Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 319 (2008) 3810–3853
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Rhetorical biset functors, rational p-biset functors and their semisimplicity in characteristic zero

Laurence Barker

Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey

Received 27 October 2006
Available online 1 February 2008
Communicated by Michel Broué

Abstract

Rhetorical biset functors can be defined for any family of finite groups that is closed under subquotients up to isomorphism. The rhetorical p-biset functors almost coincide with the rational p-biset functors. We show that, over a field with characteristic zero, the rhetorical biset functors are semisimple and, furthermore, they admit a character theory involving primitive characters of automorphism groups of cyclic groups.

Keywords: Biset functor; Representation ring; Burnside ring; Primitive character

1. Introduction and conclusions

Finite group representation theory has been based, essentially, on two methods for reduction to smaller groups. One of them, reduction to subgroups, is usually effected by means of induction and restriction. The other, reduction to quotient groups, is sometimes effected by means of inflation and, when it exists, deflation. Isogation is even more important than induction, restriction, inflation and deflation. In fact, it is so ubiquitous that it normally passes without mention. By isogation, we mean transport of structure through a group isomorphism. Mackey functors capture the notions of induction, restriction and isogation. Biset functors, introduced by Bouc [2], capture all five notions: induction, restriction, isogation, inflation and deflation. It can be said that, in the theory of biset functors, reduction to subgroups and reduction to quotient groups are unified within a more general method: reduction to subquotients.

E-mail address: barker@fen.bilkent.edu.tr.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.12.018

Throughout this paper, we let R be a commutative unital ring and we let $\mathbb{K}$ be a field with characteristic zero. We let G be a finite group. We let $\mathcal{X}$ be a non-empty set of finite groups that is closed under subquotients up to isomorphism. That is to say, if G is in $\mathcal{X}$, then any subquotient of G is isomorphic to some group in $\mathcal{X}$. We let p be a prime.

We shall speak of biset functors for $\mathcal{X}$ over R. A case of especial concern to us will be that where R is replaced by $\mathbb{K}$. A local scenario: we understand a biset functor for G to be a biset functor for $\mathcal{X}(G)$, where $\mathcal{X}(G)$ is a set of representatives of the isomorphism classes of subquotients of G. A global scenario: we understand a p-biset functor to be a biset functor for $\mathcal{X}_p$, where $\mathcal{X}_p$ is a set of representatives of the isomorphism classes of finite p-groups.

A biset functor L for $\mathcal{X}$ over R can be seen as a family of R-modules together with five kinds of R-maps. For each group G in $\mathcal{X}$, there is an R-module $L(G)$. There are two "upward" maps, namely, a transfer map and an inflation map

$$\mathrm{tra}^{\nu}_{G,H} : L(G) \leftarrow L(H), \qquad \mathrm{inf}^{\mu}_{F,G} : L(F) \leftarrow L(G)$$

where $\nu : G \leftarrow H$ is a group monomorphism and $\mu : F \leftarrow G$ is a group epimorphism. There are two "downward" maps, namely a restriction map and a deflation map

$$\mathrm{res}^{\nu}_{H,G} : L(H) \leftarrow L(G), \qquad \mathrm{def}^{\mu}_{G,F} : L(G) \leftarrow L(F).$$

The fifth kind of map is an isogation map

$$\mathrm{iso}^{\theta}_{G,G'} : L(G) \leftarrow L(G')$$

where $\theta : G \leftarrow G'$ is a group isomorphism. Some relations are imposed on these five kinds of map. For instance, the isogation map associated with an inner automorphism of G is required to be the identity map on $L(G)$. We also require that

$$\mathrm{iso}^{\theta}_{G,G'} \, \mathrm{iso}^{\theta'}_{G',G''} = \mathrm{iso}^{\theta\theta'}_{G,G''}.$$

Those two relations ensure that $L(G)$ is an $R\,\mathrm{Out}(G)$-module. We shall be regarding the biset functors for $\mathcal{X}$ over R as modules of the R-algebra $R\Gamma^{\mathcal{X}}$ generated by these five kinds of map, with all the relations accommodated. (Concerning the fifth element, "For if the natural motion is upward, it will be fire or air, and if downward, water or earth ... It necessarily follows that circular movement, being unnatural to these bodies, is the natural movement of some other," Aristotle, *On the Heavens, I.2.*) We call $R\Gamma^{\mathcal{X}}$ the alchemic algebra for $\mathcal{X}$ over R. The rationale for the terminology is that the alchemic algebra is composed of five kinds of elements (two moving upwards, two moving downwards, one moving in circles) just as the alchemic theory proposes five kinds of elements (two moving upwards, two moving downwards and one moving in circles).

In close analogy with the Thévenaz–Webb classification of the simple Mackey functors, Bouc showed how the simple biset functors $S^{\mathcal{X},R}_{H,\nu}$ for $\mathcal{X}$ over R are parameterized by the pairs (H, ν) where H is a group in $\mathcal{X}$ well-defined up to isomorphism and ν is a simple $R\,\mathrm{Out}(H)$-module up to isomorphism. See Section 2. A celebrated theorem of Thévenaz and Webb asserts that every Mackey functor over $\mathbb{K}$ is semisimple. Alas, as a negative result, we have the following necessary and sufficient criterion for semisimplicity of biset functors over $\mathbb{K}$. The theorem was established independently and with priority by Bouc (personal communication).

Theorem 1.1 *(Bouc). Every biset functor for $\mathcal{X}$ over $\mathbb{K}$ is semisimple if and only if every group in $\mathcal{X}$ is cyclic.*

We shall see that one direction is easy. The necessity of the criterion will become evident from a glance at the biset functor $\mathbb{K}B^{\mathcal{X}}$ associated with the $\mathbb{K}$-linear extension $\mathbb{K}B(G)$ of the Burnside ring $B(G)$. See Corollary 2.7. The sufficiency of the criterion will be demonstrated, in Section 5, by an argument involving a calculation of dimensions.

This paper is concerned with two classes of biset functors, called rhetorical biset functors and rational p-biset functors. The rhetorical biset functors are defined for arbitrary $\mathcal{X}$, whereas the rational p-biset functors are defined only when all the groups in $\mathcal{X}$ are p-groups. For p-groups, the two classes are very similar to each other and, for some coefficient rings, they coincide with each other. The new term *rhetorical* has been chosen because the term *rational* has already been used by Bouc (and also because, in *Elements, Book 10*, Euclid uses *rhetos* to refer to certain ratios that are close to being rational).

The definitions of the two classes of biset functor will be presented in Section 3. The definitions are very difficult to express, even vaguely, without the prerequisite background machinery. For now, let us attempt only a very sketchy indication. Something akin to both of the concepts was implicitly introduced by Hambleton, Taylor and Williams [9]. Their "group ring functors" differ from biset functors in several ways. One of the differences is that their functors are constructed using bimodules, whereas biset functors are constructed using bisets. The notion of a rhetorical biset functor captures something of this bimodule construction. We shall construct a quotient algebra $R\Upsilon^{\mathcal{X}}$ of $R\Gamma^{\mathcal{X}}$, and we shall realize the rhetorical biset functors as precisely those biset functors that can be inflated from $R\Upsilon^{\mathcal{X}}$.

Hambleton et al. showed that, for the class of hyperelementary groups, their functors have, as they called it, "detection" and "generation" properties. These two properties allow for reduction to the subclass consisting of the hyperelementary groups whose normal abelian subgroups are all cyclic. The "generation" condition roughly says that the whole functor can be obtained by induction and inflation from that subclass. The "detection" property roughly says that the functor is determined by its deflations and restrictions to that subclass. The rational p-biset functors of Bouc [4] are defined to be the p-biset functors which satisfy a version of the "generation" property.

Something the history of this paper can now be narrated. The notion of a rhetorical biset functor, as presented in Section 3, arose initially from some Bilkent seminars by Yalçın, in 2004, concerning some speculative applications of [9] to the study of Dade groups. It was Yalçın who noticed the connection between the work of Hambleton et al. and the work of Bouc. In April 2006, a few days before Bouc came to visit us in Bilkent, I sent Bouc an incomplete version of the present paper. At that time, I was presuming that the rhetorical p-biset functors are the same as the rational p-biset functors, but I had neglected to confirm it. Bouc queried this gap. Upon his arrival, we found that each of the three of us had a different preference as to the explanation of the easy direction. In Section 8, the following result will be demonstrated using the classification of the rhetorical biset functors. Another proof of the result appears in Bouc [7]. Yalçın—who had already recognized the result back in 2004—noted that it can be derived from the proof of [9, 1.A.11, 1.A.12].

Theorem 1.2 *(Hambleton, Taylor, Williams). Supposing that every group in $\mathcal{X}$ is a p-group, then every rhetorical biset functor for $\mathcal{X}$ is rational. In particular, every rhetorical p-biset functor is rational.*

Counterexamples abound for the converse to the main part of the theorem, even in characteristic zero. See Non-example 3.E below. The issue is more subtle in the case of p-biset functors. Eventually, Bouc [7] found the counterexample in Non-example 3.F, and he also obtained the following resolution of the matter.

Theorem 1.3 *(Bouc). The rhetorical p-biset functors over a field $\mathbb{F}$ coincide with the rational p-biset functors over $\mathbb{F}$ if and only if* $\mathrm{char}(\mathbb{F}) \neq 2$ *or* $p \neq 2$. *In particular, the rhetorical p-biset functors over $\mathbb{K}$ coincide with the rational p-biset functors over $\mathbb{K}$.*

Let us sketch an example. We introduce another field $\mathbb{J}$ with characteristic zero. Throughout this paper, we shall tend to use $\mathbb{J}$ as a coefficient field for modules of group algebras, whereas we shall tend to use $\mathbb{K}$ as a coefficient ring for biset functors. The $\mathbb{J}$-representation algebra $RA_{\mathbb{J}}(G)$, also called the $\mathbb{J}$-character algebra, is an R-module freely generated by the $\mathbb{J}G$-irreps; we understand a $\mathbb{J}G$-irrep to be a simple $\mathbb{J}G$-module, in other words, an irreducible $\mathbb{J}G$-character, or equivalently, an irreducible $\mathbb{J}G$-representation. The $\mathbb{J}$-representation biset functor for $\mathcal{X}$ over R is defined to be the biset functor

$$RA_{\mathbb{J}}^{\mathcal{X}} = \bigoplus_{G \in \mathcal{X}} RA_{\mathbb{J}}(G).$$

In Example 3.A, we shall show that $RA_{\mathbb{J}}$ is a rhetorical biset functor. We defer to Section 3 for a more detailed discussion.

The term "rational p-biset functor" testifies to some more history. As usual, the concept was introduced in a difficult context; easy contexts emerge through subsequent work of pedagogues. Bouc [4] introduced the term in connection with the rational p-biset functor $D_{\mathrm{tors}}^{\Omega}$, where $D_{\mathrm{tors}}^{\Omega}(G)$ is the torsion subgroup of the group generated by the relative syzygies in the Dade group of G. The significance of "rational" is that $D_{\mathrm{tors}}^{\Omega}$ has a property that was already familiar from Bouc's study [3] of the rational representation functor $RA_{\mathbb{Q}}$ as a p-biset functor.

Theorem 1.4. *Every rhetorical biset functor over $\mathbb{K}$ is semisimple.*

In particular, the biset functor $\mathbb{K}A_{\mathbb{J}}$ is semisimple. We shall also classify the simple rhetorical biset functors over $\mathbb{K}$.

Theorem 1.5. *The simple rhetorical biset functors for $\mathcal{X}$ over $\mathbb{K}$ are precisely the simple biset functors over $\mathbb{K}$ that have the form $S_{C,\sigma}^{\mathcal{X},\mathbb{K}}$ where C is a cyclic group in $\mathcal{X}$ and σ is a primitive $\mathbb{K}\,\mathrm{Aut}(C)$-module.*

Precisely those simple biset functors have already made an appearance in the following result of Bouc [2, 10.3].

Theorem 1.6 *(Bouc). As a direct sum of rhetorical biset functors over $\mathbb{K}$, we have*

$$\mathbb{K}A_{\mathbb{C}} \cong \bigoplus_{C,\sigma} S_{C,\sigma}^{\mathcal{X},\mathbb{K}}$$

where C runs over representatives of the isomorphism classes of cyclic groups in $\mathcal{X}$ and σ runs over the isomorphism classes of primitive $\mathbb{K}\,\mathrm{Aut}(C)$-modules.

In other words, each simple rhetorical biset functor occurs exactly once in the semisimple biset functor $\mathbb{K}A_{\mathbb{C}}$. At the other extreme, the semisimple biset functor $\mathbb{K}A_{\mathbb{Q}}$ is actually simple; the following result is Bouc [2, 7.2.1].

Theorem 1.7 *(Bouc). As an isomorphism of rhetorical biset functors,* $\mathbb{K}A_{\mathbb{Q}} \cong S_{1,1}$.

The notion of primitivity will be recalled in Section 7. A proof of Theorem 1.5 and a new proof of Theorem 1.6 will be given in Section 8. The following generalization of Theorems 1.6 and 1.7 will be proved in Section 9.

Theorem 1.8. *As a direct sum of rhetorical biset functors,*

$$\mathbb{K}A_{\mathbb{J}} \cong \bigoplus_{C,\sigma} S_{C,\sigma}^{\mathcal{X},\mathbb{K}}.$$

Here, C still runs over the representatives of the isomorphism classes of cyclic groups in $\mathcal{X}$. Letting $\mathbb{J}_C$ be the extension field obtained from $\mathbb{J}$ by adjoining a primitive $|C|$th root of unity, then the Galois group $\mathrm{Gal}(\mathbb{J}_C/\mathbb{J})$ *embeds in* $\mathrm{Aut}(C)$. *The index σ runs over those primitive* $\mathbb{K}\,\mathrm{Aut}(C)$*-modules whose kernel contains* $\mathrm{Gal}(\mathbb{J}_C/\mathbb{J})$.

The proof will be an application of the character-theoretic principle expressed in the following theorem. As we shall explain in Section 9, the nearest analogues of this theorem for Mackey functors involve recursion or a Möbius inversion formula. For Mackey functors in characteristic zero, the multiplicity of a given simple factor cannot be determined just by examining a single coordinate module.

Theorem 1.9. *Given a rhetorical biset functor L over $\mathbb{K}$, then*

$$L \cong \bigoplus_{C,\sigma} m_{C,\sigma} S_{C,\sigma}^{\mathcal{X},\mathbb{K}}.$$

For any cyclic group C in $\mathcal{X}$ and any primitive $\mathbb{K}\,\mathrm{Aut}(C)$*-module σ, the multiplicity $m_{C,\sigma}$ of $S_{C,\sigma}$ in L is equal to the multiplicity of σ in the* $\mathbb{K}\,\mathrm{Aut}(C)$*-module* $L(C)$.

2. Some recollections concerning biset functors

In essential content, this section is just a summary of some prerequisite material from Bouc [2]. However, by treating biset functors as modules of the alchemic algebra $R\Upsilon^{\mathcal{X}}$ and by allowing the possibility that $\mathcal{X}$ is finite, we shall be able to make use of the theory of unital algebras that have finite rank over their coefficient rings. This will be convenient when we discuss semisimplicity.

The passage from finite $\mathcal{X}$ to arbitrary $\mathcal{X}$ will be plain sailing, but this is only because it will follow in the wake of a little ring-theoretic tug. Let us quickly admit the ring theory.

Recall that ring is said to be **unital** provided it has a unity element. A homomorphism of unital rings is said to be **unital** provided the unity element is preserved. A module of a unital ring is said to be **unital** provided the unity element acts as the identity map. But we shall be needing something slightly more general than that. Consider a ring Λ containing a set of mutually orthogonal idempotents $\mathcal{E}$ which is complete in the sense that $\Lambda = \bigoplus_{e,f \in \mathcal{E}} e\Lambda f$. Such a ring Λ is said to be **locally unital**. A homomorphism of locally unital rings is said to be **locally unital** provided the image of some complete $\mathcal{E}$—and then necessarily all complete $\mathcal{E}$—is itself complete. A Λ-module M is said to be **locally unital** provided, for some $\mathcal{E}$—and then all $\mathcal{E}$—we have $M = \bigoplus_{e \in \mathcal{E}} eM$. This is equivalent to the condition that the representation $\Lambda \to \mathrm{End}(M)$ is locally unital. Another equivalent condition is that, for all $m \in M$ there exists some $\lambda \in \Lambda$ satisfying $\lambda m = m$. For any Λ-module, there is a maximum among the locally unital submodules.

A module, recall, is semisimple provided it is a sum—then necessarily a direct sum—of simple modules. (The sums may be infinite or empty.) But a ring is said to be semisimple provided every module is semisimple, in which case the ring can have only finitely many isomorphism classes of simple modules. Let us list some easy exercises which deal with such matters. Given a finite subset $\mathcal{F} \subseteq \mathcal{E}$, then the sum of the elements of $\mathcal{F}$, denoted $1_{\mathcal{F}}$, is an idempotent of Λ. Let M and N be locally unital Λ-modules. The following three conditions are mutually equivalent: $M \cong N$ as Λ-modules; for all idempotents i of Λ, we have $iM \cong iN$ as $i\Lambda i$-modules; for all finite subsets $\mathcal{F} \subseteq \mathcal{E}$, we have $1_{\mathcal{F}}M \cong 1_{\mathcal{F}}N$ as $1_{\mathcal{F}}\Lambda 1_{\mathcal{F}}$-modules. The next three conditions are mutually equivalent: M is semisimple; each iM is semisimple; each $1_{\mathcal{F}}M$ is semisimple. Up to isomorphism, there is a bijective correspondence $iS \leftrightarrow S$ between the simple $i\Lambda i$-modules iS and those simple Λ-modules S that are not annihilated by i. And the next three conditions are equivalent: every locally unital Λ-module is semisimple; each ring $i\Lambda i$ is semisimple; each ring $1_{\mathcal{F}}\Lambda 1_{\mathcal{F}}$ is semisimple. In that case, we say that Λ is **locally semisimple**.

The ring theory has passed through, and we can now open our eyes again. Consider finite groups I, J, K. An **I-J-biset** is defined to be a finite $I \times J$-set (a finite permutation set for the group $I \times J$) with the action on the left. Let X be an I-J-biset and let Y be a J-K-biset. For $(i, j) \in I \times J$ and $x \in X$, we write $ixj^{-1} = (i, j)x$. The **cross product** of X and Y over J, denoted $X \times_J Y$, is defined to be the I-K-biset consisting of the J-orbits of the $I \times J \times K$-set $X \times Y$. Writing $x \times_J y$ for the J-orbit of an element $(x, y) \in X \times Y$, then $(i, k)(x \times_J y) = ix \times_J yk^{-1}$.

Recall that the transitive G-sets have the form G/H where $H \leqslant G$. We use square brackets to indicate an isomorphism class. The isomorphism classes $[G/H]$ comprise a $\mathbb{Z}$-basis for the Burnside ring $B(G)$ and, more generally, they comprise an R-basis for the Burnside algebra $RB(G) = R \otimes_{\mathbb{Z}} B(G)$. As an R-module, we define

$$R\Gamma(I, J) = RB(I \times J).$$

The cross product over J gives rise to a binary operation, called multiplication,

$$R\Gamma(I, J) \times R\Gamma(J, K) \to R\Gamma(I, K).$$

We shall give two equivalent definitions of a biset functor. According to one definition, which we shall refer to as the theological definition, the set $\mathcal{X}$ is to be regarded as an R-preadditive category such that, given objects $I, J \in \mathcal{X}$, then $R\Gamma(I, J)$ is the hom-set to I from J. A **biset functor** for $\mathcal{X}$ over R is defined to be an R-additive functor from $\mathcal{X}$ to the category of R-modules. The morphisms of these biset functors are defined to be the R-additive natural transformations.

The other definition of a biset functor, which we shall refer to as the occult definition, is expressed in terms of the R-algebra

$$R\Gamma^{\mathcal{X}} = \bigoplus_{I, J \in \mathcal{X}} R\Gamma(I, J)$$

which we call the **alchemic algebra** for $\mathcal{X}$ over R. The multiplication operation on $R\Gamma^{\mathcal{X}}$ comes from the multiplication operation defined above. In a moment, we shall explain why $R\Gamma^{\mathcal{X}}$ is locally unital. A **biset functor** for $\mathcal{X}$ over R is defined to be a locally unital $R\Gamma^{\mathcal{X}}$-module. The morphisms of these biset functors are defined to be the module homomorphisms.

Throughout this paper, we shall be working exclusively with the occult definition; we shall always understand a biset functor to a locally unital module of the alchemic algebra. Soon, we shall be checking that the two definitions are equivalent, but first let us introduce a little more notation to facilitate the discussion.

The **biset algebra** for I over R is defined to be the R-subalgebra $R\Gamma(I,I)$ of $R\Gamma^{\mathcal{X}}$. The R-algebra $R\Gamma(I,I)$ is unital, and its unity element is

$$\mathrm{iso}_I = \big[(I \times I)/\Delta(I)\big]$$

where $\Delta(I)$ denotes the diagonal subgroup $\{(i,i)\colon\ i \in I\}$ of $I \times I$. We have

$$R\Gamma(I,J) = \mathrm{iso}_I \, R\Gamma^{\mathcal{X}} \, \mathrm{iso}_J \, .$$

So $\{\mathrm{iso}_I\colon\ I \in \mathcal{X}\}$ is a complete set of mutually orthogonal idempotents of $R\Gamma^{\mathcal{X}}$. We have now shown that $R\Gamma^{\mathcal{X}}$ is locally unital. Evidently, the elements of $R\Gamma(I,J)$ act as maps $L(I) \leftarrow L(J)$.

Let L be a biset functor for $\mathcal{X}$ over R (a locally unital $R\Gamma^{\mathcal{X}}$-module). The **coordinate module** for L at I is defined to be $L(I) = \mathrm{iso}_I \, L$. Since L is a locally unital module,

$$L = \bigoplus_{I \in \mathcal{X}} L(I)$$

as a direct sum of R-modules. (We did need to insist that biset functors are locally unital. Without that hypothesis, we would not have the decomposition as a direct sum of the coordinate modules.) Consider a morphism $\lambda : L \to L'$ of biset functors for $\mathcal{X}$ over R. Since λ commutes with all the idempotents iso_I, we have $\lambda = \bigoplus_I \lambda_I$ as a direct sum of R-linear maps $\lambda_I : L(I) \to L'(I)$. We call λ_I the **coordinate map** of λ at I.

We can now see why the two definitions of a biset functor are equivalent. Consider $\theta : L \to L'$ as above, and write $\rho : R\Gamma^{\mathcal{X}} \to \mathrm{End}_R(L)$ and $\rho' : R\Gamma^{\mathcal{X}} \to \mathrm{End}_R(L')$ for the representations. Since L is locally unital, it is determined by the coordinate modules $L(I)$ together with the restrictions of ρ to the R-linear maps between hom-sets $R\Gamma(I,J) \to \mathrm{Hom}_R(L(J), L(I))$. Thus, L and L' give rise to R-additive functors $\mathcal{X} \to R$-Mod, and θ is a natural transformation between these two functors. These constructions can be reversed, and thus we obtain an equivalence between the category of biset functors over $\mathcal{X}$ and the category of R-additive functors $\mathcal{X} \to R$-Mod.

Up to equivalence, the category of biset functors for $\mathcal{X}$ over R depends only on R and on the isomorphism classes of groups that appear in $\mathcal{X}$ (and not on the multiplicity of each isomorphism class). Indeed, this is already clear from the theological definition of a biset functor, but the occult definition allows us to express the point a little more precisely. Let $\mathcal{Y}$ be a set of groups such that the isomorphism classes appearing in $\mathcal{X}$ and in $\mathcal{Y}$ coincide. Consider the $R\Gamma^{\mathcal{X}}$-$R\Gamma^{\mathcal{Y}}$-bimodule ${}^{\mathcal{X}}R\Gamma^{\mathcal{Y}} = \bigoplus_{I \in \mathcal{X}, J \in \mathcal{Y}} R\Gamma(I,J)$ and the similarly defined bimodule ${}^{\mathcal{Y}}R\Gamma^{\mathcal{X}}$. Tensoring over $R\Gamma^{\mathcal{Y}}$, there is an evident bimodule isomorphism ${}^{\mathcal{X}}R\Gamma^{\mathcal{Y}} \otimes {}^{\mathcal{Y}}R\Gamma^{\mathcal{X}} \cong R\Gamma^{\mathcal{X}}$. So the functors ${}^{\mathcal{X}}R\Gamma^{\mathcal{Y}} \otimes -$ and ${}^{\mathcal{Y}}R\Gamma^{\mathcal{X}} \otimes -$ yield mutually inverse Morita equivalences between the category of biset functors for $\mathcal{X}$ over R and the category of biset functors for $\mathcal{Y}$ over R.

Since $R\Gamma(I, J)$ has an R-basis consisting of the isomorphism classes of transitive I-J-bisets, and since these bisets have the form $[(I \times J)/S]$ where S runs over the conjugacy classes of subgroups of $I \times J$, we had better recall Goursat's Theorem concerning the subgroups of a direct product. For any subgroup $S \leqslant I \times J$, let the subgroup ${\uparrow}S \leqslant I$ be the image of the projection from S to I. Let the normal subgroup ${\downarrow}S \trianglelefteq {\uparrow}S$ be such that ${\downarrow}S \times 1$ is the kernel of the projection from S to J. We define subgroups $S{\downarrow} \trianglelefteq S{\uparrow} \leqslant J$ similarly. The two evident epimorphisms ${\uparrow}S/{\downarrow}S \leftarrow S \rightarrow S{\uparrow}/S{\downarrow}$ both have kernel $S/({\downarrow}S \times S{\downarrow})$. Hence we obtain two group isomorphisms

$$\frac{{\uparrow}S}{{\downarrow}S} \cong \frac{S}{{\downarrow}S \times S{\downarrow}} \cong \frac{S{\uparrow}}{S{\downarrow}}.$$

We let θ_S be the composite isomorphism ${\uparrow}S/{\downarrow}S \leftarrow S{\uparrow}/S{\downarrow}$. By recollection or by easy exercise, we obtain the following venerable theorem.

Theorem 2.1 *(Goursat's Theorem). For arbitrary groups I and J, there is a bijective correspondence between the subgroups $S \leqslant I \times J$ and the quintuples $(I_1, I_2, \theta, J_2, J_1)$ such that $I_2 \trianglelefteq I_1 \leqslant I$ and $J_2 \trianglelefteq J_1 \leqslant J$ and θ is an isomorphism $I_1/I_2 \leftarrow J_1/J_2$. The correspondence is such that $S \leftrightarrow ({\uparrow}S, {\downarrow}S, \theta_S, S{\downarrow}, S{\uparrow})$.*

For subgroups $S \leqslant I \times J$ and $T \leqslant J \times K$, the **join** of S and T is defined to be the subgroup $S * T \leqslant I \times K$ such that $(i, k) \in S * T$ if and only if $(i, j) \in S$ and $(j, k) \in T$ for some j. The next result, due to Bouc [2, 3.2], amounts to an explicit formula for the cross product of two transitive bisets (but we express the formula as the product of two elements of the alchemic algebra).

Theorem 2.2 *(Generalized Mackey Product Theorem, Bouc). Given finite groups I, J, K and subgroups $S \leqslant I \times J$ and $T \leqslant J \times K$, then*

$$\left[\frac{I \times J}{S}\right]\left[\frac{J \times K}{T}\right] = \sum_{S{\uparrow}.j.{\uparrow}T \subseteq J} \left[\frac{I \times K}{S * {}^{(j,1)}T}\right]$$

where the notation indicates that j runs over representatives of the double cosets of $S{\uparrow}$ and ${\uparrow}T$ in J. The isomorphism class of the $I \times K$-set $(I \times K)/(S * {}^{(j,1)}T)$ depends only on the double coset $S{\uparrow} . j . {\uparrow}T$.

We can now discuss transfer, inflation, isogation, deflation and restriction. Let $H \leqslant G \trianglerighteq N$, and suppose that the groups H, G, G/N belong to $\mathcal{X}$. Writing $\Delta(H, G) = \{(h, h)\colon h \in H\} = \Delta(G, H)$, we define, respectively, a **transfer map** and a **restriction map**

$$\mathrm{tra}_{G,H} = \left[\frac{G \times H}{\Delta(G, H)}\right] \in R\Gamma(G, H), \qquad \mathrm{res}_{H,G} = \left[\frac{H \times G}{\Delta(H, G)}\right] \in R\Gamma(H, G).$$

Writing $\Delta(G, G/N) = \{(g, gN)\colon\ g \in G\}$ and $\Delta(G/N, G) = \{(gN, g)\colon\ g \in G\}$, we define an **inflation map** and a **deflation map**

$$\mathrm{inf}_{G,G/N} = \left[\frac{G \times G/N}{\Delta(G, G/N)}\right] \in R\Gamma(G, G/N), \qquad \mathrm{def}_{G/N,G} = \left[\frac{G/N \times G}{\Delta(G/N, G)}\right] \in R\Gamma(G/N, G).$$

Given an isomorphism $\theta : G \leftarrow G'$ between two groups in $\mathcal{X}$, we write $\Delta(G, \theta, G') = \{(\theta(x), x)\colon\ x \in G'\}$, and we define an **isogation map**

$$\mathrm{iso}^{\theta}_{G,G'} = \left[\frac{G \times G'}{\Delta(G, \theta, G')}\right] \in R\Gamma(G, G').$$

Of course, we are referring to these elements as maps because, given a biset functor L, then each of these elements acts as a map between two coordinate modules of L. Note that the unity element iso_G of the biset algebra $R\Gamma(G, G)$ can be variously written as $\mathrm{iso}_G = \mathrm{tra}_{G,G} = \mathrm{inf}_{G,G} = \mathrm{iso}^1_{G,G} = \mathrm{def}_{G,G} = \mathrm{res}_{G,G}$. Also note that, given an inner automorphism ϕ of G, then $\Delta(G, \phi, G)$ is conjugate to the diagonal subgroup $\Delta(G) = \Delta(G, G)$, hence $\mathrm{iso}^{\phi}_{G,G} = \mathrm{iso}_G$. The next result is Bouc [2, 3.3].

Theorem 2.3 *(Butterfly Decomposition Theorem, Bouc). Let I and J be finite groups, and let $S \leqslant I \times J$. Assuming that all the groups involved belong to $\mathcal{X}$ then, in the notation of Goursat's Theorem above,*

$$\left[\frac{I \times J}{S}\right] = \mathrm{tra}_{I,\uparrow S}\, \mathrm{inf}_{\uparrow S, \uparrow S/\downarrow S}\, \mathrm{iso}^{\theta_S}_{\uparrow S/\downarrow S, S\uparrow/S\downarrow}\, \mathrm{def}_{S\uparrow/S\downarrow, S\uparrow}\, \mathrm{res}_{S\uparrow, J}\,.$$

The five kinds of maps that we have defined admit fifteen commutation relations, which can be determined laboriously but easily using the latest two results. We shall not write them all down, but we comment on just a few of them. In Section 1, we already discussed the product of two isogations, and we also explained how the coordinate module $L(G)$ becomes an $R\,\mathrm{Out}(G)$-module. Letting $H \leqslant G$ and $H' \leqslant G'$ be such that the isomorphism $\theta : G \leftarrow G'$ restricts to an isomorphism $\theta : H \leftarrow H'$, then

$$\mathrm{iso}^{\theta}_{G,G'}\, \mathrm{tra}_{G',H'} = \mathrm{tra}_{G,H}\, \mathrm{iso}^{\theta}_{H,H'}\,.$$

The isogation maps have similar commutation relations with the inflation, deflation and restriction maps. Two more commutation relations are, with the evident notation,

$$\mathrm{def}_{G/M,G}\, \mathrm{inf}_{G,G/N} = \mathrm{inf}_{G/M,G/MN}\, \mathrm{def}_{G/MN,G/N},$$

$$\mathrm{def}_{G/N,G}\, \mathrm{tra}_{G,H} = \mathrm{tra}_{G/N,HN/N}\, \mathrm{iso}_{HN/N,H/(H\cap N)}\, \mathrm{def}_{H/(H\cap N),H}\,.$$

Among the fifteen commutation relations, there is only one where the corresponding sum in Theorem 2.2 has more than one term: it is the Mackey relation for the product of a restriction map and a transfer map.

Dropping the assumption that H is a subgroup of G, and replacing it with the assumption that H is isomorphic to a subgroup of G, then we can consider a group monomorphism $\nu : H \to G$, and we can define a generalized version of the transfer map

$$\mathrm{tra}^{\nu}_{G,H} = \mathrm{tra}_{G,\nu(H)}\, \mathrm{iso}^{\nu}_{\nu(H),H}\,.$$

Similarly, one can define generalized versions of the inflation, deflation and restriction maps. In view of Theorem 2.3, the alchemic algebra $R\Gamma^{\mathcal{X}}$ is generated by the generalized transfer, inflation, isogation, restriction and deflation maps. Whether or not the groups mentioned in the proposition belong to $\mathcal{X}$, the isomorphism class of a transitive biset can still be expressed as the product of a transfer, an inflation, an isogation, a deflation and a restriction, in that order. It is now clear that, for arbitrary $\mathcal{X}$, the alchemic algebra $R\Gamma^{\mathcal{X}}$ could be defined as the R-algebra generated by the five kinds of maps. There are seventeen kinds of relations on the maps: the fifteen commutation relations; the relation expressing the triviality of any isogation associated with an inner automorphism; the relation asserting that a product is zero when the domain of the left-hand map is distinct from the codomain of the right-hand map.

The simple biset functors were classified by Bouc [2, Section 4], as follows. Note that any simple biset functor S for $\mathcal{X}$ over R is annihilated by some maximal ideal M of R, and hence S becomes a biset functor for $\mathcal{X}$ over the field R/M.

Theorem 2.4 *(Simple Biset Functor Classification Theorem, Bouc). Consider the pairs (H, ν) such that H is a group in $\mathcal{X}$ and ν is a simple $R\,\mathrm{Out}(H)$-module. Two such pairs (H, ν) and (H', ν') are deemed to be equivalent provided $H \cong H'$ and the isomorphism $H \to H'$ transports ν to ν'. There is a bijective correspondence $(H, \nu) \leftrightarrow S_{H,\nu}$ between the equivalence classes of pairs (H, ν) and the isomorphism classes of simple biset functors $S_{H,\nu}$. The correspondence is characterized by the condition that, with respect to the subquotient relation, H is minimal among the groups J in $\mathcal{X}$ satisfying $S_{H,\nu}(J) \neq 0$, and furthermore, $S_{H,\nu}(H) \cong \nu$ as $R\,\mathrm{Out}(H)$-modules.*

We sometimes write $S_{H,\nu}$ less briefly as $S^{R}_{H,\nu}$ or as $S^{\mathcal{X},R}_{H,\nu}$. However, part (3) of the following remark reveals a sense in which the simple biset functor $S^{\mathcal{X},R}_{H,\nu}$ is essentially independent of $\mathcal{X}$. The remark is an easy consequence of the comments on locally unital modules at the beginning of this section. It will be of much use to us as a means for reducing to the case of finite $\mathcal{X}$.

Remark 2.5 *(Finite Reduction Principle for Biset Functors).* Let L and L' be biset functors for $\mathcal{X}$ over R. Let $\mathcal{F}$ run over those finite subsets of $\mathcal{X}$ such that, up to isomorphism, $\mathcal{F}$ is closed under subquotients. Write $1_{\mathcal{F}}$ to denote the unity element of $R\Gamma^{\mathcal{F}}$. Note that $R\Gamma^{\mathcal{F}} = 1_{\mathcal{F}} R\Gamma^{\mathcal{X}} 1_{\mathcal{F}}$ as a subalgebra of $R\Gamma^{\mathcal{X}}$.

(1) $L \cong L'$ if and only if, for all $\mathcal{F}$, we have $1_{\mathcal{F}} L \cong 1_{\mathcal{F}} L$ as biset functors for $\mathcal{F}$ over R.
(2) L is semisimple if and only if the biset functor $1_{\mathcal{F}} L$ is semisimple for all $\mathcal{F}$.
(3) For $H \in \mathcal{F}$ and a simple $R\,\mathrm{Out}(H)$-module ν, we have $S^{\mathcal{F},R}_{H,\nu} \cong 1_{\mathcal{F}} S^{\mathcal{X},R}_{H,\nu}$. In particular, given $K \in \mathcal{F}$, then $S^{\mathcal{F},R}_{H,\nu}(K) \cong S^{\mathcal{X},R}_{H,\nu}(K)$ as $R\,\mathrm{Out}(K)$-modules and as $R\Gamma(K, K)$-modules.

Let us end this section with some comments about three example, two of them quite classical. The biset functor

$$RB = RB^{\mathcal{X}} = \bigoplus_{G \in \mathcal{X}} RB(G)$$

is called the **Burnside biset functor** for $\mathcal{X}$ over R. The action of $R\Gamma^{\mathcal{X}}$ is such that, given an I-J-set X and a J-set Y, then the element $[X] \in R\Gamma(I, J)$ sends the element $[Y] \in RB(J)$ to the element $[X \times_J Y] \in RB(I)$.

The following result is widely-known, but we draw attention to it because of Proposition 3.6 below, which is an analogous result for rhetorical biset functors.

Proposition 2.6. *Suppose that R is a field. Then, as biset functors, the projective cover of the simple functor $S_{1,1}$ is the Burnside functor RB.*

Proof. Theorem 2.4 tells us that $S_{1,1}$ is the unique simple biset functor with a non-zero co-ordinate module at the trivial group. So iso_1 annihilates all the simple biset functors except for $S_{1,1}$. Furthermore, $\mathrm{iso}_1\, S_{1,1} \cong S_{1,1}(1) \cong R$ as R-modules. Therefore iso_1 maps to a primitive idempotent of the semisimple quotient $R\Gamma^{\mathcal{X}}/J(R\Gamma^{\mathcal{X}})$. By idempotent lifting theorems, iso_1 is a primitive idempotent of $R\Gamma^{\mathcal{X}}$. In fact, it must belong to the conjugacy class of primitive idempotents associated with $S_{1,1}$. Therefore $R\Gamma^{\mathcal{X}}\,\mathrm{iso}_1$ is the projective cover of $S_{1,1}$. By regarding I-sets as I-1-bisets, we can make an identification $RB(I) = R\Gamma(I, 1)$, whence $RB^{\mathcal{X}} = R\Gamma^{\mathcal{X}}\, \mathrm{iso}_1$. □

Let us consolidate the sketch we made in Section 1 concerning the biset functor $RA_{\mathbb{J}}$. All $\mathbb{J}G$-modules, let us agree, are deemed to be finite-dimensional. We may neglect to distinguish between the $\mathbb{J}G$-characters and isomorphism classes of $\mathbb{J}G$-modules. Given a $\mathbb{J}G$-module M, we write $[M]$ to denote the character of M, in other words, the isomorphism class of M. When M is simple, we call $[M]$ a $\mathbb{J}G$-irrep. The $\mathbb{J}G$-irreps comprise a $\mathbb{Z}$-basis for the $\mathbb{J}$-representation ring $A_{\mathbb{J}}(G)$ and they comprise an R-basis for the $\mathbb{J}$-representation algebra $RA_{\mathbb{J}}(G) = R \otimes_{\mathbb{Z}} A_{\mathbb{J}}(G)$. The **$\mathbb{J}$-representation biset functor** for $\mathcal{X}$ over R is defined to be the biset functor

$$RA_{\mathbb{J}} = RA_{\mathbb{J}}^{\mathcal{X}} = \bigoplus_{G \in \mathcal{X}} RA_{\mathbb{J}}(G).$$

Given $F, G \in \mathcal{X}$, an F-G-set X and a $\mathbb{J}G$-module M, then the element $[X] \in R\Gamma(F, G)$ sends the element $[M] \in RA_{\mathbb{J}}(G)$ to the element $[\mathbb{J}X \otimes_{\mathbb{J}G} \mathbb{J}M] \in RA_{\mathbb{J}}(F)$. It is easy to check that this action gives the usual induction, restriction and inflation maps of character theory. The isogation is, of course, the evident transport of structure. For $N \trianglelefteq G$, we have $\mathrm{def}_{G/N,G}[M] = [M^N] = [M_N]$ where M^N and M_N denote subspaces of M consisting of the N-fixed points and the N-cofixed points, respectively.

We define the **linearization morphism** to be the morphism of biset functors

$$\mathrm{lin}_{\mathbb{J}}^{\mathcal{X}} : RB^{\mathcal{X}} \to RA_{\mathbb{J}}^{\mathcal{X}}$$

whose coordinate map at G is the linearization map $\mathrm{lin}_{\mathbb{J},G} : RB(G) \to RA_{\mathbb{J}}(G)$ which sends the isomorphism class of a G-set to the isomorphism class of the corresponding $\mathbb{J}G$-module.

The following is another well-known result. We record it because it implies the easy half of Theorem 1.1.

Corollary 2.7. *Suppose that R is a field and that some group in $\mathcal{X}$ is non-cyclic. Then the biset functor RB is not semisimple.*

Proof. By part (2) of Remark 2.5, we may assume that $\mathcal{X}$ is finite. When G is non-cyclic, the linearization map $\mathrm{lin}_{\mathbb{J},G}$ is non-zero and non-injective. Since $\mathcal{X}$ owns a non-cyclic group, the linearization morphism $\mathrm{lin}_{\mathbb{J}}^{\mathcal{X}}$ is non-zero and non-injective. Hence RB is non-simple. But the latest proposition implies that RB is indecomposable. $\square$

Actually, Theorem 1.1 will imply the converse. Thus, when R is a field, RB is semisimple if and only if every group in $\mathcal{X}$ is cyclic.

Let us signal especial interest in less classical example of a biset functor. The kernel $K = K^{\mathcal{X}} = \mathrm{Ker}(\mathrm{lin}_{\mathbb{J}}^{\mathcal{X}} : B^{\mathcal{X}} \to A_{\mathbb{J}}^{\mathcal{X}})$ is independent of $\mathbb{J}$. Indeed, the coordinate module $K(G) = \mathrm{Ker}(\mathrm{lin}_{\mathbb{J},G} : B(G) \to A_{\mathbb{J}}(G))$ consists of those elements $x \in B(G)$ such that $\mathrm{res}_{C,G}(x) = 0$ for every cyclic subgroup C of G. The biset functor K has been studied by, for instance, Bouc [5,7] and Yalçın [12]. We also mention that the maps considered by Tornehave [11]—parameterized by an automorphism of $\mathbb{C}$—can be realized as morphisms of biset functors from K to the unit group of the Burnside ring. For arbitrary p, there are some analogous morphisms from RK to the dual RB^* of the Burnside ring. The author intends to discuss these morphisms in a future paper.

3. Rhetorical biset functors

Rhetorical biset functors were implicitly introduced by Hambleton, Taylor and Williams [9, 1.A.4, 1.A.12], and the notion was further consolidated mainly in some seminars by Yalçın in 2004. We shall define the hermetic algebra $R\Upsilon^{\mathcal{X}}$ as a quotient of the alchemic algebra $R\Gamma^{\mathcal{X}}$, and we shall define the rhetorical biset functors for $\mathcal{X}$ over R to be the biset functors that are inflated from $R\Upsilon^{\mathcal{X}}$.

Rational p-biset functors were introduced by Bouc [4]. Recall that a **p-biset functor** is defined to be a biset functor for $\mathcal{X}_p$ where $\mathcal{X}_p$ is set of representatives of the isomorphism classes of p-groups. In order to make a comparison with rhetorical biset functors, we shall find it convenient to work generally with rational biset functors for any class of p-groups that is closed under subquotients up to isomorphism. However, Non-example 3.E suggests that these more general kinds of rational biset functors are unlikely to be of fundamental significance.

In this section, we shall discuss some characterizations of rhetorical and rational biset functors, we shall give some examples, and we shall observe some useful closure properties.

For finite groups I and J, we understand a $\mathbb{J}I$-$\mathbb{J}J$-bimodule to be a $\mathbb{J}(I \times J)$-module; given $i \in I$ and $j \in J$, then the bimodule action of i on the left and j^{-1} on the right coincides with the module action of (i, j) on the left. Consider the R-module $R\Gamma(I, J) = RB(I \times J)$. Let $R\mathcal{K}(I, J)$ be the R-submodule of $R\Gamma(I, J)$ spanned by the elements having the form $[X] - [X']$ where X and X' are I-J-bisets such that $\mathbb{J}X \cong \mathbb{J}X'$ as an isomorphism of $\mathbb{J}I$-$\mathbb{J}J$-bimodules. Observe that the condition is independent of $\mathbb{J}$. Thus, $R\mathcal{K}(I, J) = RK(I \times J)$ where K is the biset functor discussed at the end of the previous section. The quotient

$$R\Upsilon(I, J) = R\Gamma(I, J)/R\mathcal{K}(I, J)$$

is a free R-module. Consider another finite group K (there will be no confusion of notation) and let Y and Y' be J-K-bisets such that $\mathbb{J}Y \cong \mathbb{J}Y'$. As an isomorphism of $\mathbb{J}I$-$\mathbb{J}K$-bimodules,

$$\mathbb{J}(X \times_J Y) \cong \mathbb{J}X \otimes_{\mathbb{J}J} \mathbb{J}Y \cong \mathbb{J}X' \otimes_{\mathbb{J}J} \mathbb{J}Y' \cong \mathbb{J}(X' \times_J Y').$$

So the multiplication on $R\Gamma^{\mathcal{X}}$ gives rise to a multiplication operation

$$R\Upsilon(I, J) \times R\Upsilon(J, K) \to R\Upsilon(I, K).$$

We give two equivalent definitions of a rhetorical biset functor. One of the definitions is as follows. Regarding $\mathcal{X}$ as an R-additive category such that $\mathrm{Hom}(J, I) = R\Upsilon(I, J)$, we (could, if we so wished) define a **rhetorical biset functor** to be an R-additive functor from $\mathcal{X}$ to the category of R-modules. We shall have no use for this characterization, but we mention that it is easily shown to be equivalent to the definition in the next paragraph.

As an ideal in $R\Gamma^{\mathcal{X}}$, we define

$$R\mathcal{K}^{\mathcal{X}} = \bigoplus_{I,J \in \mathcal{X}} R\mathcal{K}(I, J).$$

We define the **hermetic algebra** to be the locally unital R-algebra

$$R\Upsilon^{\mathcal{X}} = R\Gamma^{\mathcal{X}} / R\mathcal{K}^{\mathcal{X}} = \bigoplus_{I,J \in \mathcal{X}} R\Upsilon(I, J).$$

A biset functor L for $\mathcal{X}$ over R is said to be **rhetorical** provided L is annihilated by $\mathcal{K}^{\mathcal{X}}$. In other words, L is rhetorical provided, for all $I, J \in \mathcal{X}$ and all I-J-bisets X and X' satisfying $\mathbb{J}X \cong \mathbb{J}X'$, the elements $[X], [X'] \in R\Gamma(I, J)$ act as the same R-linear maps $L(I) \leftarrow L(J)$. The rhetorical biset functors are precisely the biset functors that are inflated from the hermetic algebra $R\Upsilon^{\mathcal{X}}$. So we can regard the rhetorical biset functors as the locally unital $R\Upsilon^{\mathcal{X}}$-modules.

Let us write the canonical R-algebra epimorphism from the alchemic algebra to the hermetic algebra as

$$\mathrm{lin}^{\mathcal{X}} : R\Gamma^{\mathcal{X}} \to R\Upsilon^{\mathcal{X}}.$$

This is a dangerous abuse of notation because, for arbitrary R, the (I, J)-component of $\mathrm{lin}^{\mathcal{X}}$ is the canonical R-module epimorphism

$$\mathrm{lin}_{I,J} : R\Gamma(I, J) \to R\Upsilon(I, J)$$

whereas the (I, J)-component of the linearization map $\mathrm{lin}^{\mathcal{X}}_{\mathbb{Q}} : RB^{\mathcal{X}} \to RA^{\mathcal{X}}_{\mathbb{Q}}$ is the possibly non-surjective R-module map

$$\mathrm{lin}_{I \times J, \mathbb{Q}} : RB(I \times J) \to RA(I \times J).$$

However, the linearization map $\mathrm{lin}_G : RB(G) \to RA_{\mathbb{Q}}(G)$ is surjective when R is a field with characteristic zero. The Ritter–Segal theorem says that the same conclusion holds for arbitrary R when G is a p-group. So, if R is a field of characteristic zero or if I and J are p-groups, then we can identify $\mathrm{lin}_{I,J}$ with $\mathrm{lin}_{I\times J,\mathbb{Q}}$ and, in fact, we can make an identification of short exact sequences as indicated in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & RK(I\times J) \arrow[r] \arrow[d, equal] & RB(I\times J) \arrow[r, "\mathrm{lin}_{I\times J,\mathbb{Q}}"] \arrow[d, equal] & RA_{\mathbb{Q}}(I\times J) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R\mathcal{K}(I,J) \arrow[r] & R\Gamma(I,J) \arrow[r, "\mathrm{lin}_{I,J}"] & R\Upsilon(I,J) \arrow[r] & 0
\end{tikzcd}$$

In the two special cases that we have indicated, these identifications yield another characterization of the rhetorical biset functors. To elucidate the point, let us write $RA_{\mathbb{Q}}(I,J) = RA_{\mathbb{Q}}(I\times J)$ and let us define a multiplication operation

$$RA_{\mathbb{Q}}(I,J)\times RA_{\mathbb{Q}}(J,K)\to RA_{\mathbb{Q}}(I,K)$$

such that, given a $\mathbb{Q}I$-$\mathbb{Q}J$-bimodule U and a $\mathbb{Q}J$-$\mathbb{Q}K$-bimodule V, then the product of the elements $[U]\in RA_{\mathbb{Q}}(I,J)$ and $[V]\in RA_{\mathbb{Q}}(J,K)$ is $[U][V]=[U\otimes_{\mathbb{Q}J}V]$. The following proposition is clear from the above comments.

Proposition 3.1. *Suppose that R is a field with characteristic zero or that every group in $\mathcal{X}$ is a p-group. Then, for all $I,J\in\mathcal{X}$, there is an R-module isomorphism $R\Upsilon(I,J)\cong RA_{\mathbb{Q}}(I,J)$ such that, given an I-J-biset X, then the image of $[X]$ in $R\Upsilon(I,J)$ corresponds to the element $[\mathbb{Q}X]$ in $RA_{\mathbb{Q}}(I,J)$. These R-module isomorphisms preserve multiplication and give rise to an isomorphism of R-algebras*

$$R\Upsilon^{\mathcal{X}}\cong\bigoplus_{I,J\in\mathcal{X}} RA_{\mathbb{Q}}(I,J)$$

*where the multiplication operation on the right-hand side is the multiplication operation defined above. In other words, the category of rhetorical biset functors for $\mathcal{X}$ over R is equivalent to the category of R-additive functors $\mathcal{X}\to R$-*Mod *where $\mathcal{X}$ is regarded as an R-preadditive category with* $\mathrm{Hom}(J,I)=RA_{\mathbb{Q}}(I,J)$.

Let us quickly review the notion of a rational biset functor for $\mathcal{X}$, where $\mathcal{X}$ is such that every group in $\mathcal{X}$ is a p-group. Let L be a biset functor for $\mathcal{X}$ over R, and let $G\in\mathcal{X}$. As an R-submodule of $L(G)$, we define

$$\partial L(G)=\bigcap_{N:\ 1<N\trianglelefteq G}\mathrm{Ker}(\mathrm{def}_{G/N,G})=f_1^G L(G).$$

Here, f_1^G is the idempotent of $R\Gamma(G,G)$ given by the formula

$$f_1^G=\sum_{Z\subseteq\Omega_1(Z(G))}(-1)^n p^{n(n-1)/2}\,\mathrm{inf}_{P,P/Z}\,\mathrm{def}_{P/Z,P}$$

where $|Z|=p^n$.

A p-group F is said to be a **Roquette p-group** provided every normal abelian subgroup of F is cyclic. Still assuming that G is a p-group, let ψ be a $\mathbb{J}G$-irrep. A subquotient H/K of G is called a **genetic subquotient** for ψ provided H/K is Roquette and there exists a faithful $\mathbb{J}H/K$-irrep ϕ such that $\psi = \mathrm{ind}_{G,H}(\mathrm{inf}_{H,H/K}(\phi))$ and ϕ occurs only once in $\mathrm{def}_{H/K,H}(\mathrm{res}_{H,G}(\psi))$. The Genotype Theorem [1, 1.1] asserts that every $\mathbb{J}G$-irrep ψ has a genetic subquotient H/K, furthermore, H/K is unique up to isomorphism and, upon fixing a choice of H/K, the $\mathbb{J}H/K$-irrep ϕ is unique. As a group well-defined up to isomorphism, H/K is called the **genotype** of ψ. We mention that the essential content of the Genotype Theorem is due to Bouc [4, 1.7], [5, 2.6], who considered the special case $\mathbb{J} = \mathbb{Q}$. The extension to arbitrary $\mathbb{J}$ is a fairly straightforward application of a field-changing principle [1, 3.5].

We define a **genetic basis** to be a set of representatives of the equivalence classes of genetic subquotients of the p-group G. Given a genetic basis $\mathcal{G}$ for G, we define an R-linear map

$$\mathcal{I}_{L,\mathcal{G}} = \bigoplus_{H/K \in \mathcal{G}} \mathrm{ind}_{G,H}\,\mathrm{inf}_{H,H/K} : \bigoplus_{H/K \in \mathcal{G}} \partial L(H/K) \to L(G).$$

Bouc [4, 3.2] has shown that the map $\mathcal{I}_{L,\mathcal{G}}$ is always a split injection. The biset functor L is said to be **rational** provided, for all $G \in \mathcal{X}$, there exists a genetic basis $\mathcal{G}$ for G such that $\mathcal{I}_{L,\mathcal{G}}$ is an isomorphism. In that case, [4, 7.3] tells us that, for all G and for all genetic bases $\mathcal{G}$ of G, the map $\mathcal{I}_{L,\mathcal{G}}$ is an isomorphism.

We now compare some closure properties of rhetorical and rational biset functors. With one exception, all the closure properties in the next lemma are immediate from the definition of a rhetorical biset functor. Only the closure under duality requires some explanatory comments. Recall that the **opposite** of an I-J-biset X is the J-I-biset X^{op} such that $X = X^{\mathrm{op}}$ as sets and the action of $I \times J$ on X commutes with the action of $J \times I$ on X^{op} via the group isomorphism $I \times J \ni (i,j) \leftrightarrow (j,i) \in J \times I$. The correspondence $[X] \leftrightarrow [X^{\mathrm{op}}]$ gives rise to a self-inverse R-module isomorphism $R\Gamma(I,J) \leftrightarrow R\Gamma(J,I)$ and that, in turn, gives rise to an anti-automorphism $\gamma \leftrightarrow \gamma^{\mathrm{op}}$ on $R\Gamma^{\mathcal{X}}$. Thus, the opposite map allows us to identify $R\Gamma^{\mathcal{X}}$ with its opposite algebra. Plainly, the opposite map restricts to self-inverse isomorphisms $R\mathcal{K}(I,J) \leftrightarrow R\mathcal{K}(J,I)$, so it gives rise to an anti-isomorphism on $R\Upsilon^{\mathcal{X}}$. For a biset functor L over R, the dual biset functor $L^* = \mathrm{Hom}_R(L,R)$ has coordinate modules $L^*(J) = \mathrm{Hom}_R(L(J),R)$ and the action of $R\Gamma^{\mathcal{X}}$ is such that $([X]\xi_J)(x_I) = \xi_J([X^{\mathrm{op}}]x_I)$ for all $\xi_J \in L^*(J)$ and $x_J \in L(H)$. If $\mathcal{K}(J,I)$ annihilates $L(I)$ then $\mathcal{K}(I,J)$ annihilates $L^*(J)$. So if $\mathcal{K}^{\mathcal{X}}$ annihilates L then $\mathcal{K}^{\mathcal{X}}$ annihilates L^*. That is to say, if L is rhetorical, then L^* is rhetorical.

Lemma 3.2. *The rhetorical biset functors for $\mathcal{X}$ over R are closed under the taking of subfunctors, quotient functors, direct sums and duals.*

A significant advantage of rational biset functors is that they admit a stronger variant of the lemma, as follows. It is obvious that the rational biset functors are closed under direct sums. The rest of the following lemma was expressed in Bouc [4, 7.4] only for the case $\mathcal{X} = \mathcal{X}_p$, but the argument carries through to the general case without change.

Lemma 3.3 *(Bouc). Suppose that every group in $\mathcal{X}$ is a p-group. Then the rational biset functors are closed in all the senses specified in the previous lemma. Furthermore, given a subfunctor L_1 of a biset functor L_2 such that L_1 and L_2/L_1 are rational, then L_2 is rational.*

Bouc has shown that closure property in the rider fails for rhetorical biset functors. See Non-example 3.F below.

Lemma 3.4. *Let R' be a unital subring of R and let L' be a biset functor for $\mathcal{X}$ over R'. Consider the R-linear extension to the biset functor $L = RL'$ for $\mathcal{X}$ over R.*

(1) *L is rhetorical if and only if L' is rhetorical.*
(2) *Suppose that every group in $\mathcal{X}$ is a p-group. Then L is rational if and only if L' is rational.*

Proof. If $R'\mathcal{K}^{\mathcal{X}}$ annihilates L' then $R\mathcal{K}^{\mathcal{X}}$ annihilates L. For the converse, we regard L' and $R'\mathcal{K}^{\mathcal{X}}$ as R'-submodules of L and $R\mathcal{K}^{\mathcal{X}}$. If $R\mathcal{K}^{\mathcal{X}}$ annihilates L then $R'\mathcal{K}^{\mathcal{X}}$ annihilates L, hence $R'\mathcal{K}^{\mathcal{X}}$ annihilates L'. Part (1) is established.

The R-module $\partial L(G) = f_1^G L(G)$ is the R-linear extension of the R'-module $\partial L'(G) = f_1^G L'(G)$. So the domain of $\mathcal{I}_{L,\mathcal{G}}$ is the R-linear extension of the domain of $\mathcal{I}_{L',\mathcal{G}}$. It is now clear that the map $\mathcal{I}_{L,\mathcal{G}}$ is the R-linear extension of the map $\mathcal{I}_{L',\mathcal{G}}$. A split injection of R'-modules is surjective if and only if its R-linear extension is surjective. So part (2) holds. $\square$

Lemma 3.5 *(Another Finite Reduction Principle). Let L be a biset functors for $\mathcal{X}$. Let $\mathcal{F}$ run over those finite subsets of $\mathcal{X}$ such that, up to isomorphism, $\mathcal{F}$ is closed under subquotients. Consider the biset functors $1_{\mathcal{F}}L$ for $\mathcal{F}$* (*in the notation of Remark* 2.5).

(1) *L is rhetorical if and only if each $1_{\mathcal{F}}L$ is rhetorical.*
(2) *Suppose that every group in $\mathcal{X}$ is a p-group. Then L is rational if and only if each $1_{\mathcal{F}}L$ is rational.*

Let us comment on some examples of rhetorical biset functors; and on two examples of rational biset functors that are not rhetorical.

Example 3.A. Plainly, the $\mathbb{J}$-representation functor $RA_{\mathbb{J}}^{\mathcal{X}}$ is rhetorical. In view of the above comments concerning genotypes of $\mathbb{J}$-irreps, the discussion in [4, 7.2] can easily be extended to show that, if every group in $\mathcal{X}$ is a p-group, then $RA_{\mathbb{J}}^{\mathcal{X}}$ is rational.

The rationality of $RA_{\mathbb{J}}^{\mathcal{X}}$ can also be deduced from Theorem 1.2. However, in Section 8, Theorem 1.2 will be deduced from the rationality of $\mathbb{C}A_{\mathbb{C}}^{\mathcal{X}}$.

Example 3.B. Let $B^\times(G)$ denote the unit group of the Burnside ring $B(G)$. Recall that $B^\times(G)$ is an elementary abelian 2-group. The realization of $B^\times$ as a biset functor over the field $\mathbb{F}_2 = \mathbb{Z}/2$ goes back to a result of Yoshida [13, Lemma 3.5], which describes the tom Dieck map as a morphism of biset functors $\mathbb{F}_2 A_{\mathbb{R}}^{\mathcal{X}} \to (B^\times)^{\mathcal{X}}$. Tornehave's Unit Theorem [11] asserts that, when G is a 2-group, the tom Dieck map $\mathbb{F}_2 A_{\mathbb{R}}(G) \to B^\times(G)$ is surjective. Yalçın [12] gave a new proof of Tornehave's Unit Theorem by first showing that the 2-biset functor $(B^\times)^{\mathcal{X}_2}$ is rational. In response, Bouc [6] showed that $(B^\times)^{\mathcal{X}_2}$ is isomorphic to a subfunctor of the dual functor $(\mathbb{F}_2 A_{\mathbb{Q}}^*)^{\mathcal{X}_2}$ of the $\mathbb{Q}$-representation functor $(\mathbb{F}_2 A_{\mathbb{Q}})^{\mathcal{X}_2}$. This implies Yalçın's rationality result because $\mathbb{F}_2 A_{\mathbb{Q}}^*$ is rational and the rational biset functors are closed under the taking of subfunctors and duals. Alternatively, granted Tornehave's Unit Theorem, then $(B^\times)^{\mathcal{X}_2}$ must be rational because it is a quotient of the rational biset functor $(\mathbb{F}_2 A_{\mathbb{R}})^{\mathcal{X}_2}$. But, in fact, $\mathbb{F}_2 A_{\mathbb{Q}}^{\mathcal{X}_2}$ and $\mathbb{F}_2 A_{\mathbb{R}}^{\mathcal{X}_2}$ are rhetorical, so these last two arguments both show that $(B^\times)^{\mathcal{X}_2}$ is rhetorical.

Example 3.C. Suppose that G is a p-group. Let $D(G)$ denote the Dade group of G and let $D^\Omega(G)$ denote the subgroup generated by those elements of $D(G)$ that correspond to the kernels of the augmentation maps on the permutation modules. For details, see Bouc [5]. When p is odd, [5, Theorem 7.7] says that $D = D^\Omega$. When $p = 2$, the quotient group $D(P)/D^\Omega(P)$ is an elementary abelian 2-group. Bouc realized it as a rational 2-biset functor over $\mathbb{F}_2$, and in [5, 10.4], he explicitly described it as a subfunctor of $(\mathbb{F}_2 A_{\mathbb{Q}})^{\mathcal{X}_2}$. Via the closure properties again, we deduce that D/D^Ω is a rhetorical 2-biset functor.

Example 3.D. Returning to the case where the prime p is arbitrary, the torsion subgroup $D^\Omega_{\mathrm{tors}}(P)$ is a finite abelian 2-group with exponent at most 4. Bouc realized $(D^\Omega_{\mathrm{tors}})^{\mathcal{X}_p}$ as a p-biset functor over $\mathbb{Z}$. In [4, 7.5], he argued that D^Ω_{tors} must be rational because it is a quotient of $A^*_{\mathbb{Q}}$. Again, the argument can equally well be read as a proof that D^Ω_{tors} is rhetorical.

Non-example 3.E. Let G be a Roquette p-group and let ω be any $\mathbb{J}\,\mathrm{Out}(G)$-irrep. Consider the simple biset functor $S = S^{\mathcal{X}(G),\mathbb{K}}_{G,\omega}$ for $\mathcal{X}(G)$ over $\mathbb{K}$. We have $\partial S(G) = S(G)$ and $\partial S(F) = S(F) = 0$ for any strict subquotient F of G. Letting $\mathcal{G}$ be a genetic basis for G, then $\mathcal{I}_{S,\mathcal{G}}$ is the identity map on $S(G)$. On the other hand, letting $\mathcal{F}$ be a genetic basis for F, then $\mathcal{I}_{S,\mathcal{F}}$ is the unique R-linear endomorphism of the zero R-module. Thus, somewhat trivially, S is a rational biset functor. However, by Theorem 1.5 (to be proved in Section 8), S is not rhetorical unless G is cyclic and ω is primitive.

The latest example indicates that our general definition of a rational biset functor is somewhat artificial. Our motive for that definition comes from part (2) of Lemma 3.5. However, Theorems 1.3 and 1.5 together imply that, if G is non-cyclic or ω is non-primitive, then the simple p-biset functor $S^{\mathcal{X}_p,\mathbb{K}}_{G,\omega}$ is non-rational. The next example of a rational but non-rhetorical biset functor is more substantial.

Non-example 3.F. Bouc [7] has shown that the Burnside p-biset functor B has a unique subfunctor B_δ which is minimal subject to B/B_δ being rational. On the other hand, K is the unique p-biset functor which is minimal subject to B/K being rhetorical. Bouc [7] also showed that $K \geqslant B_\delta$ and the quotient K/B_δ is isomorphic to the cokernel of the exponential morphism $\exp : B \to B^\times$. In particular, $K \neq B_\delta$ if and only if $p = 2$. In that case, we have a non-split exact sequence of biset functors

$$0 \to K/B_\delta \to B/B_\delta \to B/K \to 0$$

such that all three terms are rational and the two end terms are rhetorical but the middle term is non-rhetorical.

The next two results concern the $\mathbb{Q}$-representation biset functor.

Proposition 3.6. *Suppose that R is a field. Also suppose that R has characteristic zero or that all the groups in $\mathcal{X}$ are p-groups. Then, in the category of rhetorical biset functors for $\mathcal{X}$ over R, the $\mathbb{Q}$-representation functor $RA_{\mathbb{Q}}$ is the projective cover of the simple functor $S_{1,1}$.*

Proof. We use some observations that were made in the proof of Proposition 2.6. Since $S_{1,1}$ is the unique simple biset functor with non-zero coordinate module at the trivial group, and since $RA_{\mathbb{Q}}(1)$ is non-zero, $S_{1,1}$ must be a composition factor of $RA_{\mathbb{Q}}^{\mathcal{X}}$. But $RA_{\mathbb{Q}}^{\mathcal{X}}$ is rhetorical. So, by Lemma 3.2, $S_{1,1}$ must be rhetorical. Since iso_1 is a primitive idempotent of $R\Gamma^{\mathcal{X}}$ which does not annihilate $S_{1,1}$, the image of iso_1 in $R\Upsilon^{\mathcal{X}}$ is a primitive idempotent of $R\Upsilon^{\mathcal{X}}$ (which still does not annihilate $S_{1,1}$). Therefore, in the category of rhetorical biset functors, $R\Upsilon^{\mathcal{X}}\,\mathrm{iso}_1$ is the projective cover of $S_{1,1}$. The hypotheses that we have imposed allow us to invoke Proposition 3.1, whose isomorphisms yield the identifications

$$R\Upsilon^{\mathcal{X}}\,\mathrm{iso}_1 = \bigoplus_{I\in\mathcal{X}} R\Upsilon(I,1) = \bigoplus_{I\in\mathcal{X}} RA_{\mathbb{Q}}(I,1).$$

By regarding $\mathbb{Q}I$-modules as $\mathbb{Q}I$-$\mathbb{Q}1$-bimodules, we can also make the identifications $RA_{\mathbb{Q}}(I) = RA_{\mathbb{Q}}(I,1)$ and $RA_{\mathbb{Q}} = R\Upsilon^{\mathcal{X}}\,\mathrm{iso}_1$. $\square$

The latest proposition and Theorem 1.7 together imply that, for arbitrary $\mathcal{X}$, the simple functor $S_{1,1}^{\mathcal{X},\mathbb{K}}$ is projective in the category of rhetorical biset functors. But that conclusion will be swallowed by Theorem 1.4. In the case where R has characteristic p, the proposition is more informative, and it yields the following corollary. I do not know whether the converse to the rider of the corollary holds.

Corollary 3.7. *Suppose that R is a field with characteristic p and that every group in $\mathcal{X}$ is a p-group. Then the simple functor $S_{1,1}$ is projective in the category of rhetorical biset functors for $\mathcal{X}$ over R if and only if every non-trivial group in $\mathcal{X}$ has exponent p. In particular, if the category is semisimple, then every non-trivial group in $\mathcal{X}$ has exponent p.*

Proof. A classic theorem of Roquette—see, for instance, the generalization in Hambleton, Taylor and Williams [9, 3.A.6]—asserts that the abelian Roquette p-groups are precisely the cyclic p-groups; the non-abelian Roquette p-groups exist only when $p = 2$, and they are the quaternion groups with order at least 8 and the dihedral and semidihedral groups with order at least 16. Evidently, the only Roquette p-groups with exponent dividing p are the two groups C_p and C_1. So if G has exponent p, then every $\mathbb{Q}G$-irrep has genotype C_p or C_1. On the other hand, the faithful $\mathbb{Q}C_{p^2}$-irreps have genotype C_{p^2}.

Of course, the dimension of $RA_{\mathbb{Q}}(G)$ is equal to the number of $\mathbb{Q}G$-irreps. Bouc [3, 1.4] tells us that the dimension of $S_{1,1}(G)$ is equal to the number of $\mathbb{Q}G$-irreps that have genotype C_p or C_1. So, if G has exponent p, then $RA_{\mathbb{Q}}(G) = S_{1,1}(G)$, but $RA_{\mathbb{Q}}(C_{p^2}) \neq S_{1,1}(C_p^2)$. $\square$

4. A light interlude

In this section, we collect together some easy observations concerning criteria for semisimplicity and dimensions of the alchemic and hermetic algebras. We shall also present a kind of Chinese Remainder Remark for those two algebras.

The following two abstract criteria for semisimplicity are stated in forms which are suited to our applications. We leave the proofs as easy exercises; and we leave the generalizations as easy irrelevant exercises.

Remark 4.1. Let Λ be a finite-dimensional algebra over $\mathbb{Q}$. If the $\mathbb{C}$-linear extension $\mathbb{C}\Lambda$ is semisimple, then the $\mathbb{K}$-linear extension $\mathbb{K}\Lambda$ is semisimple.

Remark 4.2. Let Λ be a finite-dimensional algebra over $\mathbb{C}$, and let $S_1, \dots, S_r$ be Λ-modules such that $\mathrm{Hom}_\Lambda(S_i, S_j) = 0$ for all $i \neq j$. Then $\dim_R(\Lambda) \geqslant \sum_i \dim_{\mathbb{F}}(S_i)^2$. If the equality holds, then Λ is semisimple and $S_1, \dots, S_r$ comprise a set of representatives of the isomorphism classes of simple Λ-modules.

For the next obvious remark, we need a couple of little items of notation. The number of conjugacy classes of subgroups of G is equal to the R-rank of $RB(G)$; we write this number as $s_*(G)$. The number of conjugacy classes of cyclic subgroups of G is equal to the number of isomorphism classes of simple $\mathbb{Q}G$-modules, in other words, the R-rank of $RA_{\mathbb{Q}}(G)$; we write this number as $k_*(G)$.

Remark 4.3. Given finite groups I and J, then the free R-modules $R\Gamma(I, J)$ and $R\Upsilon(I, J)$ have R-ranks $s_*(I \times J)$ and $k_*(I \times J)$, respectively.

The next result, too, may seem to be virtually obvious, but we give cautious proof because an analogous assertion for representation rings can fail. As a widely-known counterexample, it can be shown that the canonical monomorphism $A_{\mathbb{R}}(Q_8) \otimes_{\mathbb{Z}} A_{\mathbb{R}}(C_3) \to A_{\mathbb{R}}(Q_8 \times C_3)$ has cokernel with order 2. Hint: let Q_8 act on $\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ by left multiplication and let a generator of C_3 act as right multiplication by $(1 + i\sqrt{3})/2$.

The set $\mathcal{X}(G)$ was defined in Section 1, but let us note that the definition can be broken up into two conditions: firstly, the isomorphism classes of groups in $\mathcal{X}(G)$ are precisely the isomorphism classes of subquotients of G; secondly, $\mathcal{X}(G)$ has only one copy of each isomorphism class. The second condition ensures that the alchemic algebra $R\Gamma^G = R\Gamma^{\mathcal{X}(G)}$ and the hermetic algebra $R\Upsilon^G = R\Upsilon^{\mathcal{X}(G)}$ are determined by G up to isomorphism (and not merely up to Morita equivalence).

Lemma 4.4 *(Chinese Remainder Lemma). Let G_1 and G_2 be finite groups whose orders are coprime. Then*

(1) $R\Gamma^{G_1 \times G_2} \cong R\Gamma^{G_1} \otimes_R R\Gamma^{G_2}$,
(2) $R\Upsilon^{G_1 \times G_2} \cong R\Upsilon^{G_1} \otimes_R R\Upsilon^{G_2}$.

Proof. Put $G = G_1 \times G_2$. We may assume that $\mathcal{X}(G)$ and $\mathcal{X}(G_1)$ and $\mathcal{X}(G_2)$ are such that each element $I \in \mathcal{X}(G)$ decomposes as $I = I_1 \times I_2$ where $I_i \in \mathcal{X}(G_i)$. Given $I, J \in \mathcal{X}(G)$, then any transitive I-J-biset X decomposes as a direct product $X \cong X_1 \times X_2$ where X_i is an I_i-J_i-biset. Since $\Gamma(I, J) = B(I \times J)$, there is an R-module isomorphism

$$\Theta_{I,J} : R\Gamma(I_1, J_1) \otimes_R R\Gamma(I_2, J_2) \to R\Gamma(I, J)$$

such that $[X_1] \otimes [X_2] \mapsto [X]$. Letting I and J run over the elements of $\mathcal{X}(G)$, then the maps $\Theta_{I,J}$ combine to form an R-algebra isomorphism

$$\Theta : R\Gamma^{G_1} \otimes_R R\Gamma^{G_2} \to R\Gamma^G.$$

Part (1) is established. To demonstrate part (2), we shall show that Θ gives rise to an R-algebra isomorphism for the hermetic algebras. Since the hermetic algebra $\Upsilon^G = \mathbb{Z}\Upsilon^G$ is a free $\mathbb{Z}$-module, we may assume that $R = \mathbb{Z}$. It is clear that Θ gives rise to a ring homomorphism

$$\Phi : \Upsilon^{G_1} \otimes_{\mathbb{Z}} \Upsilon^{G_2} \to \Upsilon^G.$$

We must show that Φ is a linear isomorphism. Recall that the $\mathbb{Z}$-module $\mathcal{K}(I, J) = \mathrm{Ker}(\mathrm{lin}_{I,J} : \Gamma(I, J) \to \Upsilon(I, J))$ coincides with the $\mathbb{Z}$-module $K(I \times J) = \mathrm{Ker}(\mathrm{lin}_{I\times J,\mathbb{Q}} : B(I \times J) \to A_{\mathbb{Q}}(I \times J))$. Hence, as a $\mathbb{Z}$-module, we can identify $\Upsilon(I, J)$ with the full sublattice $\mathrm{lin}(B(I \times J))$ of $A_{\mathbb{Q}}(I, J)$. The (I, J)-component of Φ is a linear map

$$\Phi_{I,J} : \Upsilon(I_1, J_1) \otimes_{\mathbb{Z}} \Upsilon(I_2, J_2) \to \Upsilon(I, J).$$

In other words, it is a linear map

$$\Phi_{I,J} : \mathrm{lin}\big(B(I_1 \times J_1)\big) \otimes_{\mathbb{Z}} \mathrm{lin}\big(B(I_2 \times J_2)\big) \to \mathrm{lin}\big(B(I \times J)\big).$$

As permutation bimodules, $\mathbb{Q}(X_1 \times X_2) \cong \mathbb{Q}X_1 \otimes_{\mathbb{Q}} \mathbb{Q}X_2$. So $\Phi_{I,J}$ extends to the monomorphism

$$A_{\mathbb{Q}}(I_1 \times J_1) \otimes_{\mathbb{Z}} A_{\mathbb{Q}}(I_2 \times J_2) \to A_{\mathbb{Q}}(I \times J)$$

such that $[M_1] \otimes_{\mathbb{Z}} [M_2] \mapsto [M_1 \otimes_{\mathbb{Q}} M_2]$ where M_i is a $\mathbb{Q}I_i$-$\mathbb{Q}J_i$-bimodule. Perforce, $\Phi_{I,J}$ is injective. On the other hand, every element of $\mathrm{lin}(B(I \times J))$ lifts to an element of $B(I \times J)$ which, in turn, corresponds to an element of γ of $B(I_1 \times J_1) \otimes_{\mathbb{Z}} B(I_2 \times J_2)$. Thus, every element of $\mathrm{lin}(B(I, J))$ has the form $\mathrm{lin}(\Theta_{I,J}(\gamma)) = \Phi_{I,J}(\mathrm{lin}(\gamma))$. Therefore $\Phi_{I,J}$ is surjective. □

5. The negative theorem on semisimplicity

We shall prove Theorem 1.1. The easy direction—the necessity of the criterion for semisimplicity—follows immediately from Corollary 2.7.

Let us start on the proof of the theorem in the harder direction. Assuming that every group in $\mathcal{X}$ is cyclic, we are required to show that every biset functor for $\mathcal{X}$ over $\mathbb{K}$ is semisimple. By part (2) of Remark 2.5, we may also assume that $\mathcal{X}$ is finite. Our task, now, is to show that the alchemic algebra $\mathbb{K}\Gamma^{\mathcal{X}}$ is semisimple.

Applying the Morita equivalence discussed in Section 2, we reduce to the case where $\mathcal{X}$ has only one representative of each isomorphism class. Letting ℓ be the lowest common multiple of the orders of the groups in $\mathcal{X}$, then $\mathbb{K}\Gamma^{\mathcal{X}} \cong i\mathbb{K}\Gamma^{C_\ell}i$ where the idempotent i is the sum of those elements iso_I such that the group $I \in \mathcal{X}(C_\ell)$ is isomorphic to a group in $\mathcal{X}$. So it suffices to show that $\mathbb{K}\Gamma^{C_\ell}$ is semisimple. By decomposing ℓ as a product of powers of distinct primes, and applying part (1) of the Chinese Remainder Lemma 4.4, we reduce to the case where ℓ is a power of a prime. By Remark 4.1, we may assume that $\mathbb{K} = \mathbb{C}$. Thus, to complete the proof of Theorem 1.1, we need only show that $\mathbb{C}\Gamma^{C_\ell}$ is semisimple when $\ell = p^\alpha$ for some natural number α. We shall do this by calculating the dimensions of the simple modules and then applying Remark 4.2.

For integers x and y, we write $x \equiv_\alpha y$ when x and y are congruent modulo p^α, and we let $[x]_\alpha$ denote the congruence class of x modulo p^α. We write the additive group of integers modulo p^α as ${}_p\alpha = \{[x]_\alpha\colon x \in \mathbb{Z}\}$. Thus, ${}_p\alpha$ is a cyclic group with order p^α. (We shall be employing notation that is peculiar to this particular group, so ${}_p\alpha$ is not to be confused with the generic representative C_{p^α} of the isomorphism class.) We choose $\mathcal{X}({}_p\alpha) = \{{}_p\beta\colon\ 0 \leqslant \beta \leqslant \alpha\}$ as our set of representatives of isomorphism classes of subquotients of ${}_p\alpha$. By Theorem 2.4, the simple biset functors for ${}_p\alpha$ over $\mathbb{C}$ have the form

$$S^\alpha_{\gamma,\sigma} = S^{\mathcal{X}({}_p\alpha),\mathbb{C}}_{p\gamma,\sigma},$$

the index γ running over the natural numbers not greater than α, the index σ running over the $\mathbb{C}$-irreps of $\mathrm{Aut}({}_p\gamma)$. For a natural number β not greater than α, we write

$$S_{\gamma,\sigma}(\beta) = S^\alpha_{\gamma,\sigma}({}_p\beta).$$

The notation makes sense by part (3) of Remark 2.5. We mean to say that, if we fix β and vary α such that $\alpha \geqslant \beta$, then the coordinate module of $S^\alpha_{\gamma,\sigma}$ at ${}_p\beta$ is independent of α. We are about to see that the dimension of $S_{\gamma,\sigma}(\beta)$ is also independent of σ.

Lemma 5.1. *Let β and γ be natural numbers, and let σ be a $\mathbb{C}$-irrep of* $\mathrm{Aut}({}_p\gamma)$. *Then*

$$\dim_{\mathbb{C}}\bigl(S_{\gamma,\sigma}(\beta)\bigr) = \begin{cases} \beta - \gamma + 1 & \text{if } \gamma \leqslant \beta,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The case $\gamma = 0$ has to be examined separately. Theorem 1.7 says that the simple biset functor $\mathbb{C}A^{\mathcal{X}({}_p\alpha)}_{\mathbb{Q}}$ is isomorphic to $S^\alpha_{0,1}$. (In the present context, we must avoid the ambiguous notation $S_{1,1}$.) Hence $\dim(S_{0,1}(\beta)) = \dim(\mathbb{C}A_{\mathbb{Q}}({}_p\beta)) = k_*({}_p\beta) = \beta + 1$.

Now let us consider the case $\gamma \geqslant 1$. We may assume that $\beta \geqslant \gamma$, since otherwise the assertion is trivial. Given two more natural numbers $\mu \leqslant \beta - \gamma \geqslant \nu$, we define

$$\mathrm{Des}^\nu_{\gamma,\beta} = \bigl\{\bigl([x]_\gamma, [p^\nu x]_\beta\bigr)\colon\ x \in \mathbb{Z}\bigr\}, \qquad \mathrm{Tin}^\mu_{\beta,\gamma} = \bigl\{\bigl([p^\mu y]_\beta, [y]_\gamma\bigr)\colon\ y \in \mathbb{Z}\bigr\}$$

as subgroups of ${}_p\gamma \times {}_p\beta$ and ${}_p\beta \times {}_p\gamma$ respectively. We define

$$\mathrm{des}^\nu_{\gamma,\beta} = \left[\frac{{}_p\gamma \times {}_p\beta}{\mathrm{Des}^\nu_{\gamma,\beta}}\right], \qquad \mathrm{tin}^\mu_{\beta,\gamma} = \left[\frac{{}_p\beta \times {}_p\gamma}{\mathrm{Tin}^\mu_{\beta,\gamma}}\right]$$

as elements of $\mathbb{C}\Gamma({}_p\gamma, {}_p\beta)$ and $\mathbb{C}\Gamma({}_p\beta, {}_p\gamma)$. Writing $D = \mathrm{Des}^\nu_{\gamma,\beta}$, we have

$$|D{\uparrow}| = p^{\beta-\nu}, \qquad |D{\downarrow}| = p^{\beta-\gamma-\nu}, \qquad |{\uparrow}D| = p^\gamma, \qquad |{\downarrow}D| = 1.$$

Therefore—making use of some terminology introduced by Olcay Coşkun—$\mathrm{des}^\nu_{\gamma,\beta}$ is a destriction (deflation and restriction) map,

$$\mathrm{des}^\nu_{\gamma,\beta} = \mathrm{iso}^\phi_{p\gamma,C}\,\mathrm{def}_{C,B}\,\mathrm{res}_{B,{}_p\beta}$$

where ϕ is some group isomorphism (whose specification will not be needed), B is the subgroup of ${}_p\beta$ with order $p^{\beta-\nu}$ and C is the quotient group of B with order p^γ. Similarly, $\mathrm{tin}^\mu_{\beta,\gamma}$ is a inflation (transfer and inflation) map

$$\mathrm{tin}^\mu_{\beta,\gamma} = \mathrm{tra}_{p\beta,B'}\,\mathrm{inf}_{B',C'}\,\mathrm{iso}^{\phi'}_{C',{}_p\gamma}$$

where $|B'| = p^{\beta-\mu}$ and $|C'| = p^{\gamma}$. By the Generalized Mackey Product Theorem 2.2,

$$\mathrm{des}^{\nu}_{\gamma,\beta}\mathrm{tin}^{\mu}_{\beta,\gamma} = p^{\mu\nu}\left[\frac{{}_p\gamma \times {}_p\gamma}{H_{\nu,\mu}}\right]$$

where $H_{\nu,\mu} = \mathrm{Des}^{\nu}_{\gamma,\beta} * \mathrm{Tin}^{\mu}_{\beta,\gamma} = \{[x]_\gamma, [y]_\gamma\colon\ p^\nu x \equiv_\beta p^\mu y\}$. If $\nu < \mu$, then each x is divisible by p, so $\uparrow H_{\nu,\mu}$ is strictly contained in ${}_p\gamma$. Similarly, if $\nu > \mu$, then $H_{\nu,\mu}\uparrow$ is strictly contained in ${}_p\gamma$. Either way, the quotient group $H_{\nu,\mu}/(\downarrow H_{\nu,\mu} \times H_{\nu,\mu}\downarrow)$ that appears in the scenario of Goursat's Theorem 2.1 has order strictly smaller than p^γ. Via the Butterfly Decomposition Theorem 2.3, we deduce that, if $\mu \neq \nu$, then $\mathrm{des}^{\nu}_{\gamma,\beta}\,\mathrm{tin}^{\mu}_{\beta,\gamma}$ factorizes though an isogation in $\mathbb{C}\Gamma({}_p\delta, {}_p\delta)$ where $\delta < \gamma$, hence $\mathrm{des}^{\nu}_{\gamma,\beta}\,\mathrm{tin}^{\mu}_{\beta,\gamma} S_{\gamma,\sigma} = 0$. On the other hand, $\beta - \mu \geqslant \gamma$, so $H_{\mu,\mu}$ is the diagonal subgroup of ${}_p\gamma$ and $\mathrm{des}^{\mu}_{\gamma,\beta}\,\mathrm{tin}^{\mu}_{\beta,\gamma} = p^{\mu^2}\,\mathrm{iso}_{p\gamma}$.

Plainly, $S_{\gamma,\sigma}(\gamma)$ is 1-dimensional. Let s be a non-zero element of $S_{\gamma,\sigma}(\gamma)$. Since s generates the simple biset functor $S^{\alpha}_{\gamma,\sigma}$, the Butterfly Decomposition Theorem implies that $S_{\gamma,\sigma}(\beta)$ is spanned by the elements having the form $\mathrm{tra}_{p\beta,U}\,\mathrm{inf}_{U,P}\,\mathrm{iso}^{\psi}_{P,Q}\,\mathrm{def}_{Q,V}\,\mathrm{res}_{V,{}_p\gamma}(s)$. But any such element is zero unless $Q = V = {}_p\gamma$. Therefore $S_{\gamma,\sigma}(\beta)$ is spanned by the elements $s_\mu = \mathrm{tin}^{\mu}_{\beta,\gamma}(s)$ where μ runs over the natural numbers with $\mu \leqslant \beta - \gamma$. By observations in the previous paragraph, $\mathrm{des}^{\nu}_{\gamma,\beta}(s_\mu)$ is $p^{\mu^2}s$ or 0 when $\mu = \nu$ or $\mu \neq \nu$, respectively. So the elements s_μ are $\mathbb{C}$-linearly independent. We have shown that the $\beta - \gamma + 1$ elements s_μ comprise a $\mathbb{C}$-basis for $S_{\gamma,\sigma}(\beta)$. $\square$

It is worth commenting on the peculiar relation that appears in the argument. Let us understand a one-step transfer to be a transfer from a coordinate module $S_{\gamma,\sigma}(\epsilon)$ to the next coordinate module $S_{\gamma,\sigma}(\epsilon+1)$; likewise for inflation, deflation and restriction. These one-step maps are well-defined up to isogation factors. One-step transfer and one step inflation commute with each other up to isogation. A similar comment holds for deflation and restriction. Suppose that $\gamma \geqslant 1$. Starting at the lowest non-zero coordinate module $S_{\gamma,\sigma}(\gamma)$, if we apply some one-step transfers and inflations, and then apply some one-step deflations and restrictions to arrive back at $S_{\gamma,\sigma}(\gamma)$, then the result will be zero unless the number of inflations is equal to the number of deflations. That relation fails for representation functors, and in particular, it fails for the representation functor $\mathbb{C}A^{p^\alpha}_{\mathbb{Q}} \cong S^{\alpha}_{0,1}$. This is why we had to deal separately with the case $\gamma = 0$.

For natural numbers $\gamma \leqslant \alpha$, the latest lemma yields

$$\dim\bigl(S^{\alpha}_{\gamma,\sigma}\bigr)^2 = \sum_{\delta,\epsilon\in[\gamma,\alpha]} \dim\bigl(S_{\gamma,\sigma}(\delta)\bigr)\dim\bigl(S_{\gamma,\sigma}(\epsilon)\bigr) = \sum_{\delta,\epsilon}(\delta-\gamma+1)(\epsilon-\gamma+1)$$

where $[\gamma,\alpha]$ denotes the set of integers λ in the range $\gamma \leqslant \lambda \leqslant \alpha$. Meanwhile, by Remark 4.3,

$$\dim\bigl(\mathbb{C}\Gamma^{p^\alpha}\bigr) = \sum_{\delta,\epsilon\in[0,\alpha]} s_*({}_p\delta \times {}_p\epsilon).$$

By Remarks 4.1 and 4.2, the proof of Theorem 1.1 will be complete when we have shown that

$$\dim\bigl(\mathbb{C}\Gamma^{p^\alpha}\bigr) = \sum_{\gamma\in[0,\alpha],\sigma} \dim\bigl(S^{\alpha}_{\gamma,\sigma}\bigr)^2$$

where σ runs over the $\mathbb{C}$-irreps of $\mathrm{Aut}({}_p\gamma)$. The calculation is an application of Goursat's Theorem 2.1.

Lemma 5.2. *Let $\delta \geqslant \gamma \leqslant \epsilon$ be natural numbers. Let $s_*({}_p\delta \times {}_p\epsilon)_\gamma$ denote the number of subgroups $S \leqslant {}_p\delta \times {}_p\epsilon$ such that the group $\uparrow S/\downarrow S \cong S\uparrow/S\downarrow$ is isomorphic to ${}_p\gamma$. Then*

$$s_*({}_p\delta \times {}_p\epsilon)_\gamma = (\delta - \gamma + 1)(\epsilon - \gamma + 1)\phi(p^\gamma) = \sum_\sigma \dim(S_{\gamma,\sigma}(\delta)) \dim(S_{\gamma,\sigma}(\epsilon))$$

where ϕ denotes the Euler function from classical number theory.

Proof. To choose a subgroup S satisfying the specified condition, we make three independent choices: the subquotient $\uparrow S/\downarrow S$ of ${}_p\delta$ such that $\uparrow S/\downarrow S \cong {}_p\gamma$; the subquotient $S\uparrow/S\downarrow$ of ${}_p\epsilon$ such that $S\uparrow/S\downarrow \cong {}_p\gamma$; the group isomorphism θ_S between the two subquotients. The numbers of choices for the first subquotient, the second subquotient and the isomorphism are, respectively, $(\delta - \gamma + 1)$ and $(\epsilon - \gamma + 1)$ and $\phi(p^\gamma)$. The first asserted equality is established. By the previous lemma, the sum in the assertion is a sum of $\phi(p^\gamma)$ terms that are all equal to $(\delta - \gamma + 1)(\epsilon - \gamma + 1)$. $\square$

Summing over δ, γ, ϵ such that $\alpha \geqslant \delta \geqslant \gamma \leqslant \epsilon \leqslant \alpha$, we obtain the required equality. The proof of Theorem 1.1 is complete.

Corollary 5.3. *Let r be the function $\mathbb{N} \to \mathbb{N}$ such that $r(n,m) = r(n)r(m)$ when n and m are coprime and $r(p^\alpha) = (\alpha+1)^3 + \alpha^3 p + \cdots + 2^3 p^{\alpha-1} + p^\alpha$. Then the R-rank of the alchemic algebra for the cyclic group with order n is* $\mathrm{rank}_R(R\Gamma^{C_n}) = r(n)$.

Proof. Applying the latest lemma and using the equality $\phi(p^\gamma) = p^\gamma - p^{\gamma-1}$ for $\gamma \geqslant 1$, we have

$$s_*({}_p\delta \times {}_p\epsilon) = \sum_{\gamma \in [0,\min(\delta,\epsilon)]} (\delta - \gamma + 1)(\epsilon - \gamma + 1)\phi(p^\gamma) = \sum_{\gamma \in [0,\min(\delta,\epsilon)]} (\delta + \epsilon + 1 - 2\gamma)p^\gamma.$$

Summing now over δ and ϵ to obtain the dimension of $\mathbb{C}\Gamma^{p^\alpha}$, then the coefficient of p^0 is

$$\sum_{\delta,\epsilon \in [0,\alpha]} (\delta + \epsilon + 1) = (\alpha + 1)^3.$$

Using the identity $\epsilon + \delta + 1 - 2\gamma = (\epsilon - \gamma) + (\delta - \gamma) + 1$, we find that $\dim_{\mathbb{C}}(\mathbb{C}\Gamma^{p^\alpha}) = r(p^\alpha)$. Remark 4.3 tells us that the R-rank of $R\Gamma^{p^\alpha}$ is independent of R. The general case follows from part (1) of the Chinese Remainder Lemma 4.4. $\square$

Let us mention a connection with another result of Bouc [2, Proposition 23]. We review the constructions (changing some of the notation). Given a G-set X, let $\beta(X) = X \times G$ as a G-G-biset with action $g_1(x,g)g_2 = (g_1x, g_1gg_2)$. By [2, Lemme 13], there is an algebra map $\beta : RB(G) \to R\Gamma(G,G)$ such that $\beta[X] = [\beta(X)]$. Let e_G^G be the primitive idempotent of $\mathbb{K}B(G)$ associated with the species $s_G^G : \mathbb{K}B(G) \to \mathbb{K}$ such that $s_G^G[X]$ is the number of G-fixed points in X. Then $\beta(e_G^G)$ is an idempotent of $\mathbb{K}\Gamma(G,G)$, and [2, Proposition 23] tells us that the $\mathbb{K}$-algebra $\mathcal{E}(G) = \beta(e_G^G)\mathbb{K}\Gamma(G, G)\beta(e_G^G)$ is semisimple if and only if G is cyclic. But if i is an idempotent of a semisimple ring Λ, then the ring $i\Lambda i$ is semisimple. So the cited proposition yields another proof of the easy half of Theorem 1.1; the harder half of Theorem 1.1 yields another proof that $\mathcal{E}(G)$ is semisimple when G is cyclic.

6. The affirmative theorem on semisimplicity

Let $\mathcal{F}$ be a finite non-empty set of finite groups that is closed under isomorphism. That is to say, $\mathcal{F}$ satisfies the hypothesis on $\mathcal{X}$ and, furthermore, $\mathcal{F}$ is finite. Throughout this section, we shall work with $\mathcal{F}$ in place of $\mathcal{X}$.

We shall prove Theorem 1.4. Thanks to part (2) of Remark 2.5, the task of proving the theorem reduces to the case where $\mathcal{X}$ is finite; in other words, we can put $\mathcal{X} = \mathcal{F}$. Curtesy of Lemma 4.1, the task further reduces to the case where $\mathbb{K} = \mathbb{C}$. So Theorem 1.4 will follow when we have shown that the hermetic algebra $\mathbb{C}\Upsilon^{\mathcal{F}}$ is semisimple.

Since $\mathcal{F}$ is finite, there exists a positive integer ℓ such that every cyclic group in $\mathcal{F}$ has order dividing ℓ. The unit group $(\mathbb{Z}/\ell)^\times$ of the ring $\mathbb{Z}/\ell = \mathbb{Z}/\ell\mathbb{Z}$ can be identified with the automorphism group $\mathrm{Aut}(C_\ell)$ of the cyclic group C_ℓ. Each element $a \in (\mathbb{Z}/\ell)^\times$ is identified with the automorphism $\alpha \in \mathrm{Aut}(C_\ell)$ such that $\alpha(c) = c^a$ for $c \in C_\ell$. As a finite abelian group, we define

$$\mathcal{A} = (\mathbb{Z}/\ell)^\times = \mathrm{Aut}(C_\ell).$$

Given any cyclic group C with order dividing ℓ, then $\mathcal{A}$ acts as automorphisms on C by $a : c \mapsto c^a$ where now $c \in C$. By identifying the isomorphism class of a $\mathbb{Q}G$-module with its character $G \to \mathbb{Q}$, we can regard $\mathbb{C}A_{\mathbb{Q}}(G)$ as a subspace of the $\mathbb{C}$-vector space consisting of the functions $G \to \mathbb{C}$.

Lemma 6.1. *Suppose that the exponent of G divides ℓ. Let $G \times \mathcal{A}$ act on G such that an element $(u, a) \in G \times \mathcal{A}$ sends an element $g \in G$ to the element ${}^u g^a$. (We have ${}^u(g^a) = ({}^u g)^a$ so the notation ${}^u g^a$ is unambiguous.) By linear extension, $\mathbb{C}G$ becomes a $\mathbb{C}(G \times \mathcal{A})$-module. Let $(\mathbb{C}G)^{G\times\mathcal{A}}$ denote the subspace of $\mathbb{C}G$ fixed by $G \times \mathcal{A}$. Then there is a $\mathbb{C}$-linear isomorphism*

$$\Xi_G : \mathbb{C}A_{\mathbb{Q}}(G) \ni \psi \mapsto \sum_{g\in G} \psi(g)g \in (\mathbb{C}G)^{G\times\mathcal{A}}.$$

Proof. Recall that two elements f and g of G are said to be **$\mathbb{Q}G$-conjugate** provided the cyclic groups generated by f and g are G-conjugate to each other. It is well known that the $\mathbb{Q}G$-characters are constant on the $\mathbb{Q}G$-conjugacy classes, the irreducible $\mathbb{Q}G$-characters are linearly independent and the number of irreducible $\mathbb{Q}G$-characters is equal to the number of G-conjugacy classes of cyclic groups. Therefore $\mathbb{Q}A_{\mathbb{Q}}(G)$ is the $\mathbb{Q}$-vector space consisting of the functions $G \to \mathbb{Q}$ that are constant on the $\mathbb{Q}G$-conjugacy classes of G. It follows that $\mathbb{C}A_{\mathbb{Q}}(G)$ is the $\mathbb{C}$-vector space of functions $G \to \mathbb{C}$ that are constant on the $\mathbb{Q}G$-conjugacy classes. The assertion holds because the $\mathbb{Q}G$-conjugacy classes are precisely the orbits of $G \times \mathcal{A}$ on G. $\square$

Given a subgroup $H \leqslant G$ and a $\mathbb{C}G$-module M, we let M^H denote the H-fixed subspace of M. Recall that the H-relative trace map for M is defined to be the $\mathbb{C}$-linear map

$$\mathrm{tr}_H^G : M^H \ni m \mapsto \sum_{gH\subseteq G} gm \in M^G.$$

The map tr_H^G is surjective, because it acts on M^G as multiplication by $|G:H|$.

Writing $H^\dagger$ to denote the sum of the elements of H, we define a $\mathbb{C}$-linear map

$$\widetilde{\mathrm{lin}}_G : \mathbb{C}B(G) \to (\mathbb{C}G)^{G\times\mathcal{A}}, \qquad [G/H] \mapsto \mathrm{tr}_H^G\big(H^\dagger\big) = \mathrm{tr}_1^G\big(H^\dagger\big)/|H|.$$

The next result relates $\widetilde{\mathrm{lin}}_G$ to the linearization map $\mathrm{lin}_G : \mathbb{C}B(G) \to \mathbb{C}A_{\mathbb{Q}}(G)$.

Lemma 6.2. *Suppose that the exponent of G divides ℓ. Then $\Xi_G \circ \mathrm{lin}_G = \widetilde{\mathrm{lin}}_G$. Furthermore, the maps lin_G and $\widetilde{\mathrm{lin}}_G$ are surjective.*

Proof. For $H \leqslant G$, let $\chi_{G/H}$ denote the character of the permutation $\mathbb{C}G$-module $\mathbb{C}G/H$. In other words, $\chi_{G/H} = \mathrm{lin}_G[G/H]$. Given $g \in G$, then

$$\chi_{G/H}(g) = \big|\{xH \subseteq G:\ gxH = xH\}\big| = \big|\big\{y \in G:\ {}^y g \in H\big\}\big|/|H|.$$

We have $\Xi_G \circ \mathrm{lin}_G = \widetilde{\mathrm{lin}}_G$ because

$$\Xi_G\big(\mathrm{lin}_G[G/H]\big) = \frac{1}{|H|}\sum_{g\in G}\big|\big\{y\in G:\ {}^y g \in G\big\}\big| g = \frac{1}{|H|}\sum_{y\in G, h\in H} {}^y h = \widetilde{\mathrm{lin}}_G[G/H].$$

Of course, lin_G is surjective because the ring of coefficients, in the present context, is the field $\mathbb{C}$, which has characteristic zero. By the previous lemma, $\widetilde{\mathrm{lin}}_G$ is surjective. □

We sketch an alternative proof of the surjectivity of $\widetilde{\mathrm{lin}}_G$. In fact, a slightly stronger conclusion will emerge. Let C run over the cyclic subgroups of G. Write $C^\#$ denote the set of generators of C, and write $(C^\#)^\dagger$ denote the sum of the elements of $C^\#$. The sets $C^\#$ are precisely the $\mathcal{A}$-orbits of G. By the surjectivity of the relative trace map, $(\mathbb{C}G)^{G\times\mathcal{A}}$ is spanned by the elements having the form $\mathrm{tr}_1^G((C^\#)^\dagger) = \mathrm{tr}_{1\times\mathcal{A}}^{G\times\mathcal{A}}((C^\#)^\dagger)$. It is now easy to deduce that $(\mathbb{C}G)^{G\times\mathcal{A}}$ is spanned by the elements having the form $\widetilde{\mathrm{lin}}_G[G/C]$. We omit further details, because we shall not be making use of this conclusion.

It is worth drawing a diagram to summarize the latest two lemmas. The depicted triangle is commutative. The notation indicates that the horizontal map is an isomorphism and that the other two maps are epimorphisms.

$$\begin{tikzcd} & \mathbb{C}B(G) \arrow[dl, two heads, "\mathrm{lin}_G"'] \arrow[dr, two heads, "\widetilde{\mathrm{lin}}_G"] & \\ \mathbb{C}A_{\mathbb{Q}}(G) \arrow[rr, "\sim", "\Xi_G"'] & & (\mathbb{C}G)^{G\times\mathcal{G}} \end{tikzcd} \qquad \begin{tikzcd} & {[G/H]} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mathrm{lin}[G/H] \arrow[rr, mapsto] & & \mathrm{tr}_H^G(H^\dagger) \end{tikzcd}$$

Given elements $f, g \in G$, we write $f =_G g$ when f and g are G-conjugate. We let $[g]_G$ denote the G-conjugacy class of g. But we shall now be working not with a fixed G, but with all the groups in $\mathcal{F}$. Consider the pairs (k, K) such that $k \in K \in \mathcal{F}$. Two such pairs (k, K) and (k', K') are deemed equivalent provided $K = K'$ and $k =_K k'$. Let $[k, K]$ denote the equivalence class of (k, K).

We let $\mathrm{Mat}^{\mathcal{F}}$ and $\mathrm{mat}^{\mathcal{F}}$ be the full matrix algebras over $\mathbb{C}$ such that the rows and columns of the former are indexed by the pairs (k, K) while the rows and columns of the latter are indexed by the equivalence classes $[k, K]$. Fixing $I, J \in \mathcal{F}$, let $\mathrm{Mat}_{I,J}$ be the subspace of $\mathrm{Mat}_{\mathcal{F}}$ such that the rows of $\mathrm{Mat}_{I,J}$ are indexed by the pairs (i, I) with $i \in I$ and the columns of $\mathrm{Mat}_{I,J}$ are indexed by the pairs (j, J) with $j \in J$. In other words, the rows of $\mathrm{Mat}_{I,J}$ are indexed by the elements of I and the columns are indexed by the elements of J. Similarly, we define $\mathrm{mat}_{I,J}$ as a subspace of $\mathrm{mat}^{\mathcal{F}}$. The rows of $\mathrm{mat}_{I,J}$ are indexed by the conjugacy classes in I and the columns are indexed by the conjugacy classes in J. Thus,

$$\mathrm{Mat}^{\mathcal{F}} = \bigoplus_{I,J\in\mathcal{F}} \mathrm{Mat}_{I,J}, \qquad \mathrm{mat}^{\mathcal{F}} = \bigoplus_{I,J\in\mathcal{F}} \mathrm{mat}_{I,J}.$$

Letting i and j run over the elements of I and J, respectively, we write $\epsilon_{I,J}(i, j)$ to denote the (i, j)th elementary matrix in $\mathrm{Mat}_{I,J}$. Now letting i and j run over representatives of the conjugacy classes in I and J, we write $\epsilon_{I,J}[i, j]$ to denote the $([i]_I, [j]_J)$th elementary matrix in $\mathrm{mat}_{I,J}$. The multiplication operations on $\mathrm{Mat}_{I,J}$ are given by

$$\epsilon_{I,J}(i, j)\epsilon_{J,K}(j', k) = \delta_{j,j'}\epsilon_{I,K}(i, k), \qquad \epsilon_{I,J}[i, j]\epsilon_{J,K}[j', k] = \delta_{[j],[j']}\epsilon_{I,K}[i, k].$$

Here, $\delta_{x,y}$ is the Kronecker delta symbol, with value 1 when $x = y$ and with value 0 when $x \neq y$. We let $I \times J$ act as algebra automorphisms on $\mathrm{Mat}_{I,J}$ such that an element $(u, v) \in I \times J$ sends $\epsilon_{I,J}(i, j)$ to $\epsilon_{I,J}({}^u i, {}^v j)$.

Lemma 6.3. *For each I and J in $\mathcal{F}$, let $\nu_{I,J}$ be the $\mathbb{C}$-linear map* $\mathrm{mat}_{I,J} \to (\mathrm{Mat}_{I,J})^{I\times J}$ *such that*

$$\nu_{I,J}\big(\epsilon_{I,J}[i, j]\big) = \frac{1}{\sqrt{|[i]_I| \cdot |[j]_J|}} \sum_{i'\in[i]_I,\, j'\in[j]_J} \epsilon_{I,J}(i', j')$$

where $i \in I$ and $j \in J$. Let $\nu = \bigoplus_{I,J\in\mathcal{F}} \nu_{I,J}$ as a $\mathbb{C}$-linear map $\mathrm{mat}^{\mathcal{F}} \to \mathrm{Mat}^{\mathcal{F}}$. *Then each $\nu_{I,J}$ is a $\mathbb{C}$-linear isomorphism and ν is a $\mathbb{C}$-algebra monomorphism.*

Proof. The formula for $\nu_{I,J}$ can be rewritten as

$$\nu_{I,J}\big(\epsilon_{I,J}[i, j]\big) = \frac{1}{\sqrt{|[i]_I| \cdot |[j]_J|}} \mathrm{tr}^{I\times J}_{C_I(i)\times C_J(j)}\big(\epsilon_{I,J}(i, j)\big) = \frac{\sqrt{|[i]_I| \cdot |[j]_J|}}{|I| \cdot |J|} \mathrm{tr}^{I\times J}_1\big(\epsilon_{I,J}(i, j)\big).$$

By the surjectivity of the relative trace map, $(\mathrm{Mat}_{I,J})^{I\times J}$ is spanned by the elements having the form $\mathrm{tr}^{I\times J}_1(\epsilon_{I,J}(i, j))$, Furthermore, $\mathrm{tr}^{I\times J}_1(\epsilon_{I,J}(i, j)) = \mathrm{tr}^{I\times J}_1(\epsilon_{I,J}(i', j'))$ if and only if (i, j) is $I \times J$-conjugate to (i', j'), in other words, $\epsilon_{I,J}[i, j] = \epsilon_{I,J}[i', j']$. Therefore $\nu_{I,J}$ is a linear isomorphism. Using the above formulas for the product of two elementary matrices, it is easy to check that $\nu(\epsilon_{I,J}[i, j])\nu(\epsilon_{J,K}[j', k]) = \nu(\epsilon_{I,J}[i, j]\epsilon_{J,K}[j', k])$. Therefore ν is an algebra monomorphism. □

Via the embedding ν, we regard $\mathrm{mat}^{\mathcal{F}}$ as a subalgebra of $\mathrm{Mat}^{\mathcal{F}}$. The first part of the lemma now says that $\mathrm{mat}_{I,J} = (\mathrm{Mat}_{I,J})^{I\times J}$.

We allow $I \times J \times \mathcal{A}$ to act on $I \times J$ and on $\mathrm{Mat}_{I,J}$ such that

$$(u, v, a)(i, j) = \left({}^u i^a, {}^v j^a\right), \qquad (u, v, a)\epsilon_{I,J}(i, j) = \epsilon_{I,J}\left({}^u i^a, {}^v j^a\right)$$

for $(u, v, a) \in I \times J \times \mathcal{A}$. (As in Lemma 6.1, the notation ${}^u i^a$ is unambiguous.) This gives rise to an action of $\mathcal{A}$ on $\mathrm{mat}_{I,J}$ such that a sends $\epsilon_{I,J}[i, j]$ to $\epsilon_{I,J}[i^a, j^a]$. Plainly,

$$\mathrm{mat}_{I,J}^{\mathcal{A}} = (\mathrm{Mat}_{I,J})^{I \times J \times \mathcal{A}}.$$

All of the $\mathbb{C}$-vector spaces and some of the $\mathbb{C}$-linear maps in the following diagram have now been defined. We shall define the other maps in the diagram, and we shall show that the diagram commutes. We shall also show that all of the vertical and horizontal maps in the diagram are $\mathbb{C}$-linear isomorphisms and that the other four maps are $\mathbb{C}$-linear epimorphisms.

$$\begin{tikzcd}
\mathbb{C}A_{\mathbb{Q}}(I \times J) \arrow[rrr, "\alpha"] \arrow[ddd, "\Xi_{I \times J}"'] & & & \mathbb{C}\Upsilon(I, J) \arrow[ddd, "\Xi_{I,J}"] \\
& & & \\
& \mathbb{C}B(I \times J) \arrow[uul, "\mathrm{lin}_{I \times J}"'] \arrow[r, "\beta"] \arrow[dl, "\widetilde{\mathrm{lin}}_{I \times J}"] & \mathbb{C}\Gamma(I, J) \arrow[uur, "\mathrm{lin}_{I,J}"] \arrow[dr, "\widetilde{\mathrm{lin}}_{I,J}"'] & \\
\mathbb{C}(I \times J)^{(I \times J) \times \mathcal{A}} \arrow[rrr, "\gamma"] & & & (\mathrm{mat}_{I,J})^{\mathcal{A}}
\end{tikzcd}$$

Lemmas 6.1 and 6.2 tell us that the left-hand triangle commutes, the map $\Xi_{I \times J}$ is an isomorphism and the other two maps in the left-hand triangle are epimorphisms.

Recall that, as a $\mathbb{C}$-vector space, $\mathbb{C}\Gamma(I, J) = \mathbb{C}B(I \times J)$. As explained in Section 3, we can make the identification $\mathbb{C}\Upsilon(I, J) = \mathbb{C}A_{\mathbb{Q}}(I \times J)$ (because $\mathbb{C}$ is a field with characteristic zero). We can also identify the two maps $\mathrm{lin}_{I,J} : \mathbb{C}\Gamma(I, J) \to \mathbb{C}\Upsilon(I, J)$ and $\mathrm{lin}_{I \times J} : \mathbb{C}B(I \times J) \to \mathbb{C}A_{\mathbb{Q}}(I \times J)$. Let $\alpha : \mathbb{C}A_{\mathbb{Q}}(I \times J) \to \mathbb{C}\Upsilon(I, J)$ and $\beta : \mathbb{C}B(I \times J) \to \mathbb{C}\Gamma(I, J)$ be the identity maps. Trivially, the top square in the diagram commutes, its two horizontal maps are isomorphisms and the other two maps are epimorphisms.

Given a subgroup $S \subseteq I \times J$, we define

$$S^+ = \sum_{(i,j) \in S} \epsilon_{I,J}(i, j)$$

as an element of $(\mathrm{Mat}_{I,J})^S$. We define

$${}_I S_J = \frac{1}{\sqrt{|I| \,.\, |J|}} \mathrm{tr}_S^{I \times J}\left(S^+\right) = \frac{1}{|S|\sqrt{|I| \,.\, |J|}} \mathrm{tr}_1^{I \times J}\left(S^+\right)$$

as an element of $\mathrm{mat}_{I,J}$. Actually, $S^+ \in (\mathrm{Mat}_{I,J})^{S \times \mathcal{A}}$, so ${}_I S_J \in (\mathrm{mat}_{I,J})^{\mathcal{A}}$. Observe that ${}_I S_J$ depends only on the $I \times J$-conjugacy class of S. As S runs over the conjugacy classes of subgroups of $I \times J$, the elements $[(I \times J)/S]$ run over the elements of a basis of the $\mathbb{C}$-vector space $\mathbb{C}\Gamma(I, J) = \mathbb{C}B(I \times J)$. So we have a $\mathbb{C}$-linear map

$$\widetilde{\mathrm{lin}}_{I,J} : \mathbb{C}\Gamma(I, J) \to (\mathrm{mat}_{I,J})^{\mathcal{A}}, \quad \left[\frac{I \times J}{S}\right] \mapsto {}_I S_J.$$

There is a $\mathbb{C}$-linear isomorphism

$$\hat{\gamma} : \mathbb{C}(I \times J) \to \mathrm{Mat}_{I,J}, \quad (i, j) \mapsto \frac{1}{\sqrt{|I| \,.\, |J|}} \epsilon_{I,J}(i, j).$$

By direct calculation, $\hat{\gamma}(\mathrm{tr}_1^{I \times J}(S^\dagger)) = {}_I S_J$. Plainly, $\hat{\gamma}$ commutes with the actions of $I \times J \times \mathcal{A}$. Therefore, $\hat{\gamma}$ restricts to an isomorphism

$$\gamma : \mathbb{C}(I \times J)^{I \times J \times \mathcal{A}} \to (\mathrm{mat}_{I,J})^{\mathcal{A}}, \quad \mathrm{tr}_S^{I \times J}\big(S^\dagger\big) \mapsto {}_I S_J.$$

Since $\mathbb{C}B(I \times J)$ is spanned by the elements having the form $[(I \times J)/S]$, the calculation

$$\gamma \circ \widetilde{\mathrm{lin}}_{I \times J}\left[\frac{I \times J}{S}\right] = \gamma\big(\mathrm{tr}_S^{I \times J}\big(S^\dagger\big)\big) = {}_I S_J = \beta \circ \widetilde{\mathrm{lin}}_{I,J}\left[\frac{I \times J}{S}\right]$$

shows that the lower square in the diagram commutes. We have already seen, in Lemma 6.2, that $\widetilde{\mathrm{lin}}_{I \times J}$ is an epimorphism. Since β and γ are isomorphisms, we deduce that $\widetilde{\mathrm{lin}}_{I,J}$ is an epimorphism.

We have not yet defined the map $\Xi_{I,J}$ but, aside from the assertions concerning $\Xi_{I,J}$, we have proved everything else that we stated about the latest diagram: the left-hand triangle and the top and bottom square are commutative, the maps $\alpha, \beta, \gamma, \Xi_{I \times J}$ are isomorphisms, the four maps denoted by decorations of the symbol lin are epimorphisms. It follows that there exists a unique isomorphism $\Xi_{I,J} : \mathbb{C}\Upsilon(I, J) \to (\mathrm{mat}_{I,J})^{\mathcal{A}}$ such that α, β, γ together comprise an isomorphism in the category of commutative triangles of $\mathbb{C}$-linear maps. In other words, there exists a unique isomorphism $\Xi_{I,J}$ such that $\Xi_{I,J} \circ \alpha = \gamma \circ \Xi_{I \times J}$ and $\Xi_{I,J} \circ \mathrm{lin}_{I,J} = \widetilde{\mathrm{lin}}_{I,J}$. Everything we stated about the diagram has now been proved. Let us collect some of those statements into a lemma.

Lemma 6.4. *There exists a well-defined $\mathbb{C}$-linear isomorphism*

$$\Xi_{I,J} : \mathbb{C}\Upsilon(I, J) \to (\mathrm{mat}_{I,J})^{\mathcal{A}}, \qquad \Xi_{I,J}\left(\mathrm{lin}\left[\frac{I \times J}{S}\right]\right) = {}_I S_J.$$

We have $\Xi_{I,J} \circ \mathrm{lin}_{I,J} = \widetilde{\mathrm{lin}}_{I,J}$. The maps $\mathrm{lin}_{I,J}$ and $\widetilde{\mathrm{lin}}_{I,J}$ are surjective.

The actions of $\mathcal{A}$ on each $\mathbb{C}$-vector space $\mathrm{mat}_{I,J}$ combine to give an action of $\mathcal{A}$ on the $\mathbb{C}$-vector space $\mathrm{mat}^{\mathcal{F}} = \bigoplus_{I,J} \mathrm{mat}_{I,J}$. Using the formula for the product of two elementary matrices, it is easy to see that each element of $\mathcal{A}$ acts as an automorphism of $\mathrm{mat}^{\mathcal{F}}$. Therefore, the $\mathcal{A}$-fixed subspace $(\mathrm{mat}^{\mathcal{F}})^{\mathcal{A}}$ is a unital subalgebra of $\mathrm{mat}_{\mathcal{F}}$.

The isomorphisms $\Xi_{I,J}$ combine to form a $\mathbb{C}$-linear isomorphism

$$\Xi^{\mathcal{F}} = \bigoplus_{I,J} \Xi_{I,J} \colon \mathbb{C}\Upsilon^{\mathcal{F}} = \bigoplus_{I,J} \mathbb{C}\Upsilon_{I,J} \to \big(\mathrm{mat}^{\mathcal{F}}\big)^{\mathcal{A}} = \bigoplus_{I,J} (\mathrm{mat}_{I,J})^{\mathcal{A}}.$$

Soon, we shall show that $\Xi^{\mathcal{F}}$ is an algebra isomorphism. For that purpose, we first need an obvious remark and a lemma.

Remark 6.5. Let I and J be finite groups and let $S \leqslant I \times J$. Then, in the notation of Goursat's Theorem 2.1, $|{\uparrow}S| \,.\, |S{\downarrow}| = |S| = |S{\uparrow}| \,.\, |{\downarrow}S|$.

Lemma 6.6. *Let I, J, K be finite groups. Let $S \leqslant I \times J$ and $T \leqslant J \times K$. Then*

(1) $|S * T| \,.\, |S{\downarrow} \cap {\downarrow}T| \,.\, |S{\uparrow} \,.\, {\uparrow}T| = |S| \,.\, |T|$.
(2) $|S{\downarrow} \cap {\downarrow}T|(S * T)^+ = S^+T^+$.

Proof. Let $U = S{\uparrow} \cap {\uparrow}T$ and $V = S{\downarrow} \cap {\downarrow}T$. We claim that

$$|S * T| \,.\, |V| = |U| \,.\, |{\downarrow}S| \,.\, |T{\downarrow}|.$$

To demonstrate the claim, we shall count, in two different ways, the elements of the set

$$\mathcal{L} = \big\{(i, j, k) \in I \times J \times K\colon\ (i, j) \in S,\ (j, k) \in T\big\}.$$

Fixing $(i, k) \in I \times K$, then $(i, k) \in S * T$ if and only if the set

$$\mathcal{J}_{i,k} = \big\{j \in J\colon\ (i, j) \in S,\ (j, k) \in T\big\}$$

is non-empty. In that case, choosing an element $j_0 \in \mathcal{J}_{i,k}$, then

$$\mathcal{J}_{i,k} = \{j \in U\colon\ jS{\downarrow} = j_0S{\downarrow},\ j{\downarrow}T = j_0{\downarrow}T\} = \{j \in U\colon\ jV = j_0V\}$$

and, in particular, $|\mathcal{J}_{i,k}| = |V|$. Therefore $|\mathcal{L}| = |S * T| \,.\, |V|$.

Now let us count the elements of $\mathcal{L}$ in a different way. Fixing an element $j \in J$, then $j \in U$ if and only if the set

$$\mathcal{M}_j = \big\{(i, j) \in I \times K\colon\ (i, j) \in S,\ (j, k) \in T\big\}$$

is non-empty. In that case, choosing an element $(i_0, k_0) \in M_j$, we have

$$\mathcal{M}_j = \big\{(i, k) \in {\uparrow}S \times T{\uparrow}\colon\ i{\downarrow}S = i_0{\downarrow}S,\ kT{\downarrow} = k_0T{\downarrow}\big\}$$

which has size $|{\downarrow}S| \,.\, |T{\downarrow}|$. Therefore $|\mathcal{L}| = |U| \,.\, |{\downarrow}S| \,.\, |T{\downarrow}|$. The claim is now established.

For any two finite subgroups P and Q of a group, we have $|P \cap Q| \,.\, |PQ| = |P| \,.\, |Q|$. This observation, together with the latest remark, allows us to rewrite the right-hand side of the established claim as

$$|S{\uparrow} \cap {\uparrow}T| \,.\, |{\downarrow}S| \,.\, |T{\downarrow}| = \frac{|S{\uparrow}| \,.\, |{\uparrow}T|}{|S{\uparrow} \,.\, {\uparrow}T|}.\frac{|S|}{|S{\uparrow}|}.\frac{|T|}{|T{\uparrow}|} = \frac{|S| \,.\, |T|}{|S{\uparrow} \,.\, {\uparrow}T|}.$$

Part (1) now follows. Part (3) holds because

$$S^+T^+ = \sum_{(i,j,k)\in\mathcal{L}} \epsilon_{I,J}(i, j)\epsilon_{J,K}(j, k) = \sum_{(i,k)\in S*T} |\mathcal{J}_{i,k}|\epsilon_{I,K}(i, k) = |V|(S * T)^+. \qquad \square$$

Lemma 6.7. *The map $\Xi^{\mathcal{F}}: \mathbb{C}\Upsilon^{\mathcal{F}} \to (\mathrm{mat}^{\mathcal{F}})^{\mathcal{A}}$ is a unital $\mathbb{C}$-algebra isomorphism.*

Proof. First, let us check that $\Xi^{\mathcal{F}}$ preserves the unity elements. The unity element of $\mathbb{C}\Upsilon^{\mathcal{F}}$ is $1_\Upsilon = \sum_{I \in \mathcal{F}} \mathrm{iso}_I$. As we noted in Section 2, each $\mathrm{iso}_I = [(I \times I)/\Delta(I)]$. Therefore $\Xi^{\mathcal{F}}(1_\Upsilon) = \sum_I {}_I\Delta(I)_I$. Consider elements $i, j \in I$. If $i =_I j$, then the coefficient of $\epsilon_{I,I}(i,j)$ in the matrix ${}_I\Delta(I)_I = \mathrm{tr}_1^{I\times I}(\Delta(I)^+)/|I|^2$ is $1/[i]_I$. If $i \neq_I j$, then the coefficient of $\epsilon_{I,I}(i,j)$ is zero. Therefore ${}_I\Delta(I)_I = \sum_i \epsilon_{I,I}[i,i]$ where i runs over representatives of the conjugacy classes in I. In other words, ${}_I\Delta(I)_I$ is the identity matrix in $\mathrm{mat}_{I,I}$. We have shown that $\Xi^{\mathcal{F}}(1_\Upsilon)$ is the unity element of $(\mathrm{mat}^{\mathcal{F}})^{\mathcal{A}}$.

It remains only to check that $\Xi^{\mathcal{F}}$ preserves multiplication. The necessary calculations are complicated but straightforward. Directly from the definition of S^+, it is easy to check that ${}^{(1,h)}S^+ \,.\, {}^{(h,1)}T^+ = S^+T^+$ for all $h \in J$. Using that equality, then using part (2) of Lemma 6.6,

$$\begin{aligned}
\mathrm{tr}_1^{I\times J}(S^+)\,\mathrm{tr}_1^{J\times K}(T^+) &= \sum_{i\in I,\, h\in J,\, j\in J,\, k\in K} {}^{(i,h)}S^+ \,.\, {}^{(hh^{-1}j,k)}T^+ = |J| \sum_{i,j,k} {}^{(i,k)}\big(S^+ \,.\, {}^{(j,1)}T^+\big) \\
&= |J|\,\mathrm{tr}_1^{I\times K}\big(S^+ \,.\, {}^{(j,1)}T^+\big) \\
&= \sum_{j\in J} \big|S{\downarrow} \cap {\downarrow}^{(j,1)}T\big| \,.\, |J|\,\mathrm{tr}_1^{I\times K}\big(\big(S * {}^{(j,1)}T\big)^+\big).
\end{aligned}$$

Hence, making use of part (1) of Lemma 6.6,

$$\begin{aligned}
\sqrt{|I|\,.\,|K|}\;{}_IS_J \,.\, {}_JT_K &= \frac{1}{|S|\,.\,|T|\,.\,|J|}\,\mathrm{tr}_1^{I\times J}(S^+)\,\mathrm{tr}_1^{J\times K}(T^+) \\
&= \sum_{j\in J} \frac{|S{\downarrow} \cap {\downarrow}^{(j,1)}T|}{|S|\,.\,|T|}\,\mathrm{tr}_1^{I\times K}\big(\big(S * {}^{(j,1)}T\big)^+\big) \\
&= \sum_{j\in J} \frac{1}{|S * {}^{(j,1)}T|\,.\,|S{\uparrow} \,.\, {\uparrow}^{(j,1)}T|}\,\mathrm{tr}_1^{I\times K}\big(\big(S * {}^{(j,1)}T\big)^+\big).
\end{aligned}$$

As we noted in the Generalized Mackey Product Theorem 2.2, the $I \times K$-conjugacy class of $S * {}^{(j,1)}T$ depends only on the double coset $S{\uparrow} \,.\, j \,.\, {\uparrow}T$. Since ${}^j({\uparrow}T) = {\uparrow}^{(j,1)}T$, we have $|S{\uparrow} \,.\, j \,.\, {\uparrow}T| = |S{\uparrow} \,.\, {\uparrow}^{(j,1)}T|$. Therefore,

$${}_IS_J \,.\, {}_JT_K = \sum_{S{\uparrow}.j.{\uparrow}T \subseteq J} \frac{1}{\sqrt{|I|\,.\,|K|}\,.\,|S * {}^{(j,1)}T|}\,\mathrm{tr}_1^{I\times K}\big(\big(S * {}^{(j,1)}T\big)^+\big) = \sum_{S{\uparrow}.j.{\uparrow}T \subseteq J} {}_I\big(S * {}^{(j,1)}T\big)_K.$$

Comparing with Theorem 2.2, we see that the composite map $\Xi^{\mathcal{F}} \circ \mathrm{lin}^{\mathcal{F}} : \mathbb{C}\Gamma^{\mathcal{F}} \to (\mathrm{mat}^{\mathcal{F}})^{\mathcal{A}}$ preserves multiplication. But $\mathrm{lin}^{\mathcal{F}} : \mathbb{C}\Gamma^{\mathcal{F}} \to \mathbb{C}\Upsilon^{\mathcal{F}}$ is a $\mathbb{C}$-algebra epimorphism, so $\Xi^{\mathcal{F}}$ respects multiplication. □

Let $\mathrm{htw}^{\mathcal{F}}$ be a $\mathbb{C}$-vector space with a basis indexed by the equivalence classes $[k,K]$ where $k \in K \in \mathcal{F}$. Let $\epsilon_K[k]$ denote the basis element indexed by $[k,K]$. By regarding $\mathrm{htw}^{\mathcal{F}}$ as the space of column vectors for the full matrix algebra $\mathrm{mat}^{\mathcal{F}}$, we see that there is a unital $\mathbb{C}$-algebra isomorphism $\rho = \rho^{\mathcal{F}} : \mathrm{mat}^{\mathcal{F}} \to \mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$ such that

$$\rho\big(\epsilon_{I,J}[i,j]\big)\epsilon_K[k] = \delta_{[j,J],[k,K]}\epsilon_I[i].$$

(Again, we are employing the Kronecker delta symbol.) We make $\mathrm{htw}^{\mathcal{F}}$ become a $\mathbb{C}\mathcal{A}$-module via the representation $\theta = \theta^{\mathcal{F}} : \mathbb{C}\mathcal{A} \to \mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$ such that

$$\theta(a)\epsilon_K[k] = \epsilon_K\big[k^a\big].$$

We let $\mathcal{A}$ act as automorphisms of $\mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$ such that a acts as conjugation by $\theta(a)$. The reason for the notation $\mathrm{htw}^{\mathcal{F}}$ will become apparent in the next section.

Lemma 6.8. *The unital $\mathbb{C}$-algebra isomorphism $\rho^{\mathcal{F}} : \mathrm{mat}^{\mathcal{F}} \to \mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$ commutes with the actions of $\mathcal{A}$. In particular, $\rho^{\mathcal{F}}$ restricts to a unital $\mathbb{C}$-algebra isomorphism $(\mathrm{mat}^{\mathcal{F}})^{\mathcal{A}} \to \mathrm{End}_{\mathbb{C}\mathcal{A}}(\mathrm{htw}^{\mathcal{F}})$.*

Proof. We have $\rho({}^a\epsilon_{I,J}[i,j]) = \rho(\epsilon_{I,J}(i^a, j^a)) = \theta(a)\rho(\epsilon_{I,J}[i,j])\theta(a)^{-1}$. □

We can now complete the proof of Theorem 1.4. At the beginning of this section, we explained why it suffices to show that $\mathbb{C}\Upsilon^{\mathcal{F}}$ is semisimple. Lemmas 6.7 and 6.8 tell us that the composite map

$$\rho^{\mathcal{F}} \circ \Xi^{\mathcal{F}} : \mathbb{C}\Upsilon^{\mathcal{F}} \to \mathrm{End}_{\mathbb{C}\mathcal{A}}\big(\mathrm{htw}^{\mathcal{F}}\big)$$

is a unital $\mathbb{C}$-algebra isomorphism. (We are abusing notation, writing $\rho^{\mathcal{F}}$ for both of the isomorphisms appearing in the statement of Lemma 6.8.) Since the group algebra $\mathbb{C}\mathcal{A}$ is semisimple, the endomorphism algebra $\mathrm{End}_{\mathbb{C}\mathcal{A}}(\mathrm{htw}^{\mathcal{F}})$ is semisimple, and it follows that the hermetic algebra $\mathbb{C}\Upsilon^{\mathcal{F}}$ is semisimple. The proof of Theorem 1.4 is now complete.

Lemma 6.9. *The unital $\mathbb{C}$-algebra isomorphism $\rho^{\mathcal{F}} \circ \Xi^{\mathcal{F}} : \mathbb{C}\Upsilon^{\mathcal{F}} \to \mathrm{End}_{\mathbb{C}\mathcal{A}}(\mathrm{htw}^{\mathcal{F}})$ restricts to a unital $\mathbb{C}$-algebra isomorphism $Z(\mathbb{C}\Upsilon^{\mathcal{F}}) \to \theta^{\mathcal{F}}(\mathbb{C}\mathcal{A})$.*

Proof. By Lemmas 6.7 and 6.8, $\rho^{\mathcal{F}} \circ \Xi^{\mathcal{F}}$ restricts to a unital $\mathbb{C}$-algebra isomorphism $Z(\mathbb{C}\Upsilon^{\mathcal{F}}) \to Z(\mathrm{End}_{\mathbb{C}\mathcal{A}}(\mathrm{htw}^{\mathcal{F}}))$. Since $\theta(\mathcal{A})$ is a commutative unital subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$, we have $Z(\mathrm{End}_{\mathbb{C}\mathcal{A}}(\mathrm{htw}^{\mathcal{F}})) = \theta(\mathcal{A})$. □

7. The biset functor $\mathrm{htw}^{\mathcal{F}}$

We continue to work with $\mathcal{F}$ in place of $\mathcal{X}$. Thus far, we have realized $\mathrm{htw}^{\mathcal{F}}$ as a $\mathbb{C}\mathcal{A}$-module. In this section, we shall realize $\mathrm{htw}^{\mathcal{F}}$ as a rhetorical biset functor over $\mathbb{C}$, in other words, as a $\mathbb{C}\Upsilon^{\mathcal{F}}$-module. Our study of $\mathrm{htw}^{\mathcal{F}}$ will yield much information about the simple rhetorical biset functors for $\mathcal{F}$ over $\mathbb{C}$. At the end of this section, we shall prove Theorems 1.5 and 1.6 in the special case where $\mathcal{X} = \mathcal{F}$ and $\mathbb{K} = \mathbb{C}$.

To simplify the discussion, it will be convenient to make several identifications. Via the isomorphism $\rho^{\mathcal{F}}$ discussed in Lemma 6.8, we make the identifications

$$\mathrm{mat}^{\mathcal{F}} = \mathrm{End}_{\mathbb{C}}\big(\mathrm{htw}^{\mathcal{F}}\big), \qquad \big(\mathrm{mat}^{\mathcal{F}}\big)^{\mathcal{A}} = \mathrm{End}_{\mathbb{C}\mathcal{A}}\big(\mathrm{htw}^{\mathcal{F}}\big).$$

Recall that the matrix algebra $\mathrm{mat}^{\mathcal{F}}$ has a basis consisting of the elements $\epsilon_{I,J}[i,j]$ and the vector space $\mathrm{htw}^{\mathcal{F}}$ has a basis consisting of the elements $\epsilon_K[k]$. In view of the identifications that we have just made, the action of $\mathrm{mat}^{\mathcal{F}}$ on $\mathrm{htw}^{\mathcal{F}}$ is given by the equality

$$\epsilon_{I,J}[i,j]\epsilon_K[k] = \delta_{[j,J],[k,K]}\epsilon_I[i].$$

Thus, we have realized $\mathrm{htw}^{\mathcal{F}}$ as the space of column vectors of the full matrix algebra $\mathrm{mat}^{\mathcal{F}}$.

Via the isomorphism $\Xi^{\mathcal{F}}$ discussed in Lemma 6.7, we make the identification

$$\mathbb{C}\Upsilon^{\mathcal{F}} = \left(\mathrm{mat}^{\mathcal{F}}\right)^{\mathcal{A}} = \mathrm{End}_{\mathbb{C}\mathcal{A}}\left(\mathrm{htw}^{\mathcal{F}}\right).$$

In this way, $\mathbb{C}\Upsilon^{\mathcal{F}}$ becomes a unital subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$. Hence $\mathrm{htw}^{\mathcal{F}}$ becomes a unital $\mathbb{C}\Upsilon^{\mathcal{F}}$-module. That is to say, $\mathrm{htw}^{\mathcal{F}}$ is now a rhetorical biset functor over $\mathbb{C}$. For fixed $K \in \mathcal{F}$, the coordinate module $\mathrm{htw}^{\mathcal{F}}(K)$ has a basis consisting of the elements $\epsilon_K[k]$ where k runs over a representatives of the conjugacy classes in K. Now that $\mathbb{C}\Upsilon^{\mathcal{F}}$ has been embedded in $\mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$, Lemma 6.9 tells us that

$$Z\left(\mathbb{C}\Upsilon^{\mathcal{F}}\right) = \theta^{\mathcal{F}}(\mathcal{A}).$$

Also recall that $\mathrm{Mat}^{\mathcal{F}}$ has a basis consisting of the elements $\epsilon_{I,J}(i,j)$. We define $\mathrm{HTW}^{\mathcal{F}}$ to be the $\mathbb{C}$-vector space consisting of the column vectors of the full matrix algebra $\mathrm{Mat}^{\mathcal{F}}$. In an evident sense, we have an identification

$$\mathrm{Mat}^{\mathcal{F}} = \mathrm{End}_{\mathbb{C}}\left(\mathrm{HTW}^{\mathcal{F}}\right).$$

Let $\{\epsilon_K(k)\colon\ k \in K \in \mathcal{F}\}$ be the basis of $\mathrm{HTW}^{\mathcal{F}}$ such that

$$\epsilon_{I,J}(i,j)\epsilon_K(k) = \delta_{(j,J),(k,K)}\epsilon_I(i).$$

Let $\mathrm{HTW}(K)$ be the subspace of $\mathrm{HTW}^{\mathcal{F}}$ spanned by the elements $\epsilon_K(k)$ where k now runs over all the elements of K. We have a direct sum decomposition $\mathrm{HTW}^{\mathcal{F}} = \bigoplus_K \mathrm{HTW}(K)$. The reason for the notation $\mathrm{HTW}^{\mathcal{F}}$ will be explained below.

The identifications that we have made are summarized in the chain of subalgebras

$$\begin{aligned} Z\left(\mathbb{C}\Upsilon^{\mathcal{F}}\right) = \theta^{\mathcal{F}}(\mathcal{A}) \leqslant \mathbb{C}\Upsilon^{\mathcal{F}} = \left(\mathrm{mat}^{\mathcal{F}}\right)^{\mathcal{A}} = \mathrm{End}_{\mathbb{C}\mathcal{A}}\left(\mathrm{htw}^{\mathcal{F}}\right) \\ \leqslant \mathrm{mat}^{\mathcal{F}} = \mathrm{End}_{\mathbb{C}}\left(\mathrm{htw}^{\mathcal{F}}\right) \leqslant \mathrm{Mat}^{\mathcal{F}} = \mathrm{End}_{\mathbb{C}}\left(\mathrm{HTW}^{\mathcal{F}}\right). \end{aligned}$$

The first three of these four algebras have the same unity element, which we shall write as 1_{mat}. Glancing at the proof of Lemma 6.3, we see that

$$1_{\mathrm{mat}} = \sum_{K,k,k'\colon\ k \in K \in \mathcal{F},\ k' \in [k]_K} \frac{1}{|[k]_K|}\epsilon_{K,K}(k,k').$$

But 1_{mat} does not coincide with the unity element $\sum_{k \in K \in \mathcal{F}} \epsilon_{K,K}(k,k)$ of the fourth algebra unless every group in $\mathcal{F}$ is abelian.

Since $\mathrm{mat}^{\mathcal{F}}$ is a subalgebra of $\mathrm{Mat}^{\mathcal{F}}$, we can regard $\mathrm{HTW}^{\mathcal{F}}$ as a $\mathrm{mat}^{\mathcal{F}}$-module. But we have just observed that $\mathrm{mat}^{\mathcal{F}}$ need not be a unital subalgebra of $\mathrm{Mat}^{\mathcal{F}}$, so $\mathrm{HTW}^{\mathcal{F}}$ need not be a unital $\mathrm{mat}^{\mathcal{F}}$-module. However, $1_{\mathrm{mat}} \,.\, \mathrm{HTW}^{\mathcal{F}}$ is a unital $\mathrm{mat}^{\mathcal{F}}$-submodule of $\mathrm{HTW}^{\mathcal{F}}$.

Lemma 7.1. *For each $K \in \mathcal{F}$, let μ_K be the $\mathbb{C}$-linear map $\mathrm{htw}^{\mathcal{F}}(K) \to 1_{\mathrm{mat}} \,.\, \mathrm{HTW}^{\mathcal{F}}(K)$ such that*

$$\mu_K\bigl(\epsilon_K[k]\bigr) = \frac{1}{\sqrt{|[k]_K|}} \sum_{k' \in [k]_K} \epsilon_K(k)$$

where $k \in K$. Let $\mu = \bigoplus_{K \in \mathcal{F}} \mu_K$ as a $\mathbb{C}$-linear map $\mathrm{htw}^{\mathcal{F}} \to 1_{\mathrm{mat}} \,.\, \mathrm{HTW}^{\mathcal{F}}$. Then each μ_K is a $\mathbb{C}$-linear isomorphism and μ is an isomorphism of $\mathrm{mat}^{\mathcal{F}}$-modules.

Proof. Using the above formula for 1_{mat}, it is easy to see that $1_{\mathrm{mat}} \,.\, \mathrm{HTW}^{\mathcal{F}}(K)$ has a basis consisting of the elements $\sum_{k' \in [k]_K} \epsilon_K(k)$ where k runs over representatives of the conjugacy classes in K. Therefore each μ_K is a $\mathbb{C}$-linear isomorphism. It follows that μ is a $\mathbb{C}$-linear isomorphism. By direct calculation,

$$\begin{aligned}
\epsilon_{I,J}[i,j]\mu\bigl(\epsilon_K[k]\bigr) &= \frac{1}{\sqrt{|[i]_I| \,.\, |[j]_J| \,.\, |[k]_K|}} \sum_{i' \in [i]_I,\, j' \in [j]_J,\, k' \in [k]_K} \epsilon_{I,J}(i',j')\epsilon_K(k') \\
&= \delta_{[j,J],[k,K]} \frac{1}{\sqrt{[i]_I}} \sum_{i' \in [i]_I} \epsilon_I(i) = \mu\bigl(\epsilon_{I,J}[i,j]\epsilon_K[k]\bigr).
\end{aligned}$$

Therefore μ is an isomorphism of $\mathrm{mat}^{\mathcal{F}}$-modules. □

Via the isomorphisms μ_K and μ, we make the identifications

$$\mathrm{htw}^{\mathcal{F}}(K) = 1_{\mathrm{mat}} \,.\, \mathrm{HTW}^{\mathcal{F}}(K), \qquad \mathrm{htw}^{\mathcal{F}} = 1_{\mathrm{mat}} \,.\, \mathrm{HTW}^{\mathcal{F}}.$$

The inclusion $\mathrm{htw}(K) \hookrightarrow \mathrm{HTW}(K)$ can be expressed in terms of the relative trace map,

$$\epsilon_K[k] = \frac{1}{\sqrt{|[k]_K|}} \mathrm{tr}^K_{C_K(k)}\bigl(\epsilon_K(k)\bigr) = \frac{\sqrt{|[k]_K|}}{|K|} \mathrm{tr}^K_1\bigl(\epsilon_K(k)\bigr).$$

At last, we can explain the reason for the notation $\mathrm{htw}^{\mathcal{F}}$ and $\mathrm{HTW}^{\mathcal{F}}$. By restriction, the $\mathrm{Mat}^{\mathcal{F}}$-module $\mathrm{HTW}^{\mathcal{F}}$ can be regarded as a $\mathbb{C}\Upsilon^{\mathcal{F}}$-module. Alas, if $\mathcal{F}$ owns a non-abelian group, then $\mathrm{HTW}^{\mathcal{F}}$ is not a biset functor because $\mathrm{HTW}^{\mathcal{F}}$ is not unital as a $\mathbb{C}\Upsilon^{\mathcal{F}}$-module. Nevertheless, there is still a functor sending G to $\mathrm{HTW}(G)$ and sending bisets to bimodules. The details of this functor are discussed in Hambleton, Taylor and Williams [9]; it is the "group ring functor" in the terminology of [9, 1.A.6]. Vaguely speaking, something of Theorem 1.2 can be gleaned from [9, 1.A.11, 1.A.12] by regarding $\mathrm{HTW}^{\mathcal{F}}$ as a kind of universal biset functor (even though it is not really a biset functor). We shall be employing something of this idea in our proof of Theorem 1.2 in the next section.

The representation $\theta = \theta^{\mathcal{F}} : \mathbb{C}\mathcal{A} \to \mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$ extends to a representation $\Theta = \Theta^{\mathcal{F}} : \mathbb{C}\mathcal{A} \to \mathrm{End}_{\mathbb{C}}(\mathrm{HTW}^{\mathcal{F}})$ such that $\Theta(a)\epsilon_K(k) = \epsilon(k^a)$. We mean to say that $\mathrm{htw}^{\mathcal{F}}$ is now a $\mathbb{C}\mathcal{A}$-submodule of $\mathrm{HTW}^{\mathcal{F}}$. Thanks to the identifications $\mathrm{mat}^{\mathcal{F}} = \mathrm{End}_{\mathbb{C}}(\mathrm{htw}^{\mathcal{F}})$ and $\mathrm{Mat}^{\mathcal{F}} = \mathrm{End}_{\mathbb{C}}(\mathrm{HTW}^{\mathcal{F}})$, we can express these two representations by the formulas

$$\theta(a) = \sum_{k \in K \in \mathcal{F}} \epsilon_{K,K}\big[k^a, k\big], \qquad \Theta(a) = \sum_{k \in K \in \mathcal{F}} \epsilon_{K,K}\big(k^a, k\big).$$

Let us recall some notions from classical number theory. For the moment, let ℓ be any positive integer. Given a divisor n of ℓ, then the canonical ring epimorphism $\mathbb{Z}/\ell \to \mathbb{Z}/n$ restricts to a group epimorphism from the group $(\mathbb{Z}/\ell)^\times = \mathrm{Aut}(C_\ell)$ to the group $(\mathbb{Z}/n)^\times = \mathrm{Aut}(C_n)$. Via this group epimorphism, we regard $\mathrm{Aut}(C_n)$ as a quotient group of $\mathrm{Aut}(C_\ell)$. Consider a $\mathbb{K}$-irrep σ of $\mathrm{Aut}(C_\ell)$. We say that σ is **primitive** provided σ is not inflated from any of those quotient groups that have the form $\mathrm{Aut}(C_n)$ where n is a strict divisor of ℓ. Using the classical Chinese Remainder Theorem, it is not hard to show that there exists a divisor $\pi(\sigma)$ such that σ is inflated from $\mathrm{Aut}(C_n)$ if and only if $\pi(\sigma)$ divides n. Evidently, $\pi(\sigma)$ is the unique divisor of ℓ such that σ is inflated from a primitive $\mathbb{K}$-irrep of $\mathrm{Aut}(C_{\pi(\sigma)})$. We call $\pi(\sigma)$ the **primitivity index** of σ.

We return to the scenario of the previous section, where ℓ is assumed to be divisible by the order of every cyclic group in $\mathcal{F}$. Again, we put $\mathcal{A} = (\mathbb{Z}/\ell)^\times = \mathrm{Aut}(C_\ell)$. We shall examine $\mathrm{htw}^{\mathcal{F}}$ as a $\mathbb{C}\mathcal{A}$-module and also as a $\mathbb{C}\Upsilon^{\mathcal{F}}$-module.

Lemma 7.2. *Given a $\mathbb{C}\mathcal{A}$-irrep σ, then the following three conditions are equivalent*:

(a) *$\pi(\sigma)$ is the order of some cyclic group belonging to $\mathcal{F}$,*
(b) *σ occurs in the $\mathbb{C}\mathcal{A}$-module* $\mathrm{htw}^{\mathcal{F}}$*,*
(c) *σ occurs in the $\mathbb{C}\mathcal{A}$-module* $\mathrm{HTW}^{\mathcal{F}}$.

Proof. Let n be a divisor of ℓ. Since $\mathrm{Aut}(C_n)$ is a quotient group of $\mathcal{A}$, we can regard $\mathrm{Aut}(C_n)$ as a transitive $\mathcal{A}$-set by left translation. Let us write Ω_n to denote $\mathrm{Aut}(C_n)$ thus regarded as an $\mathcal{A}$-set. Since $\mathcal{A}$ is abelian, the permutation $\mathcal{A}$-module $\mathbb{C}\Omega_n$ is a direct sum of mutually non-isomorphic $\mathbb{C}\mathcal{A}$-irreps. Observe that σ occurs in $\mathbb{C}\Omega_n$ if and only if the kernel of σ contains the kernel of the epimorphism $\mathcal{A} \to \mathrm{Aut}(C_n)$. This is equivalent to the condition that σ has primitivity index dividing n. Therefore, $\mathbb{C}\Omega_n$ is isomorphic to the direct sum of those $\mathbb{C}\mathcal{A}$-irreps that have primitivity index dividing n.

Consider the permutation basis $\{\epsilon_K(k)\colon k \in K \in \mathcal{F}\}$ for the permutation $\mathbb{C}\mathcal{A}$-module $\mathrm{HTW}^{\mathcal{F}}$. Let $n(k)$ denote the order of k. Let $M_{K,k}$ denote the $\mathbb{C}\mathcal{A}$-submodule of $\mathrm{HTW}^{\mathcal{F}}$ generated by $\epsilon_K(k)$. Then $M_{K,k}$ is a transitive permutation $\mathbb{C}\mathcal{A}$-module with a permutation basis $\{\epsilon_K(k^a)\colon a \in \mathcal{A}\}$. Evidently, $M_{K,k} \cong \mathbb{C}\Omega_{n(k)}$. So σ occurs in $M_{K,k}$ if and only if $\pi(\sigma)$ divides $n(k)$.

Assume (c). Since $\mathrm{HTW}^{\mathcal{F}}$ is the sum of the submodules having the form $M_{K,k}$, there must be some k and K such that σ occurs in $M_{K,k}$. So $\pi(\sigma)$ divides $n(k)$. Therefore, K contains a cyclic subgroup with order $\pi(\sigma)$. Using the closure hypothesis on $\mathcal{F}$, we deduce (a). Now assume (a). Let K be a cyclic group with order $\pi(\sigma)$ and let k be a generator of K. Then σ occurs in $M_{K,k}$. But K is abelian, so $\epsilon_K(k) = \epsilon_K[k]$ and $M_{K,k} \leqslant \mathrm{htw}^{\mathcal{F}}$. We deduce (b). Trivially, (b) implies (c). $\square$

The next result makes it clear why the rhetorical biset functor $\mathrm{htw}^{\mathcal{F}}$ plays such a central role in the study of the simple rhetorical biset functors over $\mathbb{C}$.

Lemma 7.3. *Each simple rhetorical biset functor occurs exactly once as a summand of the rhetorical biset functor* $\mathrm{htw}^{\mathcal{F}}$. *Furthermore, there are bijective correspondences between*:

(a) *the $\mathbb{C}\mathcal{A}$-irreps σ that occur in the $\mathbb{C}\mathcal{A}$-module* $\mathrm{htw}^{\mathcal{F}}$,
(b) *the primitive idempotents e of $Z(\mathbb{C}\Upsilon^{\mathcal{F}})$*,
(c) *the isomorphism classes of simple rhetorical biset functors S.*

The correspondence $\sigma \leftrightarrow e$ is characterized by the condition that, regarding $e\,.\,\mathrm{htw}^{\mathcal{F}}$ as a $\mathbb{C}\mathcal{A}$-submodule of $\mathrm{htw}^{\mathcal{F}}$, then $e\,.\,\mathrm{htw}^{\mathcal{F}}$ is the sum of those $\mathbb{C}\mathcal{A}$-submodules of $\mathrm{htw}^{\mathcal{F}}$ that are isomorphic to σ. The correspondence $e \leftrightarrow S$ is characterized by the condition that, regarding $e\,.\,\mathrm{htw}^{\mathcal{F}}$ as a biset subfunctor of $\mathrm{htw}^{\mathcal{F}}$, then $e\,.\,\mathrm{htw}^{\mathcal{F}} \cong S$.

Proof. Letting e run over the primitive idempotents of the algebra $Z(\mathbb{C}\Upsilon^{\mathcal{F}}) = \theta^{\mathcal{F}}(\mathcal{A})$, then

$$\mathrm{htw}^{\mathcal{F}} = \bigoplus_{e} e\,.\,\mathrm{htw}^{\mathcal{F}}$$

both as a direct sum of $\mathbb{C}\mathcal{A}$-modules and as a direct sum of rhetorical biset functors. The correspondence $\sigma \leftrightarrow e$ is already clear.

We have $\mathbb{C}\Upsilon^{\mathcal{F}} = \bigoplus_{e} \mathbb{C}\Upsilon^{\mathcal{F}} e$ as a direct sum of full matrix algebras over $\mathbb{C}$. So $e\,.\,\mathrm{htw}^{\mathcal{F}}$ must be a direct sum of copies of the simple $\mathbb{C}\Upsilon^{\mathcal{F}} e$-module. But $\mathbb{C}\Upsilon^{\mathcal{F}} = \mathrm{End}_{\mathbb{C}\mathcal{A}}(\mathrm{htw}^{\mathcal{F}})$ and each $e\,.\,\mathrm{htw}^{\mathcal{F}}$ is a direct sum of copies of σ. Therefore $e\,.\,\mathrm{htw}^{\mathcal{F}}$ is simple as a $\mathbb{C}\Upsilon^{\mathcal{F}}$-module. The correspondence $e \leftrightarrow S$ is now clear. $\square$

In a sense, the latest lemma already classifies the simple rhetorical biset functors over $\mathbb{C}$, since it indexes them with the parameter σ. To make the classification explicit, we must determine the isomorphism class of the simple rhetorical biset functor corresponding to σ.

Lemma 7.4. *Let σ be a $\mathbb{C}\mathcal{A}$-irrep satisfying the equivalent conditions of Lemma* 7.2. *Regarding σ as a primitive $\mathbb{C}$-irrep of the group $\mathrm{Aut}(C_{\pi(\sigma)}) = \mathrm{Out}(C_{\pi(\sigma)})$, then $\mathcal{F}$ owns a copy of the cyclic group $C_{\pi(\sigma)}$, so we can form the simple biset functor $S^{\mathcal{F},\mathbb{C}}_{C_{\pi(\sigma)},\sigma}$. Let S be the simple rhetorical biset functor corresponding to σ as in Lemma* 7.3. *Then $S \cong S^{\mathcal{F},\mathbb{C}}_{C_{\pi(\sigma)},\sigma}$.*

Proof. The proof of Lemma 7.2 shows that, given a cyclic group K, then σ occurs in $\mathrm{htw}^{\mathcal{F}}(K)$ if and only of $\pi(\sigma)$ divides $|K|$. Therefore, the cyclic group $C_{\pi(\sigma)}$ is a minimal group for S_σ in the sense of Theorem 2.4. In particular, $\mathcal{F}$ owns a copy of $C_{\pi(\sigma)}$. We have $S_\sigma(C_{\pi(\sigma)}) \cong \sigma$ as $\mathbb{C}\,\mathrm{Aut}(C_{\pi(\sigma)})$-modules. Therefore $S_\sigma \cong S_{C_{\pi(\sigma)},\sigma}$. $\square$

We can further exploit the fact that the hermetic algebra $\mathbb{C}\Upsilon^{\mathcal{F}}$ has been realized, quite tangibly, as a subalgebra of a full matrix algebra. Recall that the alchemic algebra $\mathbb{C}\Gamma^{\mathcal{F}}$ is so-named because it is generated by the elements having the form $\mathrm{iso}^{\theta}_{G',G}$, $\mathrm{res}_{H,G}$, $\mathrm{tra}_{G,H}$, $\mathrm{inf}_{G,G/N}$, $\mathrm{def}_{G/N,G}$. As an abuse of notation, the images of those elements in $\mathbb{C}\Upsilon^{\mathcal{F}}$ will still be denoted by $\mathrm{iso}^{\theta}_{G',G}$, $\mathrm{res}_{H,G}$, $\mathrm{tra}_{G,H}$, $\mathrm{inf}_{G,G/N}$, $\mathrm{def}_{G/N,G}$. Since we are regarding $\mathbb{C}\Upsilon^{\mathcal{F}}$ as a subalgebra of $\mathrm{mat}^{\mathcal{F}}$, we can write those five elements of $\mathbb{C}\Upsilon^{\mathcal{F}}$ as

$$\mathrm{iso}^{\theta}_{G,G'} = {}_G\Delta(G,\theta,G')_{G'}, \qquad \mathrm{res}_{H,G} = {}_H\Delta(H,G)_G, \qquad \mathrm{tra}_{G,H} = {}_H\Delta(H,G)_G,$$
$$\mathrm{inf}_{G,G/N} = {}_G\Delta(G,G/N)_{G/N}, \qquad \mathrm{def}_{G/N,G} = {}_{G/N}\Delta(G/N,G)_G.$$

Let us describe these five matrices more explicitly.

Lemma 7.5. *Let $H \leqslant G \trianglerighteq N$, let $\theta : G \leftarrow G'$ be a group isomorphism, and suppose that H, G, G/N, G' belong to $\mathcal{F}$. Then, as a matrix belonging to the subalgebra $\mathbb{C}\Upsilon^{\mathcal{F}}$ of $\mathrm{mat}^{\mathcal{F}}$, we have*

$$\mathrm{iso}^{\theta}_{G,G'} = \sum_{g \in_G G} \epsilon_{G,G'}\big[g, \theta^{-1}(g)\big]$$

where the notation indicates that g runs over representatives of the conjugacy classes in G. The matrices $\mathrm{res}_{G,H}$ and $\mathrm{inf}_{G,G/N}$ are the transposes of the matrices

$$\mathrm{tra}_{G,H} = \sum_{h \in_H H} \sqrt{\frac{|C_G(h)|}{|C_H(h)|}}\epsilon_{G,H}[h,h], \qquad \mathrm{def}_{G/N,G} = \sum_{g \in_G G} \sqrt{\frac{|C_{G/N}(gN)|}{|C_G(g)|}}\epsilon_{G/N,G}[gN,g].$$

Proof. For $h \in H$ and $g \in G$, let $c(h,g)$ be the coefficient of $\epsilon_{H,G}(h,g)$ in the element

$$\mathrm{res}_{H,G} = {}_H\Delta(H,G)_G = \frac{1}{|H|\sqrt{|H|\,.\,|G|}}\,\mathrm{tr}_1^{H\times G}\big(\Delta(H,G)^+\big).$$

If $h \neq_G g$ then $c(h,g) = 0$. If $h =_G g$ then $c(h,g) = c(h,h)$. We have

$$|H|\sqrt{|H|\,.\,|G|}c(h,h) = \big|\big\{(x,y,z) \in H \times G \times H\colon \big({}^xz, {}^yz\big) = (h,h)\big\}\big| = |H|\,.\,\big|C_G(h)\big|.$$

So the coefficient of $\epsilon_{H,G}[h,h]$ in $\mathrm{res}_{H,G}$ is

$$\sqrt{\big|[h]_H\big|\,.\,\big|[h]_G\big|}c(h,h) = \sqrt{\big|C_G(h) : C_H(h)\big|}.$$

We have shown that

$$\mathrm{res}_{H,G} = \sum_{h \in_H H} \sqrt{\frac{|C_G(h)|}{|C_H(h)|}}\epsilon_{H,G}[h,h].$$

The same method can be used to obtain a similar formula for the inflation or the deflation element. The rest of the argument is very easy. □

Lemma 7.6. *There is an isomorphism of biset functors $\lambda : \mathbb{C}A_{\mathbb{C}} \to \mathrm{htw}^{\mathcal{F}}$ such that, letting $G \in \mathcal{F}$, and regarding an element $\chi \in \mathbb{C}A_{\mathbb{C}}(G)$ as a class function $G \to \mathbb{C}$, then*

$$\lambda_G(\chi) = \frac{1}{|G|}\sum_{g \in G}\chi(g)\epsilon_G(g) = \sum_{g \in_G G}\frac{\chi(g)}{\sqrt{|C_G(g)|}}\epsilon_G[g].$$

Proof. It is easy to check that the two expressions for $\lambda_G(\chi)$ are equal to each other. By comparing dimensions, we see that λ_G is a $\mathbb{C}$-linear isomorphism. So it suffices to check that λ commutes with transfer, inflation, isogation, deflation and restriction. Let $H \leqslant G$ and $\psi \in \mathbb{C}A_{\mathbb{C}}(H)$. Using the formula for $\mathrm{tra}_{H,G}$ in Lemma 7.5, we obtain

$$\mathrm{tra}_{G,H}\big(\lambda_H(\psi)\big) = \sum_{g \in_G G,\, h \in_H H:\ h =_G g} \frac{\sqrt{|C_G(g)|}}{|C_G(h)|} \psi(h)\epsilon_G[g] = \lambda_G\big(\mathrm{ind}_{G,H}(\psi)\big).$$

Now let $N \trianglelefteq G$. The formula for deflation is

$$\mathrm{def}_{G/N,G}(\chi)(gN) = \frac{1}{|N|} \sum_{z \in N} \chi(zg)$$

because $\mathrm{def}_{G/N,G}$ can be regarded as the projection operator associated with the idempotent $\sum_{z \in N} z/|N|$. Taking care over the indices of the sums, we find that

$$\begin{aligned}
\mathrm{def}_{G/N,G}\big(\lambda_G(\chi)\big) &= \frac{1}{|G|} \sum_{g \in G} \sqrt{\big|C_{G/N}(gN)\big|}\,\chi(g)\epsilon_G[gN] \\
&= \frac{|N|}{|G|} \sum_{gN \in G/N} \sqrt{\big|C_{G/N}(gN)\big|}\,\mathrm{def}_{G/N,G}(\chi)(g)\epsilon_G[gN] \\
&= \lambda_{G/N}\big(\mathrm{def}_{G/N,G}(\chi)\big).
\end{aligned}$$

The commutativity with inflation, isogation and restriction are similar and easier. □

The following result implies Theorems 1.5 and 1.6 in the case where $\mathcal{X} = \mathcal{F}$ and $\mathbb{K} = \mathbb{C}$.

Proposition 7.7. *As rhetorical biset functors over $\mathbb{C}$, we have*

$$\mathbb{C}A_{\mathbb{C}} \cong \mathrm{htw}^{\mathcal{F}} \cong \bigoplus_{C,\sigma} S^{\mathcal{F},\mathbb{C}}_{C,\sigma}$$

where C runs over representatives of the isomorphism classes of cyclic groups in $\mathcal{F}$ and σ runs over the $\mathbb{C}\,\mathrm{Aut}(C)$-irreps. Furthermore, the simple rhetorical biset functors for $\mathcal{F}$ over $\mathbb{C}$ are precisely those simple biset functors for $\mathcal{F}$ over $\mathbb{C}$ that appear in the direct sum.

Proof. The second isomorphism and the rider follows easily from Lemmas 7.2, 7.3, 7.4. The first isomorphism is part of Lemma 7.6. □

8. The simple rhetorical biset functors over $\mathbb{K}$

As well as dealing with Theorems 1.2, 1.5, 1.6, we shall also present some results on the dimensions of the coordinate modules of the simple rhetorical biset functors over $\mathbb{K}$.

First of all, we need a lemma that will help us to pass from the case where $\mathbb{K} = \mathbb{C}$ to the case where $\mathbb{K}$ is arbitrary. Recall that, given a Galois extension $\mathbb{L}$ of $\mathbb{K}$ and a semisimple $\mathbb{K}$-algebra Λ which extends to a semisimple $\mathbb{L}$-algebra $\mathbb{L}\Lambda$, then there is a bijective correspondence between the simple Λ-modules S and the $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$-conjugacy classes of simple $\mathbb{L}\Lambda$-modules S_1. The correspondence is characterized by the condition that the $\mathbb{L}$-linear extension of S has the form $\mathbb{L}S \cong m(S_1 \oplus \cdots \oplus S_r)$ where $S_1, \ldots, S_r$ are the Galois conjugates of S_1 and m is a positive integer. Note that, if Λ is commutative, then $m = 1$. The following lemma is easily established by considering the coordinate module at H.

Lemma 8.1. *Let $\mathbb{L}$ be a Galois extension of $\mathbb{K}$. Let $H \in \mathcal{X}$, let ν be a simple $\mathbb{K}\,\mathrm{Out}(H)$-module, and write $\nu = m(\nu_1 \oplus \cdots \oplus \nu_r)$ where $\nu_1, \ldots, \nu_r$ are the mutually $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$-conjugate simple $\mathbb{L}\,\mathrm{Out}(H)$-modules corresponding to ν. Then*

$$\mathbb{L}S^{\mathcal{X},\mathbb{K}}_{H,\nu} = m\big(S^{\mathcal{X},\mathbb{L}}_{H,\nu_1} \oplus \cdots \oplus S^{\mathcal{X},\mathbb{L}}_{H,\nu_1}\big).$$

If H abelian, then $m = 1$. Supposing that H is cyclic, then ν is primitive if and only if one of $\nu_1, \ldots, \nu_r$ is primitive. In that case, $\nu_1, \ldots, \nu_r$ are all primitive.

Let us prove Theorem 1.5. The latest lemma reduces to the case where $\mathbb{K}$ is algebraically closed. But every algebraically closed field with characteristic zero contains a copy of the algebraic closure of $\mathbb{Q}$. Therefore, if the required conclusion holds for some algebraically closed $\mathbb{K}$, then the required conclusion holds for all algebraically closed $\mathbb{K}$. So, in fact, the lemma reduces to the case where $\mathbb{K} = \mathbb{C}$. By part (3) of Remark 2.5, we may assume that $\mathcal{X}$ is finite. Thus, we have reduced to the case that has already been established in Proposition 7.7. The proof of Theorem 1.5 is complete.

Now let us give a new proof of Bouc's result, Theorem 1.6. Using the latest lemma again, we reduce to the case where $\mathbb{K} = \mathbb{C}$. Using parts (1) and (3) of Remark 2.5, we reduce to the case where $\mathcal{X}$ is finite. Again, we have reduced to the case already handled in Proposition 7.7. The proof of Theorem 1.6 is complete.

Theorem 1.2, which originates in the work of Hambleton, Taylor and Williams [9], also merits an alternative proof. The argument will apply the two theorems that we have just now established. Assume that every group in $\mathcal{X}$ is a p-group. Let F be a free module for the ring $\Upsilon^{\mathcal{X}} = \mathbb{Z}\Upsilon^{\mathcal{X}}$. Then $\mathbb{C}F$ is a free $\mathbb{C}\Upsilon^{\mathcal{X}}$-module. Furthermore, F and $\mathbb{C}F$ are locally unital, in other words, they are rhetorical biset functors. By Theorem 1.4, $\mathbb{C}F$ is semisimple, say, $\mathbb{C}F = \bigoplus_i S_i$ as a (possibly infinite) direct sum of simple $\mathbb{C}\Upsilon^{\mathcal{X}}$-modules. By Theorems 1.5 and 1.6, each S_i is a quotient of $\mathbb{C}A_{\mathbb{C}}$, so $\mathbb{C}F$ is a quotient of a direct sum of copies of $\mathbb{C}A_{\mathbb{C}}$. As a special case of Example 3.A, $\mathbb{C}A_{\mathbb{C}}$ is rational. Lemma 3.3 tells us that the rational biset functors are closed under quotients and direct sums. Therefore $\mathbb{C}F$ is rational. By part (2) of Lemma 3.4, F is rational.

Let L be a rhetorical biset functor for $\mathcal{X}$ over R. We mean to say that L is a locally unital $R\Upsilon^{\mathcal{X}}$-module. We are to show that L is rational. But L is a quotient of a free $R\Upsilon^{\mathcal{X}}$-module, and any free $R\Upsilon^{\mathcal{X}}$-module has the form RF where F is a free $\Upsilon^{\mathcal{X}}$-module. By the closure property for quotients, it suffices to show that RF is rational. But we have already shown that F is rational so, by part (2) of Lemma 3.4 again, RF is rational, as required. The proof of Theorem 1.2 is complete.

We make two comments on the latest proof. Firstly, one could invoke Lemma 3.5 to reduce to the case where $\mathcal{X}$ is finite, but this reduction would not simplify the argument. Secondly, we did not use the condition that L is locally unital. But there is no mistake here. Allowing L to be an arbitrary $R\Upsilon^{\mathcal{X}}$-module, and writing $L(G) = \mathrm{iso}_G\, L$, then what the argument shows is that the locally unital submodule $\bigoplus_G L(G)$ is a rational biset functor.

The last two results in this section concern the dimensions of the coordinate modules of the simple rhetorical biset functors in characteristic zero.

Proposition 8.2. *Let S be a simple rhetorical biset functor for $\mathcal{X}$ over $\mathbb{C}$. Let $G \in \mathcal{X}$. Let $C \in \mathcal{X}$ be of minimal order such that $S(C) \neq 0$. Note that C is cyclic. For each cyclic subgroup Z of G, let $k_Z(G)$ be the number of conjugacy classes of G which contain a generator of Z. Let $d_C(G)$ be the number of conjugacy classes of cyclic subgroups Z of G such that $|C|$ divides $k_Z(G)$. Then $\dim_{\mathbb{C}}(S(G)) = d_C(G)$. More generally, consider a simple rhetorical biset functor over $\mathbb{K}$ and write it in the form $S_{C,\sigma}$ where σ is a primitive $\mathbb{K}\,\mathrm{Aut}(C)$-irrep. Then $\dim_{\mathbb{K}}(S_{C,\sigma}) = \dim_{\mathbb{K}}(\sigma) d_C(G)$.*

Proof. Lemma 8.1 reduces to the case where $\mathbb{K}$ is algebraically closed. As before, by considering the algebraic closure of $\mathbb{Q}$, we see that there is no loss of generality in assuming that $\mathbb{K} = \mathbb{C}$. We can now write $S = S_{C,\sigma}$ where σ is a primitive $\mathbb{C}\,\mathrm{Aut}(C)$-irrep. Part (3) of Remark 2.5 allows us to assume that $\mathcal{X}$ is finite. Let $\mathcal{A}$ be as in Sections 6 and 7. By inflation, we regard σ as a $\mathbb{C}\mathcal{A}$-irrep. Our notation here is consistent with the notation in Lemma 7.3, since S is the simple rhetorical biset functor corresponding to σ. Let e be the primitive idempotent of $Z(\mathbb{C}\Upsilon^{\mathcal{X}})$ corresponding to σ and S. Then $S \cong e \,.\, \mathrm{htw}^{\mathcal{X}}$. Lemma 7.3 tells us that $\dim_{\mathbb{C}}(S(G))$ is equal to the multiplicity of σ in $\mathrm{htw}^{\mathcal{X}}(G)$.

We now adapt the proof of Lemma 7.2. Consider the permutation basis $\{\epsilon_G[g]\colon\ g \in_G G\}$ for the permutation $\mathbb{C}\mathcal{A}$-module $\mathrm{htw}^{\mathcal{X}}(G)$. Let Z be a cyclic subgroup of G, and let M_Z be the $\mathbb{C}\mathcal{A}$-submodule of $\mathrm{htw}^{\mathcal{X}}$ such that M_Z has a basis consisting of those elements $\epsilon_G[g]$ such that $Z =_G \langle g\rangle$. Then M_Z is a transitive permutation $\mathbb{C}\mathcal{A}$-module with dimension $k_Z(G)$. In the notation of the proof of Lemma 7.2, $M_Z \cong \mathbb{C}\Omega_{k_Z(G)}$. So σ occurs in M_Z if and only if $k_Z(G)$ is divisible by the integer $\pi(\sigma) = |C|$. Now letting Z run over representatives of the conjugacy classes of cyclic subgroups of G, then $\mathrm{htw}^{\mathcal{X}}(G) = \bigoplus_Z M_Z$. Therefore, the multiplicity of σ in $\mathrm{htw}^{\mathcal{X}}(G)$ is $d_C(g)$. □

Recall that, for $g \in G$, the **class sum** $[g]_G^+$ is defined to be the sum of the elements in the conjugacy class $[g]_G$. Of course, the class sums comprise a basis for the center $Z(\mathbb{C}G)$. We can regard $Z(\mathbb{C}G)$ as a permutation $\mathbb{C}\mathcal{A}$-module such that an element $a \in \mathcal{A}$ sends the class sum $[g]_G^+$ to the class sum $[g^a]_G^+$. There is a $\mathbb{C}\mathcal{A}$-module isomorphism $\mathrm{htw}(G) \cong Z(\mathbb{C}G)$ such that $\epsilon_G[g] \leftrightarrow [g]_G^+$. So the proof of the latest proposition shows that, when $\mathbb{K} = \mathbb{C}$ and σ is a primitive $\mathbb{C}\,\mathrm{Aut}(C)$-irrep, $\dim_{\mathbb{C}}(S^{\mathbb{C}}_{C,\sigma})$ is equal to the multiplicity of σ in $Z(\mathbb{C}G)$.

Proposition 8.3. *Given $I, J \in \mathcal{X}$, then*

$$\mathrm{rank}_R\big(R\Upsilon(I,J)\big) = k_*(I,J) = \sum_{C,\sigma} \dim_{\mathbb{C}}\big(S^{\mathbb{C}}_{C,\sigma}(I)\big)\dim\big(S^{\mathbb{C}}_{C,\sigma}(J)\big)$$

where C runs over the isomorphism classes of cyclic groups and σ runs over the primitive $\mathbb{C}\,\mathrm{Aut}(C)$-irreps.

Proof. The first asserted equality is part of Remark 4.3. The sum is finite because $S_{C,\sigma}(I) = 0$ unless C is isomorphic to a subquotient of I. By part (3) of Remark 2.5 yet again, we may assume that $\mathcal{X}$ is finite. Let $I \times J \times \mathcal{A}$ act on $I \times J$ as in Section 6. Two elements (i, j) and

(i', j') of $I \times J$ belong to the same $\mathcal{A}$-orbit if and only if (i, j) and (i', j') generate the same cyclic subgroup. So the number of orbits of $I \times J \times \mathcal{A}$ on $I \times J$ is $k_*(I \times J)$.

As in Section 7, we can inflate the $\mathbb{C}\,\mathrm{Aut}(C)$-irrep σ to a $\mathbb{C}\mathcal{A}$-irrep also denoted σ. The $\mathbb{C}\mathcal{A}$-irrep σ determines C up to isomorphism, indeed, C is the cyclic group with order $\pi(\sigma)$. So we can write $S_\sigma = S_{C,\sigma}$ without ambiguity. Our task is to show that the above number of orbits is equal to $\sum_\sigma \dim(S_\sigma(I))\dim(S_\sigma(J))$ where σ now runs over those $\mathbb{C}\mathcal{A}$-irreps that satisfy the equivalent conditions of Lemma 7.2. Plainly, the number of orbits is equal to the multiplicity of the trivial $\mathbb{C}$-irrep of $I \times J \times \mathcal{A}$ in $\mathbb{C}(I \times J)$. The $I \times J$-fixed subspace of $\mathbb{C}(I \times J)$ is isomorphic to $Z(\mathbb{C}I) \times Z(\mathbb{C}J)$. By the comments above,

$$Z(\mathbb{C}I) \cong \sum_\sigma \dim\bigl(S_\sigma(I)\bigr)\sigma$$

and similarly for $Z(\mathbb{C}J)$. Given $\mathbb{C}\mathcal{A}$-irreps σ and ρ, then $\sigma \otimes \rho$ is an $\mathcal{A}$-irrep. Furthermore, $\sigma \otimes \rho$ is the trivial irrep if and only if σ and ρ are mutual duals. So the multiplicity of the trivial irrep in the $\mathbb{C}\mathcal{A}$-module $Z(\mathbb{C}I) \otimes Z(\mathbb{C}J)$ is $k_*(I, J) = \sum_\sigma \dim(S_\sigma(I))\dim(S_{\sigma^*}(J))$ where σ^* is the dual of σ. The result follows because $\dim(S_{\sigma^*}(J)) = \dim(S_\sigma(J))$. $\square$

The character-theoretic aspect of the material will be brought out in the next section, but we can indicate something of the character-theoretic flavor by indulging in an example to illustrate the above two results. Let us examine the hermetic algebra $R\Upsilon^{A_5}$ for the group A_5. For $I, J \in \mathcal{X}(A_5)$, the following values of $k_*(I, J)$ can be laboriously determined using Goursat's Theorem 2.1. Thus, for instance, it can be shown that the group $D_{10} \times A_5$ has 1, 3, 1, 4, 1, 2, 1 conjugacy classes of cyclic subgroups with orders 1, 2, 3, 5, 6, 10, 15, respectively, hence $k_*(D_{10} \times A_5) = 1 + 3 + 1 + 4 + 1 + 2 + 1 = 13$. The sum of the 81 numbers in the table is $\mathrm{rank}_R(R\Upsilon^{A_5}) = 591$.

$k_*(I \times J)$		1	C_2	C_3	C_5	V_4	S_3	D_{10}	A_4	A_5	J
I	1	1	2	2	2	4	3	3	3	4	
	C_2	2	4	4	4	8	6	6	6	8	
	C_3	2	4	5	4	8	6	6	7	8	
	C_5	2	4	4	7	8	6	7	6	9	
	V_4	4	8	8	8	16	12	12	12	16	
	S_3	3	6	6	6	12	9	9	9	12	
	D_{10}	3	6	6	7	12	9	10	9	13	
	A_4	3	6	7	6	12	9	9	10	12	
	A_5	4	8	8	9	16	12	13	12	17	

Meanwhile, Theorem 1.5 tells us that there are precisely five isomorphism classes of simple rhetorical biset functors for A_5 over $\mathbb{C}$. The dimensions of their coordinate modules, shown in the next table, can quickly be determined using Proposition 8.2.

$\dim(S_{C,\sigma}^{A_5,\mathbb{C}}(I))$	1	C_2	C_3	C_5	V_4	S_3	D_{10}	A_4	A_5	I
$S_{1,1}^G$	1	2	2	2	4	3	3	3	4	
$S_{C_3,-1}^G$	0	0	1	0	0	0	0	1	0	
$S_{C_5,-1}^G$	0	0	1	0	0	0	1	0	1	
$S_{C_5,i}^G$	0	0	1	0	0	0	0	0	0	
$S_{C_5,-i}^G$	0	0	1	0	0	0	0	0	0	

A faster way of obtaining the values in the first table is to apply Proposition 8.3 to the values in the second table. For instance, $k_*(D_{10} \times A_5) = 3.4 + 0.0 + 1.1 + 0.0 + 0.0 = 13$. The dimensions of the simple modules are the sums of the numbers in the rows: 24, 2, 3, 1, 1. We recover the equality $\mathrm{rank}_R(R\varUpsilon^{A_5}) = \dim_{\mathbb{C}}(\mathbb{C}\varUpsilon^{A_5}) = 24^2 + 2^2 + 3^2 + 1^2 + 1^2 = 591$.

These observations can be turned around to yield a variant of the proof we gave for Theorem 1.4. (The author discovered the theorem by examining the above two tables, and the similar tables for the groups A_4 and S_4 and the non-abelian group with order 21.) The hardest part of the argument we gave in Section 6 was in showing that $\varXi^{\mathcal{F}}(\mathbb{C}\varUpsilon^{\mathcal{F}})$ and $\theta^{\mathcal{F}}(\mathbb{C}\mathcal{A})$ are mutual centralizers. It is comparatively straightforward just to show that $\varXi^{\mathcal{F}}$ is a unital algebra monomorphism $\mathbb{C}\varUpsilon^{\mathcal{F}} \to \mathrm{mat}^{\mathcal{F}}$ and that $\varXi^{\mathcal{F}}(\mathbb{C}\varUpsilon^{\mathcal{F}})$ and $\theta^{\mathcal{F}}(\mathbb{C}\mathcal{A})$ centralize each other. The argument can then be completed as follows. Let e_σ be the primitive idempotent of $\theta^{\mathcal{F}}(\mathbb{C}\mathcal{A})$ corresponding to σ. We first observe that the idempotents $\theta(e_\sigma)$ belong to $\varXi^{\mathcal{X}}(\mathbb{C}\varUpsilon^{\mathcal{X}}) \cap \theta^{\mathcal{X}}(\mathbb{C}\mathcal{A})$. Therefore, the subspaces $S_\sigma = e_\sigma \,.\, \mathrm{htw}^{\mathcal{X}}$ are $\mathbb{C}\varUpsilon^{\mathcal{X}}$-modules and, furthermore, $\mathrm{Hom}_{\mathbb{C}\varUpsilon^{\mathcal{X}}}(S_\sigma, S_{\sigma'}) = 0$ for $\sigma \neq \sigma'$. By Proposition 8.3,

$$\dim_{\mathbb{C}}\big(\mathbb{C}\varUpsilon^{\mathcal{X}}\big) = \sum_{I,J} k_*(I,J) = \sum_{\sigma}\bigg(\sum_{I}\dim\big(S_\sigma(I)\big)\bigg)^2 = \sum_{\sigma}\dim(S_\sigma)^2.$$

Remark 4.2 now implies that $\mathbb{C}\varUpsilon^{\mathcal{X}}$ is semisimple and that each $\mathbb{C}\varUpsilon^{\mathcal{X}}$-module S_σ is simple. The alternative variant of the proof of Theorem 1.4 is now complete. Moreover, as a bonus, the argument has also supplied us with the essential content of Lemma 7.3 and Theorem 1.5.

9. Some character theory

We introduce a character-theoretic technique for determining the multiplicities of the simple factors of a rhetorical biset functor over $\mathbb{K}$. The technique, expressed in Theorem 1.9, is a special feature of these functors, and it uses more than the mere fact that these functors are semisimple. As we shall explain, the technique does not seem to be applicable, in any very straightforward way, to Mackey functors over $\mathbb{K}$, despite the fact that such Mackey functors are semisimple. After proving Theorem 1.9, we shall use it to prove Theorem 1.8.

In an abstract sense, any finite-dimensional algebra $\varLambda$ over $\mathbb{K}$ admits a kind of character theory. Indeed, given a finite-dimensional $\varLambda$-module N, then the composition factors of N, and their multiplicities, are determined by the character $\varLambda \ni \lambda \mapsto \mathrm{tr}_N(\lambda) \in \mathbb{K}$ where $\mathrm{tr}_M(\lambda)$ is the trace of λ as an operator on N. When $\varLambda$ is semisimple, the isomorphism class of M is uniquely determined by the character of M. Of course, from the point of view of usefulness to non-specialists, the main successes of finite group representation theory have derived from applications of this character-theoretic principle to the group algebra $\mathbb{C}G$.

For details of the theory of Mackey functors, we refer to Thévenaz and Webb [10]. Let M be a finite-dimensional Mackey functor for G over $\mathbb{K}$. In other words, M is a finite-dimensional module of the Mackey algebra $\mathbb{K}\mu(G)$. Recall that $M = \bigoplus_{H \leqslant G} M(H)$, each coordinate module $M(H)$ being a $\mathbb{K}N_G(H)/H$-module, where $\overline{N}(H) = N_G(H)/H$. Also recall that, letting J run over representatives of the conjugacy classes of subgroups of G, and letting W run over the $\mathbb{K}\overline{N}(J)$-irreps, then $S_{J,W}$ runs over the isomorphism classes of simple Mackey functors for G over $\mathbb{K}$. Each $S_{J,W}$ is characterized by the condition that J is minimal subject to the condition $S_{J,W}(J) \cong W$. It can be shown that, imposing the evident $\mathbb{K}H$-module and $\mathbb{K}N_G(J)$-module structures,

$$S_{J,W}(H) \cong \bigoplus_{HgN_G(J) \subseteq J} \mathrm{ind}_{H, H \cap {}^g N_G(J)} \left({}^g \left(W^{H^g \cap N_G(J)} \right) \right).$$

Since $\mathbb{K}\mu(G)$ is semisimple, the isomorphism class of M is determined by its simple composition factors and their multiplicities. The multiplicity $m_{H,V}$ of $S_{H,V}$ in M is equal to the multiplicity of V in the $\mathbb{K}\overline{N}(H)$-module $\overline{M}(H) = M(H)/\mathcal{I}_M(H)$. Here, $\mathcal{I}_M(H) = \sum_{J<H} \mathrm{tra}_{H,J}(M(J))$ where $\mathrm{tra}_{H,J}$ denotes the transfer map $M(H) \leftarrow M(J)$. On the other hand, the isomorphism class of M is determined by the character of M. Actually, in view of the classification of the simple Mackey functors, the isomorphism class of M is determined by the characters $[M(H)] \in A_{\mathbb{K}}(\overline{N}(H))$. Given those characters, then $m_{H,V}$ can be obtained using the following recursive algorithm.

- Letting J run over representatives of the G-conjugacy classes of subgroups of G such that some G-conjugate of J is strictly contained in H, and letting W run over the $\mathbb{K}\overline{N}(J)$-irreps, first calculate each $m_{J,W}$ using this algorithm.
- Determine the $\mathbb{K}\overline{N}(H)$-module $\overline{M}(H) = M(H) - \sum_{J,W} m_{J,W} S_{J,W}(H)$, then calculate $m_{H,V}$ as the multiplicity of V in $\overline{M}(H)$.

Using Möbius inversion on the poset of G-conjugacy classes of subgroups of H, the multiplicity $m_{H,V}$ can be expressed in terms of the characters $[M(H)]$ without mentioning the multiplicities $m_{J,W}$. Perhaps these observations could lead to some form of character theory for Mackey functors in characteristic zero. The author has not pursued the matter.

The situation is quite different for rhetorical biset functors in characteristic zero. Theorem 1.9 shows how, for these functors, the multiplicities of the simple composition factors can be read off from the coordinate modules of the cyclic groups. No recursion or Möbius inversion is needed.

Let us prove Theorem 1.9. Consider a biset functor L for $\mathcal{X}$ over $\mathbb{K}$. In the notation of Theorem 1.5, and also appealing to Theorem 1.4, we can write $L \cong \bigoplus_{C,\sigma} m_{C,\sigma} S_{C,\sigma}$. Let C' be a cyclic group in $\mathcal{X}$, and let σ' be a primitive $\mathbb{K}\,\mathrm{Aut}(C')$-irrep. By once again reducing to the case where $\mathcal{X}$ is finite and $\mathbb{K} = \mathbb{C}$, and then applying Proposition 7.7, we deduce that, if $S_{C',\sigma'}(C) \neq 0$, then $|C'|$ divides $|C|$ and the $\mathbb{K}\,\mathrm{Aut}(C)$-module $S_{C',\sigma'}(C)$ is a direct sum of copies of the inflation of σ'. But, if $|C'|$ strictly divides $|C|$, then σ' is imprimitive as a $\mathbb{K}\,\mathrm{Aut}(C)$-module. Therefore σ does not occur in $S_{C',\sigma'}(C)$ unless $C' \cong C$. We conclude that $m_{C,\sigma}$ is equal to the multiplicity of σ in $L(C)$. Theorem 1.9 is now proved.

To complete this paper, all that remains is to prove Theorem 1.8. We begin by recalling a classical description of the coordinate module $\mathbb{J}A_{\mathbb{J}}(G)$. Fixing G, let n be a multiple of the exponent of G. Let $\mathbb{L}$ be a Galois extension field of $\mathbb{J}$ such that $\mathbb{L}$ owns a primitive nth root of unity ω. Note that $\mathbb{L}$ is a splitting field for G. Let $\mathcal{G}$ denote the image of the group homomorphism

$\mathrm{Gal}(\mathbb{L}/\mathbb{J}) \ni \theta \mapsto a \in (\mathbb{Z}/n)^\times$ where $\theta(\omega) = \omega^a$. We allow $G \times \mathcal{G}$ to act on G such that an element $(u, a) \in G \times \mathcal{G}$ sends an element $g \in G$ to the element ${}^u(g^a) = ({}^u g)^a$. The orbits of $G \times \mathcal{G}$ are called the **$\mathbb{J}G$-conjugacy classes** of G. A well-known theorem of Berman and Witt asserts that, identifying the isomorphism class of a $\mathbb{J}G$-module with its character $G \to \mathbb{J}$, then $\mathbb{J}A_{\mathbb{J}}(G)$ is the $\mathbb{J}$-vector space of functions $G \to \mathbb{J}$ that are constant on each $\mathbb{J}G$-conjugacy class. See Curtis and Reiner [8, 21.3, 21.5].

We shall be using that result only in a special case, as follows. Let C be a finite cyclic group. We put $G = C$ and $n = |C|$. Then the subgroup $\mathcal{G} \leqslant (\mathbb{Z}/n)^\times$ depends only on $\mathbb{J}$ and C, not on $\mathbb{L}$. We shall consider two different values of $\mathbb{L}$. Taking $\mathbb{L}$ to be the algebraic closure $\bar{\mathbb{J}}$ of $\mathbb{J}$, we see that $\mathcal{G}$ is the group consisting of those units a in $\mathbb{Z}/n$ such that $\theta(\omega) = \omega^a$ for some $\theta \in \mathrm{Gal}(\bar{\mathbb{J}}/\mathbb{J})$. On the other hand, if we take $\mathbb{L}$ to be the field $\mathbb{J}_C = \mathbb{J}[\omega]$, then the group homomorphism $\mathrm{Gal}(\mathbb{J}_C/\mathbb{J}) \to (\mathbb{Z}/n)^\times$ is injective because the elements of $\mathrm{Gal}(\mathbb{J}_C/\mathbb{J})$ are determined by their values on ω. Via this injective group homomorphism, we can identify the domain $\mathrm{Gal}(\mathbb{J}_{\mathbb{C}}/\mathbb{J})$ with the image $\mathcal{G}$. A further identification can be made. Much as we did at the beginning of Section 6, we identify $\mathrm{Aut}(C)$ with $(\mathbb{Z}/n)^\times$ in such a way that each group automorphism α is identified with the unit a of $\mathbb{Z}/n$ such that $\alpha(c) = c^a$ for $c \in C$. To summarize these identifications,

$$\mathcal{G} = \mathrm{Gal}(\mathbb{J}_C) \leqslant \mathrm{Aut}(C) = (\mathbb{Z}/n)^\times.$$

Since C is abelian, the above theorem of Berman and Witt tells us that $\mathbb{J}A_{\mathbb{J}}(C)$ is the $\mathbb{J}$-vector space of functions $G \to \mathbb{J}$ that are constant on each $\mathcal{G}$-orbit of G. Perforce, $\bar{\mathbb{J}}A_{\mathbb{J}}(C)$ is the $\bar{\mathbb{J}}$-vector space of functions $G \to \bar{\mathbb{J}}$ that are constant on each $\mathcal{G}$-orbit.

Thanks to Lemma 8.1, it suffices to demonstrate Theorem 1.8 in the case $\mathbb{K} = \bar{\mathbb{J}}$. By embedding $\overline{\mathbb{Q}}$ in $\bar{\mathbb{J}}$ and in $\mathbb{C}$, we obtain isomorphisms of biset functors $A_{\bar{\mathbb{J}}} \cong A_{\overline{\mathbb{Q}}} \cong A_{\mathbb{C}}$, so Theorem 1.6 yields

$$\bar{\mathbb{J}}A_{\bar{\mathbb{J}}} \cong \bigoplus_{C} \bigoplus_{\sigma \in P(C)} S_{C,\sigma}^{\bar{\mathbb{J}}}$$

where C runs over representatives of the isomorphism classes of cyclic groups in $\mathcal{X}$ and $P(C)$ is the set of primitive $\bar{\mathbb{J}}\,\mathrm{Aut}(C)$-irreps. Since $A_{\mathbb{J}}$ is a biset subfunctor of $A_{\bar{\mathbb{J}}}$, there must be a subset $P(C, \mathbb{J}) \subseteq P(C)$ such that

$$\bar{\mathbb{J}}A_{\mathbb{J}} \cong \bigoplus_{C} \bigoplus_{\sigma \in P(C,\mathbb{J})} S_{C,\sigma}^{\bar{\mathbb{J}}}.$$

Fix C, let $\mathcal{G}$ be as above, and consider an element $\sigma \in P(C)$. We shall be finished when we have shown that $\sigma \in P(C, \mathbb{J})$ if and only if $\mathcal{G} \leqslant \mathrm{Ker}(\sigma)$. By Theorem 1.9, $\sigma \in P(C, \mathbb{J})$ if and only if σ occurs in the $\bar{\mathbb{J}}\,\mathrm{Aut}(C)$-module $\bar{\mathbb{J}}A_{\mathbb{J}}(C)$. Above, we noted that $\bar{\mathbb{J}}A_{\mathbb{J}}(C)$ is the $\mathbb{L}$-vector space of functions $G \to \mathbb{L}$ that are constant on each $\mathcal{G}$-orbit. So σ occurs in $\bar{\mathbb{J}}A_{\mathbb{J}}(C)$ if and only if σ is constant on each $\mathcal{G}$-orbit. This is equivalent to the condition that $\mathcal{G} \leqslant \mathrm{Ker}(\sigma)$. We have shown that $\sigma \in P(C, \mathbb{J})$ if and only if $\mathcal{G} \leqslant \mathrm{Ker}(\sigma)$, as required.

References

[1] L. Barker, Genotypes of irreducible representations of finite p-groups, J. Algebra 306 (2006) 655–681.

[2] S. Bouc, Foncteurs d'ensembles munis d'une double action, J. Algebra 183 (1996) 664–736.
[3] S. Bouc, The functor of rational representations for p-groups, Adv. Math. 186 (2004) 267–306.
[4] S. Bouc, Biset functors and genetic sections for p-groups, J. Algebra 284 (2005) 179–202.
[5] S. Bouc, The Dade group of a p-group, Invent. Math. 164 (2006) 189–231.
[6] S. Bouc, The functor of units of Burnside rings of p-groups, Comment. Math. Helv. 82 (2007) 583–615.
[7] S. Bouc, Rational p-bist functors, J. Algebra 319 (2008) 1776–1800.
[8] C.W. Curtis, I. Reiner, Methods of Representation Theory, vol. I, Wiley, New York, 1981, reprint 1990.
[9] I. Hambleton, L. Taylor, B. Williams, Detection theorems for K-theory and L-theory, J. Pure Appl. Algebra 63 (1990) 247–299.
[10] J. Thévenaz, P. Webb, The structure of Mackey functors, Trans. Amer. Math. Soc. 347 (6) (1995) 1865–1961.
[11] J. Tornehave, The unit theorem for the Burnside ring of a 2-group, Aarhaus Universitet, Preprint Series 1983/84, No. 41, May 1984.
[12] E. Yalçın, An induction theorem for the unit groups of Burnside rings of 2-groups, J. Algebra 289 (2005) 105–127.
[13] T. Yoshida, On the unit groups of Burnside rings, J. Math. Soc. Japan 42 (1990) 31–64.