ELSEVIER

An improved probability bound for the Approximate S-Lemma

Kürşad Derinkuyu, Mustafa Ç. Pınar*, Ahmet Camcı

Department of Industrial Engineering, Bilkent University, 06800 Bilkent, Ankara, Turkey

Abstract

The purpose of this note is to give a probability bound on symmetric matrices to improve an error bound in the Approximate S-Lemma used in establishing levels of conservatism results for approximate robust counterparts.
© 2007 Elsevier B.V. All rights reserved.

Keywords: Robust optimization; S-Lemma

1. Introduction

The purpose of this note is to prove the following result:

Lemma 1. *Let B denote a symmetric $n \times n$ matrix and $\xi = \{\xi_1, \ldots, \xi_n\} \in \mathbb{R}^n$. If the coordinates ξ_i of ξ are independently identically distributed random variables with*

$$\Pr(\xi_i = 1) = \Pr(\xi_i = -1) = 1/2 \tag{1}$$

then one has

$$\Pr(\xi^T B \xi \leqslant \operatorname{Tr} B) \geqslant \frac{1}{2^{\lceil \log_2(n) \rceil}} > \frac{1}{2n}. \tag{2}$$

* Corresponding author. Tel.: +90 312 290 1514;
fax: +90 312 266 4054.

E-mail addresses: kursad@mail.utexas.edu (K. Derinkuyu), mustafap@bilkent.edu.tr (M.Ç. Pınar), camci@bilkent.edu.tr (A. Camcı).

The above result improves Lemma A.4 by Ben-Tal et al. [1] which stated

$$\Pr(\xi^T B \xi \leqslant \operatorname{Tr} B) \geqslant \frac{1}{8n^2},$$

and where the authors conjectured that the right-hand side could be improved to $\frac{1}{4}$. Ben-Tal et al. [1] used Lemma A.4 to give the Approximate S-Lemma used in levels of conservatism results for approximate robust counterparts of uncertain convex programs. Our Lemma 1 above improves the error bound in the Approximate S-Lemma of [1] to

$$\rho := \left(2 \log \left(4n \sum_{k=1}^{K} \operatorname{rank} R_k \right) \right)^{1/2} \tag{3}$$

from

$$\rho := \left(2 \log \left(16n^2 \sum_{k=1}^{K} \operatorname{rank} R_k \right) \right)^{1/2}. \tag{4}$$

2. Proof of the main result

Our proof, which is based on contradiction, recursively eliminates the non-zero entries of a symmetric matrix while the proof of [1] uses moments. We arrive at the proof of Lemma 1 after giving three intermediate results.

First, since $\mathrm{Tr}\, B = \xi^{\mathrm{T}}\,\mathrm{diag}\, B\,\xi$ for any $\xi \in \{-1, 1\}^n$ it follows that

$$
\begin{aligned}
\Pr(\xi^{\mathrm{T}} B \xi \leqslant \mathrm{Tr}\, B) &= \Pr(\xi^{\mathrm{T}} B \xi - \mathrm{Tr}\, B \leqslant 0)\\
&= \Pr(\xi^{\mathrm{T}}(B - \mathrm{diag}\, B)\xi \leqslant 0).
\end{aligned}
$$

This enables us to restrict ourselves to the case that the matrix under consideration is a symmetric matrix with zero diagonal since $B - \mathrm{diag}\, B$ is a matrix with this property. Therefore, in order to prove Lemma 1 we need to show that for any symmetric matrix B with zero diagonal, and for ξ as defined in Lemma 1 we have

$$
\Pr(\xi^{\mathrm{T}} B \xi \leqslant 0) \geqslant \frac{1}{2^{\lceil \log_2(n) \rceil}}. \tag{5}
$$

Now, we will give three intermediate results which lead to the proof of Lemma 1.

Lemma 2. *Let X be a finite set. Then for any pair of subsets U and V of X, one has*

$$
|U \cap V| \geqslant |U| + |V| - |X|.
$$

Proof. Using the inclusion–exclusion principle we have $|U| + |V| - |U \cap V| = |U \cup V| \leqslant |X|$. After rearranging the right and left sides of the inequality we get the desired result. □

Lemma 3. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) = \lceil n/2 \rceil$. If $k = \lceil \log_2(n) \rceil$, then $f^k(n) = f(f(\dots(f(n))\dots)) \leqslant 1$.*

Proof. By the definition of k we have $k - 1 < \log_2(n) \leqslant k$, which implies $n \leqslant 2^k$. Since f is a non-decreasing function, we have $f^k(n) \leqslant f^k(2^k)$. It can be seen that $f^k(2^k) = 1$. Therefore the result holds. □

In the remaining part of the paper for any $q \in \mathbb{R}^n$ such that $q(i) \in \{-1, 1\}$ for any $i \in \{1, \dots, n\}$ we denote $\mathrm{diag}(q)$ by Q. Here, $q(i)$ is the ith entry of vector q. For any such Q and any symmetric matrix B

having zero diagonal entries we define

$$
B^q = \tfrac{1}{2}(B + QBQ).
$$

The matrix QBQ is a symmetric matrix with zero diagonal. Hence, B^q is a symmetric matrix with zero diagonal. Since $q(i)q(j) \in \{-1, 1\}$ and the (i, j) entry of QBQ is given by $q(i)q(j)B_{ij}$ we have

$$
B^q_{ij} = \begin{cases} B_{ij} & \text{if } q(i)q(j) = 1,\\ 0 & \text{if } q(i)q(j) = -1. \end{cases}
$$

Lemma 4. *Let ξ and B defined as in* Lemma 1. *Moreover, let $Q = \mathrm{diag}(q)$, with $q \in \mathbb{R}^n$ such that $q_i \in \{-1, 1\}$ and B^q as defined above. Then one has*

$$
\Pr(\xi^{\mathrm{T}} B \xi > 0) = \Pr(\xi^{\mathrm{T}} QBQ \xi > 0), \tag{6}
$$

and

$$
\Pr(\xi^{\mathrm{T}} B^q \xi > 0) \geqslant 2 \Pr(\xi^{\mathrm{T}} B \xi > 0) - 1. \tag{7}
$$

Proof. We have

$$
(Q\xi)^{\mathrm{T}} \cdot QBQ \cdot Q\xi = \xi^{\mathrm{T}} Q^2 B Q^2 \xi = \xi^{\mathrm{T}} B \xi,
$$

since $Q^2 = I_n$, where I_n is the $n \times n$ identity matrix. Hence

$$
\Pr(\xi^{\mathrm{T}} B \xi > 0) = \Pr((Q\xi)^{\mathrm{T}} \cdot QBQ \cdot Q\xi > 0).
$$

Since ξ and $Q\xi$ occur with the same probability this implies (6). To prove (7) we use the fact

$$
\begin{aligned}
\Pr(\xi^{\mathrm{T}} B^q \xi > 0) &= \Pr(\xi^{\mathrm{T}}(B + QBQ)\xi > 0)\\
&\geqslant \Pr(\xi^{\mathrm{T}} B \xi > 0\ \&\ \xi^{\mathrm{T}} QBQ \xi > 0).
\end{aligned}
$$

Then using Lemma 2 we get

$$
\begin{aligned}
&\Pr(\xi^{\mathrm{T}} B \xi > 0\ \&\ \xi^{\mathrm{T}} QBQ \xi > 0)\\
&\quad \geqslant \Pr(\xi^{\mathrm{T}} B \xi > 0) + \Pr(\xi^{\mathrm{T}} QBQ \xi > 0) - 1\\
&\quad = 2 \Pr(\xi^{\mathrm{T}} B \xi > 0) - 1,
\end{aligned}
$$

where the last equality follows from (6). Therefore we get inequality (7). □

At this point, using our result in Lemma 4, we are ready to prove Lemma 1.

Proof of Lemma 1. Assume to the contrary that Lemma 1 is false. Then, one can see from the derivation of inequality (5) that there exists a symmetric $n \times n$ matrix B having zero diagonal such that

$$\Pr(\xi^T B \xi \leqslant 0) < \frac{1}{2^{\lceil \log_2(n) \rceil}} \qquad (8)$$

which is equivalent to

$$\Pr(\xi^T B \xi > 0) > 1 - \frac{1}{2^{\lceil \log_2(n) \rceil}}. \qquad (9)$$

We construct a sequence of block diagonal matrices B_i having zero diagonal such that

$$B_1 = B, \quad B_{i+1} = B_i^{q_i}, \quad i = 1, 2, \ldots, k.$$

We have $k = \lceil \log_2(n) \rceil$, and q_i's are chosen according to the following process. For q_1 we take the first $\lceil n/2 \rceil$ entries as 1's and the remaining entries as -1's. Let us call these two parts of q_1 as segments of q_1. We illustrate this for $n = 13$ with two segments separated by the symbol " | ".

$$q_1 = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ | \ -1 \ -1 \ -1 \ -1 \ -1 \ -1].$$

For q_{i+1}, consider each segment of q_i. If the length of a segment is l we take the first $\lceil l/2 \rceil$ entries as 1's and the remaining entries in the segment as -1's. Let us call these two parts segments again. Note that if $l = 1$ for a segment the process will produce only one part of length 1 out of the segment. The resulting vector is q_{i+1} with its segments defined as above. To illustrate it for $n = 13$, we show q_2 obtained from q_1. Here, q_2 has four segments separated by the symbol " | " again:

$$q_2 = [1 \ 1 \ 1 \ 1 \ | \ -1 \ -1 \ -1 \ | \ 1 \ 1 \ 1 \ | \ -1 \ -1 \ -1].$$

Now, let S denote the first principal submatrix of B with size $\lceil n/2 \rceil \times \lceil n/2 \rceil$, and let T denote the last principal submatrix of B with size $\lfloor n/2 \rfloor \times \lfloor n/2 \rfloor$. Denote the remaining matrix at the upper right corner of B by R, and the remaining matrix at the lower left corner of B becomes R^T since B is symmetric. Then B^{q_1} is obtained from B by replacing all entries of R and R^T by zeros. In other words

$$B_1 = B = \begin{bmatrix} S & R \\ R^T & T \end{bmatrix} \Rightarrow Q_1 B_1 Q_1 = \begin{bmatrix} S & -R \\ -R^T & T \end{bmatrix}$$

$$\Rightarrow B_2 = B^{q_1} = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix},$$

where Q_1 is the diagonal matrix with the vector q_1 as the diagonal. Now using Lemma 4 and (9) we obtain

$$\Pr(\xi^T B_2 \xi > 0) > 2 \left(1 - \frac{1}{2^{\lceil \log_2(n) \rceil}} \right) - 1$$

$$= 1 - \frac{2}{2^{\lceil \log_2(n) \rceil}}. \qquad (10)$$

Note that the block matrices along the diagonal of B_2 have sizes $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$. Hence, the sizes do not exceed $f(n)$ of Lemma 3 which was defined as $f(n) = \lceil n/2 \rceil$. We repeat the above procedure using q_2 which was shown before. Thus we obtain $B_3 = B_2^{q_2}$ which has the form

$$B_3 = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{bmatrix},$$

where D_1, D_2, D_3 and D_4 constitute the symmetric, zero-diagonal blocks of the block diagonal matrix B_3. These block matrices have dimensions $\lceil \frac{1}{2} \lceil n/2 \rceil \rceil \times \lceil \frac{1}{2} \lceil n/2 \rceil \rceil$, $\lfloor \frac{1}{2} \lceil n/2 \rceil \rfloor \times \lfloor \frac{1}{2} \lceil n/2 \rceil \rfloor$, $\lceil \frac{1}{2} \lfloor n/2 \rfloor \rceil \times \lceil \frac{1}{2} \lfloor n/2 \rfloor \rceil$, $\lfloor \frac{1}{2} \lfloor n/2 \rfloor \rfloor \times \lfloor \frac{1}{2} \lfloor n/2 \rfloor \rfloor$, respectively.

Now, again by Lemma 4 and (10) B_3 satisfies

$$\Pr(\xi^T B_3 \xi > 0) > 2 \left(1 - \frac{2}{2^{\lceil \log_2(n) \rceil}} \right) - 1$$

$$= 1 - \frac{2^2}{2^{\lceil \log_2(n) \rceil}}. \qquad (11)$$

Note that the sizes of the block diagonal matrices along the diagonal of B_3 can be at most $\lceil \frac{1}{2} \lceil n/2 \rceil \rceil$ which does not exceed $f^2(n)$. We construct q_3 in the same way as before. For $n = 13$ this gives

$$q_3 = [1 \ 1 \ | \ -1 \ -1 \ | \ 1 \ 1 \ | \ -1 \ | \ 1 \ 1 \ | \\ -1 \ | \ 1 \ 1 \ | \ -1].$$

Again by using Lemma 4 and (11) we obtain for B_4 that

$$\Pr(\xi^T B_4 \xi > 0) > 2 \left(1 - \frac{2^2}{2^{\lceil \log_2(n) \rceil}} \right) - 1$$

$$= 1 - \frac{2^3}{2^{\lceil \log_2(n) \rceil}}. \qquad (12)$$

This time the sizes of the block diagonal matrices along the diagonal of B_4 do not exceed $f^3(n)$. Then, q_4 is constructed in the same manner, and for $n = 13$ we have

$$q_4 = [1 \mid -1 \mid 1 \mid -1 \mid 1 \mid -1 \mid 1 \mid 1 \mid$$
$$-1 \mid 1 \mid -1 \mid 1 \mid -1].$$

Hence, at the next step we get

$$\Pr(\xi^{\mathrm{T}} B_5 \xi > 0) > 2 \left(1 - \frac{2^3}{2^{\lceil \log_2(n) \rceil}} \right) - 1$$

$$= 1 - \frac{2^4}{2^{\lceil \log_2(n) \rceil}}, \tag{13}$$

and the sizes of the block diagonal matrices along the diagonal of B_5 do not exceed $f^4(n)$. Note that for $n = 13$ these block matrices all have size 1. In the general case we proceed in the same way and after k steps we obtain

$$\Pr(\xi^{\mathrm{T}} B_{k+1} \xi > 0) > 1 - \frac{2^k}{2^{\lceil \log_2(n) \rceil}}, \tag{14}$$

and the block diagonal matrices along the diagonal of B_{k+1} have sizes that do not exceed $f^k(n)$. Now Lemma 3 implies that if $k = \lceil \log_2(n) \rceil$, then $f^k(n) \leqslant 1$. In that case the right hand side of (14) is equal to 0. Also, the block diagonal matrices along the diagonal of B_{k+1} have sizes at most 1. We know from the construction procedure of B_{k+1} that it has zero diagonal. Hence, B_{k+1} becomes a matrix of zeros. But then the left-hand side of (14) is also equal to 0. Therefore, we arrive at the contradiction $0 > 0$. This completes the proof of Lemma 1.   □

Now, it suffices to observe that equipped with the result of the previous lemma, one has to solve Eq. (A.38) pp. 559 of [1] using the probability bound $1/2n$ to obtain the improved bound (3).

Although we were not able to prove the conjecture of Ben-Tal et al. in [1] that would help us remove the factor n under the logarithm altogether, we offered an improvement from n^2 to n under the logarithm. While this paper was under review, we learned of a recent result [2] where it is shown that

$$\Pr(\xi^{\mathrm{T}} B \xi \leqslant \mathrm{Tr}\, B) \geqslant \tfrac{1}{87}.$$

Our result in Lemma 1 remains better in the range $3 \leqslant n \leqslant 64$.

Acknowledgments

References

[1] A. Ben-Tal, A. Nemirovski, C. Roos, Robust solutions of uncertain quadratic and conic-quadratic problems, SIAM J. Optim. 13 (2002) 535–560.
[2] S. He, Z.-Q. Luo, J. Nie, S. Zhang, Semidefinite relaxation bounds for indefinite homogeneous quadratic optimization, Technical Report SEEM2007-01, Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, 2007.