambda_6)) + \text{Cov}(\lambda_1, \lambda_6).$$

Here, 

$$V(X) \equiv \langle (X - \langle X \rangle)^2 \rangle$$

denotes the variance (uncertainty), and the covariance 

$$\text{Cov}(X, Y) = \frac{1}{2} \langle XY + YX \rangle - \langle X \rangle \langle Y \rangle$$

describes the quantum correlation between the measurements of two observables. Thus, reduction of the symmetry SU(3) $\longrightarrow$ SU(2) with the corresponding change of basic observables leads to an increase of uncertainty in measurements.

Hereafter we call the ternary quantum system with the SU(3) symmetry and observables (2) the true qutrit, and the system with the reduced symmetry SU(2) the SU(2) or spin qutrit.

In physics, the true qutrit is usually associated with the “eightfold way” and quark theory [11]. Nevertheless, some simple objects can also be associated with true qutrits. An example is provided by three-level atoms. In turn, the spin-qutrit represents a “spin-1” object that can be associated with spin, isotop-spin, angular momentum, etc. [12]. In the context of quantum information, the true qutrit can be realized as a quantum-dot cellular automata with eight quantum dots [13]. The simplest example of a spin-qutrit is provided by a biphoton [14, 15] (a couple of photons created simultaneously and propagating in the same direction).

Our analysis is based on the following key results.

1. **States carrying complete entanglement manifest maximal amount of total variance**

$$\nabla(\psi_{\text{CE}}) = \max_{\psi \in \mathcal{H}} \nabla(\psi).$$

Here, 

$$\nabla(\psi) \equiv \sum_i V(X_i)$$

and summation is performed over all basic observables $X_i$ and over all parties in the case of multipartite system.

This condition (4) reflects the physical nature of quantum entanglement as a manifestation of quantum fluctuations at their extreme [9, 10, 16].

2. **All entangled states of a given system are equivalent to within SLOCC** (stochastic local operations assisted by classical communications). This property has been established in [17]. It has been shown in [18] that SLOCC are represented by operators $g^c$ from the complexified dynamic symmetry group $G^c = \exp(\mathcal{L} \otimes \mathbb{C})$.

It follows from these two statements that all entangled states of a given system can be constructed from a certain generic CE state by means of SLOCC. A simple algorithm for construction of generic entangled states has been proposed in [19].

Entanglement of pure states of two true qutrits was discussed in [7] and within the frameworks of the dynamic symmetry approach in [10, 12, 20]. Below, we briefly discuss some important results.

The system is defined in the Hilbert space

$$\mathcal{H} = \mathcal{H}_3 \otimes \mathcal{H}_3$$

and has the dynamic symmetry SU(3) $\times$ SU(3). The generic CE state of the system has the form [1, 19]

$$|\psi_{\text{CE}}\rangle = \frac{1}{\sqrt{3}}(|++1, +1\rangle + e^{i2k\pi/3}|0, 0\rangle + e^{i4k\pi/3}|1, -1\rangle),$$

$$k = 0, 1, 2,$$

where $|s, s\rangle \equiv |s\rangle \otimes |s\rangle$. This state can be considered as a generic entangled state of two true qutrits. This means that all other entangled states of this system can be defined as follows:

$$|\psi_{\text{E}}\rangle = g^c_{\text{loc}}|\psi_{\text{CE}}\rangle,$$

where $z_j$ is a complex parameter. The amount of entanglement carried by two true qutrits can be measured by means of concurrence, which can be expressed in terms of the total variance as follows [21]:

$$C(\psi) = \frac{\sqrt{\nabla(\psi) - \nabla_{\text{min}}}}{\sqrt{\nabla_{\text{max}} - \nabla_{\text{min}}}}.$$
from the equivalence of two qubits with basic states, forming a symmetric triplet, and single SU(2) qutrit. As an illustrative example of some considerable interest, the states of biphoton [14, 15] can be mentioned. In terms of the horizontal (H) and vertical (V) polarizations forming a qubit for each photon, the basic states are denoted as follows:

\[ |V, V\rangle, \quad \frac{1}{\sqrt{2}}(|V, H\rangle + |H, V\rangle), \quad |H, H\rangle. \quad (10) \]

The antisymmetric polarization state of a biphoton is forbidden due to the Bosonic nature of this object. It is clear that the states with parallel polarizations \(|V, V\rangle\) and \(|H, H\rangle\) can be associated with the spin-projection states \(|1\rangle\) and \(|-1\rangle\) in (1). In turn, the state with indefinite polarization in (10) corresponds to the spin-projection state \(|0\rangle\) in (1). This state is certainly completely entangled with respect to polarization of photons. In more general settings, a single SU(2) qutrit in the state \(|0\rangle\) is completely entangled with respect to certain intrinsic degrees of freedom. For physical examples, see [12, 20].

To be more rigorous, we note that existence of the single SU(2)-qutrit entanglement follows directly from the Clebsch–Gordon decomposition of the Hilbert space of two qubits:

\[ \mathcal{H}_2 \otimes \mathcal{H}_2 = \mathcal{H}_3 \otimes \mathcal{H}_A, \]

where \(\mathcal{H}_2\) denotes the single-qubit space of states and \(\mathcal{H}_A\) corresponds to the antisymmetric singlet state.

The principle difference between the complete entangled state (6) and separable entangled state \(|0, 0\rangle\) can also be illustrated by the fact that local measurements between the two parties are strongly correlated in the former case and completely uncorrelated in the latter case. In other words, both states carry complete entanglement. In state (6), it is caused by the quantum correlations between parties of the system, which corresponds to the traditional interpretation of entanglement. In the state \(|0, 0\rangle\), entanglement has a completely local nature. The intrinsic degrees of freedom are responsible for this kind of entanglement.

Besides that, it is easily seen that the states (6) and \(|0, 0\rangle\) are nonequivalent with respect to SLOCC (9).

The third class of states is formed by unentangled states that can be generated from \(|\pm 1, \pm 1\rangle\) by means of SLOCC (9). They are separable as well as \(|0, 0\rangle\) and do not manifest quantum correlation between parties either. Nevertheless, they are completely different because of the principle difference between the single SU(2) qutrit states \(|0\rangle\) and \(|\pm 1\rangle\). As we have seen, the former one is completely entangled. The latter states can be associated with the SU(2) coherent states that can be defined as follows [22]:

\[ |\alpha\rangle = \exp(i\alpha S_+ - \alpha^* S_-)|-1\rangle \]

\[ = \frac{e^{i\phi}}{2} \left(1 - \cos 2|\alpha|\right)|1\rangle + \frac{e^{i\phi}}{\sqrt{2}} \sin 2|\alpha| |0\rangle \]

\[ + \frac{1}{2} (1 + \cos 2|\alpha|)|-1\rangle, \]

where \(S_\pm = S_x \pm i S_y\), \(\alpha\) is a complex parameter, and \(\phi = \arg \alpha\). At \(|\alpha| = (2k + 1)\pi/2\), this state coincides with \(|1\rangle\), at \(|\alpha| = k\pi\), with \(|-1\rangle\). It can never coincide with \(|0\rangle\), which is the spin-squeezed rather than spin-coherent state [12].

The properties of states of two SU(2) qutrits are listed in the following table.

<table>
<thead>
<tr>
<th>Type of state</th>
<th>Generic state</th>
<th>Total variance</th>
<th>Correlation between parties</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entangled</td>
<td>Given by Eq. (7)</td>
<td>(V_{\text{max}} \geq V &gt; V_{\text{min}})</td>
<td>Strong correlation</td>
<td>No correlations</td>
</tr>
<tr>
<td>Separable entangled</td>
<td>(</td>
<td>0, 0\rangle)</td>
<td>(V_{\text{max}} \geq V &gt; V_{\text{min}})</td>
<td>No correlations</td>
</tr>
<tr>
<td>Unentangled</td>
<td>(</td>
<td>\pm 1, \pm 1\rangle)</td>
<td>(V = V_{\text{min}})</td>
<td>No correlations</td>
</tr>
</tbody>
</table>
entanglement in the three-partite sector, while is entangled in the bipartite sector [23].

The classes of entangled states of two true and SU(2) qutrits have the same generic state (6). This means that entanglement of true qutrits involves entanglement of the SU(2) qutrits, but not vice versa.

As a hint for practical distinguishability among the entangled and separable entangled states of two SU(2) qutrits, we note that both parties in the former class manifest the same amount of total variance, while the amount of the single-party variance is different for separable entangled states (except the generic state $|0, 0\rangle$).

Above, we have considered pure states. Classification of mixed states of qutrits deserves special consideration.

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