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Quantifying the value of buyer–vendor coordination: Analytical and numerical results under different replenishment cost structures ☆

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Abstract

Despite a growing interest in channel coordination, no detailed analytical or numerical results measuring its impact on system performance have been reported in the literature. Hence, this paper aims to develop *analytical* and *numerical* results documenting the system-wide cost improvement rates that are due to coordination. To this end, we revisit the classical buyer–vendor coordination problem introduced by Goyal [S.K. Goyal, An integrated inventory model for a single-supplier single-customer problem. International Journal of Production Research 15 (1976) 107–111] and extended by Toptal et al. [A. Toptal, S. Çetinkaya, C.-Y. Lee, The buyer–vendor coordination problem: modeling inbound and outbound cargo capacity and costs, IIE Transactions on Logistics and Scheduling 35 (2003) 987–1002] to consider generalized replenishment costs under centralized decision making. We analyze (i) how the counterpart centralized and decentralized solutions differ from each other, (ii) under what circumstances their implications are similar, and (iii) the effect of generalized replenishment costs on the system-wide cost improvement rates that are due to coordination. First, considering Goyal's basic setting, we show that the improvement rate depends on cost parameters. We characterize this dependency *analytically*, develop analytical bounds on the improvement rate, and identify the problem instances in which considerable savings can be achieved through coordination. Next, we analyze Toptal et al.'s [A. Toptal, S. Çetinkaya, C.-Y. Lee, The buyer–vendor coordination problem: modeling inbound and outbound cargo capacity and costs, IIE Transactions on Logistics and Scheduling 35 (2003) 987–1002] extended setting that considers generalized replenishment costs representing inbound and outbound transportation considerations, and we present detailed numerical results quantifying the value of coordination. We report significant improvement rates *with* and *without* explicit transportation considerations, and we present numerical evidence which suggests that larger rates are more likely in the former case.
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1. Introduction and related literature

The buyer–vendor coordination problem is one of the classical research areas in the multi-echelon inventory literature. A fundamental stream of research in this area, known as *centralized modeling*, recommends integrating and solving the decision problems of the buyer and the vendor together, e.g., [2,4–6,8]. Although this approach provides the best result in terms of total system-wide profit/cost, it may not be feasible or desirable in many practical cases due to incentive conflicts. The alternative approach, known as *decentralized modeling*, suggests that the buyer and the vendor solve their decision problems independently of each other. However, the total system profits resulting from the centralized approach are superior to those resulting from the corresponding decentralized approach.

In other words, decentralized models often result in lost profits for the system when compared to centralized models. As a remedy, another line of research in the literature proposes an alternative approach that relies on using the profit/cost gap between the centralized and decentralized approaches as an inducement to improve decentralized solutions, e.g., [9,10,12]. This alternative approach, known as *channel coordination*, requires the decentralized solution to be improved in a way that (i) it results in the same values for the decision variables as the centralized solution, and (ii) it suggests a mutually agreeable way of sharing the resulting profits. The sharing can be done by means of quantity discounts, rebates, refunds, and fixed payments between the parties, or some combination of these. All of these methods represent different forms of incentive schemes, or so-called *coordination mechanisms*, whose terms can be made explicit under a contract. Consequently, the output of channel coordination, i.e., the *coordinated solution*, combines the benefits of both centralized and decentralized solutions.

Despite a growing interest in channel coordination over the past few decades [1,4,9,10,15,12,14], no detailed analytical or numerical results measuring its impact on system performance have been reported in the literature. For this reason, we revisit the classical buyer–vendor coordination problem introduced by Goyal [4] (called Goyal's Problem from now on) and extended by Toptal et al. [13]. Goyal's basic setting assumes that both the buyer and the vendor operate under the assumptions of the deterministic constant demand EOQ model with the traditional inventory holding and fixed replenishment costs. Toptal et al. [13] take a broader view of this setting to consider generalized replenishment cost structures representing inbound and outbound transportation considerations. More specifically, Toptal et al. [13] first consider the case where the *vendor's replenishment cost* includes a stepwise inbound transportation cost component, representing the cargo cost (called Problem I from now on). They then extend the problem setting to consider the case where *both the vendor and the buyer* are subject to stepwise transportation costs (called Problem II from now on). Clearly, Goyal's Problem is a special case of Problems I and II, and the current paper is aimed at providing *analytical* and *numerical* results documenting the system-wide cost improvement rates that are due to coordination in all of these three problem settings . Since Toptal et al. [13] focus on centralized models *only* and Goyal [4] does not investigate channel coordination mechanisms, here we investigate the counterpart decentralized models, develop effective channel coordination mechanisms, and quantify the value of channel coordination through a comparison of the counterpart centralized and decentralized solutions of Problems I and II as well as Goyal's Problem.

Making an analytical comparison of the centralized and decentralized solutions for Goyal's Problem for certain parameter ranges, we are able to develop analytical results[1] representing the improvement rates resulting from channel coordination. These analytical results are useful in characterizing the relationship between the improvement rates and the underlying model parameters that have a direct impact on the magnitude of these improvements. Our analytical results reveal two important insights. First, the value of coordination depends on two important ratios that can be expressed in terms of the critical cost parameters. Secondly, the value of coordination does not depend on the demand rate, i.e., the demand rate is not a critical model parameter for our purposes. Furthermore, by developing bounds on the improvement rates, we identify the problem instances for which considerable savings can be achieved through coordination. However, unlike Goyal's Problem, insightful analytical results, representing the improvement rates due to channel coordination, cannot be obtained for Problems I and II, i.e., under generalized replenishment costs. Hence, in these cases, we rely on a detailed numerical study for quantifying the value of coordination.

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[1] See Corollary 1 and Proposition 5.

In summary, by analyzing (i) how the centralized and decentralized solutions differ from each other, and (ii) under what circumstances their implications are similar, we quantify the value of coordination both *analytically* and *numerically*, *with* and *without* explicit transportation considerations. We report that the maximum achievable improvement rates under coordination are greater under explicit transportation considerations, i.e., under generalized replenishment costs, and we document that our analytical results for Goyal's Problem, i.e., the case without generalized replenishment costs, prove to be useful for a careful numerical investigation.

The remainder of the paper is organized as follows. The general problem setting is discussed next in Section 2 where a summary of the notation is also presented. Section 3 revisits Goyal's Problem and provides an in-depth analysis in the context of quantifying the value of coordination. Section 4 concentrates on the extended setting with generalized replenishment cost structures, and develops specific results for quantifying the value of coordination. Section 5 presents our numerical results and a summary of their interpretation and implications. Section 6 concludes the paper.

2. General problem setting and notation

We use the index "w" to represent the parameters and decision variables of the vendor (warehouse) and "r" to represent the parameters and decision variables of the buyer (retailer). The buyer faces a constant demand rate, denoted by D, over an infinite horizon; and, given the costs of inventory replenishment and holding for both parties, the problem is to compute the minimum cost replenishment order quantities for the vendor and the buyer so that the demand can be satisfied. The vendor's and the buyer's replenishment order quantities are denoted by Q_w and Q_r, respectively. In this context, Q_w represents the size of an inbound shipment for the buyer–vendor pair whereas Q_r represents the size of an outbound shipment. The buyer's replenishment cycle length, denoted T_r, is given by Q_r/D. The vendor's replenishment cycle length, denoted T_w, is given by $T_w = nT_r$ where n is a positive integer denoting the number of buyer replenishments within a replenishment cycle of the vendor. It follows that $Q_w = nQ_r$. Notation associated with the cost parameters is introduced in Table 1 which also includes a summary of the notation introduced so far and that will be used throughout the rest of the paper.

Since the focus of the paper is on different replenishment cost structures, we denote the replenishment cost of party j, where $j = \text{w, r}$, by $\mathscr{C}_j(Q_j)$ which, naturally, is a function of Q_j, the order quantity of party j. In general terms, this function can be represented by

$$\mathscr{C}_j(Q_j) = K_j + \left\lceil \frac{Q_j}{P_j} \right\rceil R_j, \tag{1}$$

Table 1
Notation

i	Index referring to the modeling approach. $i = \text{d}$: decentralized, $i = \text{c}$: centralized
j	Index referring to the parties in the system. $j = \text{w}$: vendor, $j = \text{r}$: retailer
n	Number of buyer replenishments within a vendor replenishment cycle ($T_w = nT_r$, and thus $Q_r = Q_w/n$)
n_i^*	Optimum value of n using Modeling Approach i
Q_i^*	Buyer's optimum order quantity using Modeling Approach i
$G_w(Q_r, n)$	Vendor's average annual cost function
$G_r(Q_r)$	Buyer's average annual cost function
$G(Q_r, n)$	System-wide cost function $G(Q_r, n) = G_w(Q_r, n) + G_r(Q_r)$
$\mathscr{C}_j(Q_j)$	Replenishment cost function of party j as a function of Q_j
K_j	Fixed replenishment cost of party j
R_j	Per cargo/truck cost of party j
P_j	Per cargo/truck capacity of party j
h_j	Holding cost per-unit per-unit-time of party j
h'	Echelon holding cost ($h' = h_r - h_w > 0$)
Q_j	Order quantity of party j
T_j	Replenishment cycle length of party j
D	Buyer's/retailer's demand rate
c	Vendor's unit price without channel coordination

where K_j, R_j, and P_j denote the fixed replenishment cost, per cargo/truck cost, and per cargo/truck capacity of party j, respectively. Hence, the three settings of interest in this paper, i.e., Goyal's Problem and Problems I and II, can be represented by setting the parameters of functions $\mathscr{C}_j(Q_j)$, $j = \mathrm{w, r}$, as follows:

- In *Goyal's Problem*, both the buyer's and vendor's cargo costs are ignored, i.e., $R_j = 0$, $j = \mathrm{w,r}$, or, equivalently, $P_j \to \infty$, $j = \mathrm{w,r}$, so that each party incurs only a fixed cost given by $K_j + R_j$, $j = \mathrm{w,r}$.
- In *Problem I*, the buyer's cargo costs are ignored whereas the vendor's cargo costs are modeled explicitly, i.e., $R_\mathrm{r} = 0$ (or, equivalently, $P_\mathrm{r} \to \infty$ so that the buyer incurs only a fixed cost given by $K_r + R_r$) whereas $R_\mathrm{w} = R > 0$ and $P_\mathrm{w} = P < \infty$.
- In *Problem II*, both the buyer's and vendor's cargo costs are modeled explicitly under the assumption that the per cargo costs and capacities of the individual parties are identical, i.e., $P_j = P < \infty$ and $R_j = R > 0$, $j = \mathrm{w,r}$.

Recalling that $Q_\mathrm{w} = nQ_\mathrm{r}$, it is easy to show that the vendor's, buyer's, and system-wide average annual cost functions can be expressed as

$$G_\mathrm{w}(Q_\mathrm{r}, n) = \mathscr{C}_\mathrm{w}(nQ_\mathrm{r})\frac{D}{nQ_\mathrm{r}} + h_\mathrm{w}\frac{(n-1)Q_\mathrm{r}}{2}, \tag{2}$$

$$G_\mathrm{r}(Q_\mathrm{r}) = \mathscr{C}_\mathrm{r}(Q_\mathrm{r})\frac{D}{Q_\mathrm{r}} + h_\mathrm{r}\frac{Q_\mathrm{r}}{2}, \quad \text{and} \tag{3}$$

$$G(Q_\mathrm{r}, n) = \mathscr{C}_\mathrm{w}(nQ_\mathrm{r})\frac{D}{nQ_\mathrm{r}} + h_\mathrm{w}\frac{(n-1)Q_\mathrm{r}}{2} + \mathscr{C}_\mathrm{r}(Q_\mathrm{r})\frac{D}{Q_\mathrm{r}} + h_\mathrm{r}\frac{Q_\mathrm{r}}{2}, \tag{4}$$

respectively. In the tradition of the classical channel coordination papers, e.g., [10,15], for our decentralized models, we focus on the case where the buyer's economic order quantity problem, i.e., the buyer's subproblem, is solved first. The formulations of the corresponding decentralized and centralized models are given in Table 2 where Q_d^* is the optimal solution of the Buyer's Subproblem, as defined in Table 1.

As we have already mentioned, our analysis builds on an investigation of Goyal's Problem which is discussed in detail below. Before concluding this section, we define

$$\mathscr{IR} = \left(\frac{\text{Total decentralized costs} - \text{Total centralized costs}}{\text{Total decentralized costs}}\right) \times 100\%, \tag{5}$$

so that $\mathscr{IR}$ represents the improvement rate resulting from channel coordination, and, hence, we use it for quantifying the value of coordination for the problems considered in this paper.

3. Analysis of Goyal's problem: $R_\mathrm{w} = R_\mathrm{r} = 0$

In this case, Expressions (2)–(4) reduce to

$$G_\mathrm{r}(Q_\mathrm{r}) = \frac{K_\mathrm{r}D}{Q_\mathrm{r}} + \frac{h_\mathrm{r}Q_\mathrm{r}}{2} \quad \text{and} \tag{6}$$

Table 2
Decentralized and centralized formulations

Decentralized model		Centralized model
Buyer's subproblem	Vendor's subproblem	
min $G_\mathrm{r}(Q_\mathrm{r})$	min $G_\mathrm{w}(Q_\mathrm{d}^*, n)$	min $G(Q_\mathrm{r}, n)$
s.t. $Q_\mathrm{r} \geqslant 0$	s.t. n: a positive integer	s.t. $Q_\mathrm{r} \geqslant 0$
		n: a positive integer

$$G_w(Q_r, n) = \frac{K_w D}{n Q_r} + \frac{h_w(n-1)Q_r}{2}, \quad \text{and} \tag{7}$$

$$G(Q_r, n) = \frac{(K_w + nK_r)D}{n Q_r} + \frac{(nh_w + h')Q_r}{2} \tag{8}$$

and Goyal [4] provides the following solutions for the corresponding centralized and decentralized formulations:

Solution to the Decentralized model

$$n_d^*(n_d^* - 1) \leqslant \frac{K_w h_r}{K_r h_w} \leqslant n_d^*(n_d^* + 1), \tag{9}$$

$$Q_d^* = \sqrt{\frac{2K_r D}{h_r}}. \tag{10}$$

Solution to the Centralized model

$$n_c^*(n_c^* - 1) \leqslant \frac{K_w h'}{K_r h_w} \leqslant n_c^*(n_c^* + 1). \tag{11}$$

$$Q_c^* = \sqrt{\frac{2D(K_r + K_w/n_c^*)}{n_c^* h_w + h'}}. \tag{12}$$

Using these earlier results, we first discuss the benefits due to coordination in this simplistic setting. We assume that $cD - G_w(Q_d^*, n_d^*) > 0$ so that the decentralized transactional setting makes economical sense for the vendor, and the vendor seeks to improve his/her profit using the centralized transactional setting as a benchmark.

Utilizing Expressions 9,11,10,12, the following two propositions compare the optimum values of *n* and Q_r in the decentralized and centralized solutions of Goyal's Problem. This comparison is important in developing the analytical results for $\mathscr{IR}$ given by Expression (5) in which the centralized transactional setting is used as a benchmark.

Proposition 1. *The buyer's optimum order quantity in the decentralized solution of Goyal's Problem is less than the buyer's optimum order quantity in the counterpart centralized solution, i.e., $Q_c^* > Q_d^*$.*

Proof. All proofs are presented in the Appendix. □

Proposition 1 implies that when cargo cost and capacity are ignored, the vendor should always encourage the buyer to order more to coordinate the channel.

Proposition 2. *The optimum value of n in the decentralized solution of Goyal's Problem is greater than, or equal to, the optimum value of n in the counterpart centralized solution, i.e., $n_d^* \geqslant n_c^*$.*

Proposition 2 indicates that when cargo cost and capacity are ignored, the decentralized solution results in more frequent dispatches to the buyer during the vendor's replenishment cycle than does the centralized solution.

The results presented in Propositions 1 and 2 can be interpreted as follows. The buyer prefers smaller, and, hence, more frequent replenishments in the decentralized setting, probably because inventory holding at the buyer is costly, i.e., $h_r > h_w$. Examining Expression (8) and using its similarity to the average annual cost function under the classical EOQ model, we can interpret $(K_w/n) + K_r$ and $nh_w + h'$ as the "setup" and "per unit per unit time holding" costs of the centralized decision maker, respectively. The centralized decision maker prefers less frequent buyer replenishments, i.e., $n_c^* \leqslant n_d^*$, and, hence, we have $(K_w/n_c^*) + K_r \geqslant (K_w/n_d^*) + K_r$ and $n_c^* h_w + h' \leqslant n_d^* h_w + h'$. This implies that the "setup" cost is larger whereas the "holding" cost is smaller for the centralized decision maker so that a larger order quantity is preferable under n_c^*. That is, the discrepancy between the preferable order frequencies of the centralized decision maker and the buyer, and the impact of this discrepancy on the "setup" and holding" costs of the centralized decision maker lead to $Q_c^* > Q_d^*$.

As we show later in the paper, Propositions 1 and 2 do not necessarily hold for Problems I and II. More specifically, when the stepwise transportation costs are considered explicitly, it may be more cost effective to replenish the buyer in full cargoes so that Q_c^* is an integer multiple of P whereas Q_d^* is not so that $Q_c^* < Q_d^*$, i.e., a full-truck-load (FTL) shipment may be preferable to a larger order quantity that constitutes a less-than-truck-load (LTL) shipment. This is simply due to the fact that the underlying cost functions are discontinuous under the general replenishment cost functions.

Based on Proposition 1, the channel coordination mechanism, i.e., the coordinated solution, outlined below in Proposition 3 builds on the idea of wholesale price discounts that discourage the buyer from ordering small quantities. More specifically, under this coordinated solution, the buyer is motivated to order the centralized order quantity Q_c^* without exceeding the cost of his/her decentralized solution.

Proposition 3. *Considering Goyal's Problem, let*

$$\Delta = \frac{G_r(Q_c^*) - G_r(Q_d^*)}{D}.$$

Under a unit discount of Δ offered by the vendor for order sizes greater than or equal to Q_c^*, ordering Q_c^* units minimizes the buyer's average annual cost. Under this new pricing scheme, the buyer's average annual cost does not exceed $G_r(Q_d^*)$, the vendor's average annual profit is improved relative to the decentralized setting, and $\Delta < c$.

We note that the efficiency of similar coordination mechanisms has been investigated in the literature for Goyal's Problem, e.g., see [9,11], and its variants, e.g., the case where the inventory holding costs are ignored [10], the case where the vendor's production rate is finite [1], the case of price sensitive demand [15], and the case where information asymmetry considerations are taken into account [3]. With the exception of the results in [9], the previous work concentrates on *lot-for-lot replenishment policies* for the buyer–vendor pair, i.e., the case where $n = 1$, ignoring the impact of the vendor's replenishment decisions on coordination, whereas, here, we consider n as a decision variable. As we demonstrate in the following development, this consideration is particularly important for an analytical quantification of the value of coordination. We also note that the coordination mechanism in Proposition 3 is presented here for the sake of completeness, i.e., for comparing the coordination issues in Goyal's Problem with those in Problems I and II. More specifically, as we show later in the paper, Proposition 1 does not hold for Problems I and II for which some nontraditional observations are reported in Section 4. As a result, when cargo cost and capacity are considered explicitly, in some cases smaller orders from the buyer are more desirable for the vendor, and, unlike under the mechanism in Proposition 3, we need to discourage the buyer from ordering more.

Next, utilizing the results about the decentralized and coordinated solutions for Goyal's Problem, we provide an in-depth analysis of our main focus: the improvement rate due to coordination. Recalling Expression (5), we have

$$\mathscr{IR} = \left(1 - \frac{G_r(Q_c^*) + G_w(Q_c^*, n_c^*)}{G_r(Q_d^*) + G_w(Q_d^*, n_d^*)}\right) \times 100\%. \tag{13}$$

We begin our analysis with Proposition 4 which provides an analytical expression of $\mathscr{IR}$ in terms of the critical model parameters and optimal n values under the decentralized and centralized solutions of Goyal's Problem. In Corollary 1, we present a simplified closed form expression of $\mathscr{IR}$ over a certain parameter range that can be characterized analytically.

Proposition 4. *For Goyal's Problem, the improvement rate due to coordination is given by*

$$\mathscr{IR} = \left(1 - \frac{2\sqrt{\left(1 + \frac{1}{n_c^*}\frac{K_w}{K_r}\right)\left((n_c^* - 1)\frac{h_w}{h_r} + 1\right)}}{\left(2 + \frac{1}{n_d^*}\frac{K_w}{K_r} + (n_d^* - 1)\frac{h_w}{h_r}\right)}\right) \times 100\%. \tag{14}$$

Corollary 1. *For Goyal's Problem,*

$$\text{if} \quad 0 \leqslant \frac{K_w h_r}{K_r h_w} \leqslant 2 \quad \text{then} \quad \mathscr{IR} = \left(1 - \frac{2\sqrt{1 + \frac{K_w}{K_r}}}{2 + \frac{K_w}{K_r}} \right) \times 100\%.$$

It is important to note that under the conditions of Corollary 1, the lot-for-lot policy is, in fact, optimal under both decentralized and centralized control, i.e., $n_d^* = n_c^* = 1$ (see the proof of Corollary 1 in the Appendix.) Hence, for those problem instances where the lot-for-lot policy is optimal under both decentralized and centralized control, the resulting *IR* value does not depend on the holding costs, h_w and h_r, or the demand rate D. In fact, *for all parameter settings*, we can easily prove that *IR* depends only on the ratios K_w/K_r and h_r/h_w and that it does not depend on D because all of the demand has to be satisfied. The following lemma provides a foundation for this proof.

Lemma 1. *For a given $(\frac{K_w}{K_r}, \frac{h_w}{h_r})$ pair, the corresponding $\left(\frac{K_w h_r}{K_r h_w}, \frac{K_w h'}{K_r h_w} \right)$ pair is unique and can be obtained using the transformation*

$$f[(x,y)] = (f_1[(x,y)], f_2[(x,y)]), \quad where$$
$$f_1[(x,y)] = x/y, \quad and$$
$$f_2[(x,y)] = (x/y) - x.$$

Similarly, for a given $\left(\frac{K_w h_r}{K_r h_w}, \frac{K_w h'}{K_r h_w} \right)$ pair, the corresponding $\left(\frac{K_w}{K_r}, \frac{h_w}{h_r} \right)$ pair is unique and can be obtained using the transformation

$$g[(x,y)] = (g_1[(x,y)], g_2[(x,y)]), \quad where$$
$$g_1[(x,y)] = x - y, \quad and$$
$$g_2[(x,y)] = (x - y)/x.$$

The above lemma implies that knowing the ratios K_w/K_r and h_r/h_w is sufficient for calculating the corresponding unique values of $\frac{K_w h_r}{K_r h_w}$ and $\frac{K_w h'}{K_r h_w}$, and vice versa. Recalling Inequalities (9) and (11), we know that the optimum n value under the decentralized and centralized models of Goyal's Problem are specified by $\frac{K_w h_r}{K_r h_w}$ and $\frac{K_w h'}{K_r h_w}$ values. Therefore, for both models, vendors of two different systems having the same K_w/K_r and h_r/h_w ratios send an equal number of buyer replenishments during one replenishment cycle. Hence, we have the following corollary.

Corollary 2. *Under the assumptions of Goyal's model, the improvement rates in different systems with the same K_w/K_r and h_r/h_w ratios are equal.*

Using the formal results we have developed so far, we proceed to provide numerical lower and upper bounds on the improvement rate.

Proposition 5. *For Goyal's problem,*

- *If $0 < \frac{K_w h_r}{K_r h_w} \leqslant 2$, then $0 < \mathscr{IR} < \left(1 - \frac{\sqrt{3}}{2} \right) \times 100\%$. Furthermore, if $K_w/K_r > 1$, then $\left(1 - \frac{2\sqrt{2}}{3} \right) \times 100\% < \mathscr{IR} < \left(1 - \frac{\sqrt{3}}{2} \right) \times 100\%$.*

- *If $\frac{K_w h_r}{K_r h_w} > 2$ and $\frac{K_w h'}{K_r h_w} \geqslant 2$, then $0 < \mathscr{IR} \leqslant (\frac{1}{3}) \times 100\%$. Furthermore, if $K_w/K_r > 1$, then $0 < \mathscr{IR} < \left(1 - \frac{2\sqrt{3}}{5} \right) \times 100\%$.*

- *When $\frac{K_w h_r}{K_r h_w} > 2$ and $\frac{K_w h'}{K_r h_w} < 2$, the value of coordination can be very high such that the improvement rate $\mathscr{IR}$ is almost 100%.*

Proposition 5 provides important practical results characterizing the improvement rate $\mathscr{IR}$. That is, by simply computing the $\frac{K_w h_r}{K_r h_w}$ and $\frac{K_w h'}{K_r h_w}$ ratios, we can obtain immediate numerical bounds quantifying the value of coordination without computing the corresponding decentralized, centralized, and coordinated solutions.

These bounds can be used by managers as practical guidelines for preliminary analysis. For a given problem instance, if the condition of Corollary 1 is satisfied, then the *exact* value of $\mathscr{IR}$ can also be computed without computing the corresponding decentralized, centralized, and coordinated solutions. For other problem instances, e.g., where the condition of the second or the third item in Proposition 5 is satisfied, then the *exact* value of coordination should be computed numerically. Hence, we report detailed numerical results later in Section 5. Next, in Section 4, we extend the problem setting by analyzing Problems I and II.

4. Analysis of Problems I and II: Generalized replenishment cost problems

As we have noted earlier, by studying the decentralized models for Problems I and II and developing effective channel coordination mechanisms, we extend [13] where the counterpart centralized solutions for these two problems were first developed. According to the results in [13], obtaining the centralized solutions for Problems I and II is a challenging task. As we show in this section, a comparison of the centralized and decentralized solutions for Problem I and II reveals important analytical properties of the coordinated solutions and these properties offer new insights.

Before proceeding with a detailed analysis, we examine the properties of a specific function denoted by $\psi(n)$ and given by

$$\psi(n) = \frac{KD}{nQ} + \frac{\lceil nQ/P \rceil RD}{nQ} + \frac{h(n-1)Q}{2}. \tag{15}$$

Observe that $\psi(n)$ is a piecewise function, and, in turn, it is not differentiable. Computing the minimizer of this function for fixed and positive values of K, R, P, h and Q is important for our purposes. Let

$$n_{\min} = \max\left(1, \left\lfloor \frac{B - \sqrt{B^2 - 2KDh}}{hQ} \right\rfloor\right), \quad \text{and}$$

$$n_{\max} = \left\lceil \frac{B + \sqrt{B^2 - 2KDh}}{hQ} \right\rceil, \quad \text{where}$$

$$B = \frac{(K+R)D}{Q} + \frac{hQ}{2}.$$

Also, let n^* denote the minimizer of $\psi(n)$. The following proposition presents lower and upper bounds for n^*.

Proposition 6. $n_{\min} \leqslant n^* \leqslant n_{\max}$.

Next, recalling the formulations in Table 2 and using Proposition 6, we present the decentralized and coordinated solutions for Problems I and II.

4.1. Decentralized and coordinated solutions for Problem I

In Problem I, the buyer's individual cost is still given by Expression (6), and hence, his/her optimal decentralized order quantity is $Q_d^* = \sqrt{2K_r D/h_r}$. Along with the usual set-up and holding costs, now the vendor also incurs a cost of \$$R$ for each cargo with capacity P so the vendor's cost function is given

$$G_w(Q_r, n) = \frac{(K_w + \lceil nQ_r/P \rceil R)D}{nQ_r} + \frac{h_w(n-1)Q_r}{2}. \tag{16}$$

Consequently, given Q_d^*, the vendor's subproblem in the Decentralized Model I is to find the optimal number of buyer replenishments within one vendor replenishment cycle, i.e., the optimum value of n that minimizes $G_w(Q_d^*, n)$ where $G_w(\cdot, \cdot)$ is given by Expression (16). Observe that $G_w(Q_d^*, n)$ has the same form as $\psi(n)$ given by Expression (15) so that its minimizer can be computed using a finite enumeration algorithm based on Proposition 6. Letting $K = K_w$, $Q = Q_d^*$ and $h = h_w$ and using the result in Proposition 6, the minimizer n_d^* of $G_w(Q_d^*, n)$ is then given by $\arg\min\{G_w(Q_d^*, n) : n = n_{\min}, \ldots, n_{\max}\}$. As a result, the decentralized solution of Problem I is specified by (Q_d^*, n_d^*).

In order to develop an effective coordination mechanism for Problem I, we need to consider two cases: $Q_d^* > Q_c^*$ and $Q_d^* < Q_c^*$. This is simply because, unlike in Goyal's Problem, there are problem instances where $Q_d^* > Q_c^*$. Hence, instead of the coordination mechanism in Proposition 3 that discourages the buyer from ordering *small* quantities, a more sophisticated mechanism is needed. The proposed coordination mechanism for Problem I is presented in Proposition 7, and, when appropriate, this mechanism discourages the buyer from ordering *large* quantities. This nontraditional result is due to cargo cost and capacity considerations under which smaller orders from the buyer *may be* more desirable for efficient cargo space utilization.

Proposition 7. *Considering Problem* I, *let*

$$\Delta = \frac{G_r(Q_c^*) - G_r(Q_d^*)}{D}.$$

- If $Q_d^* < Q_c^*$, under a unit discount of Δ offered by the vendor for order sizes greater than, or equal to, Q_c^*, ordering Q_c^* minimizes the buyer's average annual cost. Under this new pricing scheme, the buyer's average annual cost does not exceed $G_r(Q_d^*)$, the vendor's average annual profit is improved relative to the decentralized setting, and $\Delta < c$.
- If $Q_d^* > Q_c^*$, under a unit discount of Δ offered by the vendor for order sizes less than, or equal to, Q_c^*, ordering Q_c^* minimizes the buyer's average annual cost. Under this new pricing scheme, the buyer's average annual cost does not exceed $G_r(Q_d^*)$, the vendor's average annual profit is improved relative to the decentralized setting, and $\Delta < c$.

For the case $Q_d^* < Q_c^*$, since the discount is valid on all items for order sizes greater than, or equal to, Q_c^*, we call the corresponding price schedule *all-units quantity pricing with economies of scale*. When $Q_d^* < Q_c^*$, since the discount is valid on all items for order sizes less than, or equal to, Q_c^*, we call the corresponding price schedule *all-units quantity pricing with diseconomies of scale*.

We note that the coordination mechanism proposed above can also be used for Goyal's Problem. Recall that the only difference between Goyal's Problem and the case considered in Problem I is the consideration of cargo cost and capacity associated with vendor's replenishments. As stated in Proposition 1, without this consideration, the optimal order quantity in the centralized solution is always greater than, or equal to, the optimal order quantity in the decentralized solution. Therefore, to coordinate the system without cargo cost and capacity, we do not need to consider the second item in Proposition 7, in which case Proposition 7 reduces to Proposition 3.

For general parameter settings, closed form expressions and analytical bounds representing the improvement rates due to channel coordination cannot be obtained for either Problem I or Problem II; however, a detailed numerical study follows in Section 5. Also, if the cargo capacity is sufficiently large so that inbound replenishments do not require more than one truck (i.e., for $P \to \infty$, we have $\lceil Q/P \rceil = 1$, $\forall 0 < Q < \infty$), then Proposition 5 can be used for computing lower and upper bounds on the improvement rate by substituting $K_w + R$ for K_w.

4.2. Decentralized and coordinated solutions for problem II

In Problem II, along with the usual set-up and holding costs, both the buyer and the vendor incur a cost of R for each cargo with capacity P. The buyer's individual cost is given by

$$G_r(Q_r) = \frac{DK_r}{Q_r} + \frac{h_r Q_r}{2} + \frac{D\lceil Q_r/P \rceil R}{Q_r} \tag{17}$$

and his/her optimal decentralized order quantity, i.e., Q_d^*, is the minimizer associated with this cost function. An algorithmic approach for computing Q_d^* is presented in [13] (see Algorithm 1 on p. 991 in [13]), and, hence, the details are omitted here. Consequently, given Q_d^*, the vendor's subproblem in the Decentralized Model II is, again, to find the optimal number of buyer replenishments within one vendor replenishment cycle, i.e., the

optimum value of n that minimizes $G_w(Q_d^*, n)$ where $G_w(\cdot, \cdot)$ is given by Expression (16). Now, letting $K = K_w$, $Q = Q_d^*$ and $h = h_w$ and using the result in Proposition 6, the minimizer n_d^* of $G_w(Q_d^*, n)$ is $n_d^* = \operatorname{argmin}\{G_w(Q_d^*, n) : n = n_{\min}, \ldots, n_{\max}\}$, and the decentralized solution of Problem II is (Q_d^*, n_d^*).

As for Problem I, in order to develop an effective coordination mechanism for Problem II, we need to consider two cases: $Q_d^* > Q_c^*$ and $Q_d^* < Q_c^*$. However, unlike the coordination mechanism in Proposition 7, the idea of wholesale pricing, with or without economies of scale, does not work in this case due to the additional difficulties for the buyer that are associated with cargo cost and capacity considerations. The proposed coordination mechanism for Problem II is presented in Proposition 8, and, when appropriate, this mechanism discourages the buyer from ordering *large* quantities using side payments. Again, this nontraditional result is due to cargo cost and capacity considerations under which smaller or larger orders from the buyer *may be* more desirable for efficient cargo space utilization.

Proposition 8. *Considering Problem* II, *let*

$$l_1 = \left\lfloor \frac{Q_c^*}{P} \right\rfloor,$$

$$l_2 = \left\lceil \frac{Q_c^*}{P} \right\rceil, \quad and$$

$$Q_{l_2} = \sqrt{\frac{2(K_r + l_2 R)D}{h_r}},$$

so that Q_{l_2} is the economic order quantity when l_2 trucks are used and l_2 is the number of trucks needed for shipping Q_c^* units. Under the following coordination mechanism, ordering Q_c^* units minimizes the buyer's average annual cost in such a way that it does not exceed $G_r(Q_d^*)$ whereas the vendor's average annual profit is improved relative to the decentralized setting.

- *If $Q_d^* < Q_c^*$:*
  - *If $Q_c^* \geqslant Q_{l_2}$, a fixed payment of $G_r(Q_c^*) - G_r(Q_d^*)$ is paid by the vendor to the buyer for order sizes larger than or equal to Q_c^*.*
  - *If $Q_c^* < Q_{l_2}$, a fixed payment of $G_r(Q_c^*) - G_r(Q_d^*)$ is paid by the vendor to the buyer for order sizes in the range $(l_1 P, Q_c^*]$.*
- *If $Q_d^* > Q_c^*$, a fixed payment of $G_r(Q_c^*) - G_r(Q_d^*)$ is paid by the vendor to the buyer for order sizes in the range $(l_1 P, Q_c^*]$.*

Under the coordination mechanism described in Proposition 8, the vendor pays fixed rewards to the buyer, which is called a *vendor-managed incentive scheme with fixed rewards to the buyer*.

Finally, we note that if the cargo capacity is sufficiently large so that inbound and outbound replenishments do not require more than one truck, then Proposition 5 can be used for computing lower and upper bounds on the improvement rate by substituting $K_w + R$ for K_w, and $K_r + R$ for K_r.

5. Numerical results

Our numerical results are based on two data sets; namely, Data Sets 1 and 2. Since the current paper is an extension of [13], Data Set 1 includes the problem instances provided therein. That is, in Data Set 1, we have $K_w = 175, 350, 700$; $K_r = 50, 100, 150$; $R = 60, 120, 240$; $P = 5, 10, 20$; $D = 2, 4, 8$; $h_w = 0.5, 1, 2$; and $h_r = 4, 8, 16$. Hence, Data Set 1 includes $3^7 = 2187$ problem instances. Data Set 2 includes 40,000 problem instances. In generating this new data set, we have focused on having a variety of values for the ratios $\frac{K_w h_r}{K_r h_w}$ and $\frac{K_w h'}{K_r h_w}$, and cargo cost parameters P and R. More specifically, for fixed values of D, K_r and h_w (i.e., $D = 10$, $K_r = 160$, $h_w = 10$), we have generated different K_w/K_r and h'/h_w ratios over $[1.01, 3)$, and $[0.01, 2)$, respectively, using a step size of 0.1. Also, in this larger data set, we have considered ten different cargo cost values, starting at 2.5 and increasing to 1280 by multiples of 2. Similarly, we have considered ten different cargo capacity val-

ues, starting at 1 and increasing to 512 by multiples of 2. Most of the results we comment on in the following discussion are based on Data Set 2; however, the same comments apply to Data Set 1 as well. Also, for illustrative purposes, we refer to some problem instances from Data Set 1, along with a couple of additional examples that do not belong to either of the data sets.

All three problem settings discussed in the paper (i.e., Goyal's Problem, Problem I, and Problem II) have been analyzed using both data sets. In examining our numerical results, we pay specific attention to the parameter ranges characterized in Proposition 5 for Goyal's Problem. These parameter ranges are:

Range 1: $0 < \frac{K_{\mathrm{w}} h_{\mathrm{r}}}{K_{\mathrm{r}} h_{\mathrm{w}}} \leqslant 2$,

Range 2: $\frac{K_{\mathrm{w}} h_{\mathrm{r}}}{K_{\mathrm{r}} h_{\mathrm{w}}} > 2$ and $\frac{K_{\mathrm{w}} h'}{K_{\mathrm{r}} h_{\mathrm{w}}} \geqslant 2$, and

Range 3: $\frac{K_{\mathrm{w}} h_{\mathrm{r}}}{K_{\mathrm{r}} h_{\mathrm{w}}} > 2$ and $\frac{K_{\mathrm{w}} h'}{K_{\mathrm{r}} h_{\mathrm{w}}} < 2$.

For each problem setting, the average, maximum, and minimum improvement rates over Ranges 1–3 are reported in Table 3.

We proceed with a discussion of important observations based on our numerical results. As expected, our numerical results indicate that, for Goyal's Problem,

- Maximum $\mathscr{IR}$ over Range 3 > Maximum $\mathscr{IR}$ over Range 1, and
- Maximum $\mathscr{IR}$ over Range 1 > Maximum $\mathscr{IR}$ over Range 2.

On the other hand, according to Table 3, for Problems I and II,

- Maximum $\mathscr{IR}$ over Range 1 > Maximum $\mathscr{IR}$ over Range 3, and
- Maximum $\mathscr{IR}$ over Range 3 > Maximum $\mathscr{IR}$ over Range 2.

Also, for Goyal's Problem, the maximum and minimum $\mathscr{IR}$ values in Table 3 provide a strong indication that the theoretical bounds of $\mathscr{IR}$ over Range 1 (given by Proposition 5) are fairly tight for Data Set 2. However, the corresponding upper bound over Range 2 is not tight for Data Set 2.

Table 3
Average, maximum, and minimum $\mathscr{IR}$ values for different ranges of the Data Set 2

Range	Goyal's Problem	Problem I	Problem II
Average $\mathscr{IR}$ values			
1	7.951	6.939	4.337
2	1.592	2.136	1.088
3	5.198	4.416	3.216
Maximum $\mathscr{IR}$ values			
1	12.743	23.454	15.467
2	2.979	13.130	9.688
3	13.147	21.055	13.938
Minimum $\mathscr{IR}$ values			
1	5.798	0.107	0
2	0.234	0.012	0
3	0.448	0.012	0

Table 4
Tightness of the bounds in Proposition 5

E.g.	K_{w}	K_{r}	h_{r}	h_{w}	D	IR (%)
1	100	50.051	1	0.999	$\forall D$	13.383
2	100	99.98	1	0.8	$\forall D$	5.721
3	180	10	1	0.9	$\forall D$	5.243

Table 4 provides three numerical examples for Goyal's Problem. These examples do not correspond to problem instances of the two data sets and are included as additional numerical evidence for our discussion of the tightness of the theoretical bounds of $\mathscr{IR}$ presented in Proposition 5. The first and second examples in Table 4 demonstrate that the theoretical upper and lower bounds over Region I are, in fact, tight. The third example in Table 4 corresponds to the problem instance representing the maximum $\mathscr{IR}$ value we have observed over Region II after extensive numerical experimentation with several problem instances including those in Data Sets 1 and 2. That is, although we have constructed a numerical example demonstrating that $\mathscr{IR}$ could be as high as 100% (see Proposition 5), we have not observed such an extreme case in our numerical study. In fact, according to Table 3, for all three problems considered in the paper, we have:

- Average $\mathscr{IR}$ over Range 1 > Average $\mathscr{IR}$ over Range 3, and
- Average $\mathscr{IR}$ over Range 3 > Average $\mathscr{IR}$ over Range 2.

Examining the maximum $\mathscr{IR}$ values in Table 3, we further conclude that maximum potential savings are particularly significant for Problems I and II. For example, the maximum $\mathscr{IR}$ values over Range I can be as high as 23.454% and 15.467% for Problems I and II, respectively, both of which exceed the 13.397% upper bound over this range for Goyal's Problem. On the other hand, the 5.719% lower bound of Goyal's Problem does not apply to Problems I and II, as the minimum $\mathscr{IR}$ values over Range I can be as small as 0% for these problems. Table 5 illustrates the specific problem instances corresponding to the maximum savings reported in Table 3 and obtained in Data Set 1.

In fact, regardless of the parameter range, i.e., Ranges 1, 2, or 3, the maximum potential impact of coordination is substantial for Problems I and II, varying between approximately 9% and 23%. Our numerical results also indicate that, although substantial savings might be achievable, they are not guaranteed in all cases. Hence, a careful analysis building on the techniques presented in the paper should be undertaken for all practical purposes.

Tables 6 and 7 illustrate the dependence of $\mathscr{IR}$ on P and R for Problem I and II, respectively, and reveal some interesting observations as discussed in the remainder of this section.

For any given P value, as R approaches 0, the impact of both R and P on $\mathscr{IR}$ diminishes. That is, for $R = 0$, the corresponding $\mathscr{IR}$ values remain constant for all P over Ranges 1, 2, and 3, for both Problems I and II. Secondly, considering Problem I, for any given R value, as P approaches 512, the impact of P on $\mathscr{IR}$ diminishes. That is, for any given R value, there exists a threshold P value after which the corresponding $\mathscr{IR}$ values remain constant. For example, in Table 6, over Range 1, the threshold P value is between 64 and 128 for $R \leqslant 80$, and it is between 128 and 256 for $R \geqslant 160$. In fact, for Problem I, over all three ranges, if $P \geqslant 256$ then the corresponding $\mathscr{IR}$ values are constant for all R. The results in Table 7 indicate that similar observations are also true for Problem II as well.

Table 5
Examples illustrating high $\mathscr{IR}$ values

E.g. #	Problem	K_w	K_r	h_r	h_w	D	P	R	IR (%)
Examples from Data Set 1									
1	Goyal	175	150	4	2	$\forall D$	–	–	4.522
2	I	175	50	4	2	2	20	240	12.947
3	II	175	50	4	2	2	20	120	10.844
Examples from Data Set 2									
4	Goyal	465.6	160	20.1	10	$\forall D$	–	–	2.979
5	Goyal	305.6	160	10.1	10	$\forall D$	–	–	12.743
6	Goyal	321.6	160	10.1	10	$\forall D$	–	–	13.147
7	I	177.6	160	1.455	0.5	10	128	1280	13.130
8	I	321.6	160	0.505	0.5	10	128	320	21.055
9	I	161.6	160	0.505	0.5	10	128	640	23.454
10	II	353.6	160	0.955	0.5	10	128	160	9.688
11	II	321.6	160	0.505	0.5	10	64	20	13.938
12	II	209.6	160	0.505	0.5	10	64	20	15.467

Table 6
The impact of P and R on $\mathscr{IR}$: Problem I

P	**R=0**	$R = 2.5$	$R = 5$	$R = 10$	$R = 20$	$R = 40$	$R = 80$	$R = 160$	$R = 320$	$R = 640$	Avg $\mathscr{IR}$
Range 1: $1 \leqslant \frac{K_w h_r}{K_r h_w} = 1.0121 \leqslant 2, K_r = 160, K_w = 161.6, h_r = 10.1, h_w = 10, h' = 0.1, D = 10$											
2	5.798	4.884	4.240	3.391	2.487	1.722	1.189	0.867	0.688	0.594	2.586
4	5.798	5.297	4.884	4.240	3.391	2.487	1.722	1.189	0.867	0.688	3.056
8	5.798	5.533	5.297	4.884	4.240	3.391	2.487	1.722	1.189	0.867	3.541
16	5.798	5.661	5.533	5.297	4.884	4.240	3.391	2.487	1.722	1.189	4.020
32	5.798	5.811	5.845	5.968	6.393	7.450	9.051	10.834	7.792	5.089	7.003
64	5.798	6.044	6.290	6.784	7.771	9.727	13.220	17.884	17.698	17.503	10.872
128	5.798	5.921	6.044	6.290	6.784	7.771	9.727	13.220	18.128	23.454	10.314
256	5.798	5.921	6.044	6.290	6.784	7.771	9.727	13.147	10.903	10.774	8.316
512	5.798	5.921	6.044	6.290	6.784	7.771	9.727	13.147	10.903	10.774	8.316
Avg $\mathscr{IR}$	5.798	5.524	5.361	5.192	5.124	5.352	6.111	7.519	7.048	7.147	6.018
Range 2: $\frac{K_w h_r}{K_r h_w} = 5.8491 > 2, \frac{K_w h'}{K_r h_w} = 3.8391 \geqslant 2, K_r = 160, K_w = 321.6, h_r = 29.1, h_w = 10, h' = 19.1, D = 10$											
		R = 2.5										
2	1.970	1.796	1.654	1.435	1.151	0.854	0.607	0.438	0.336	0.280	1.052
4	1.970	1.980	1.996	2.024	2.066	2.121	2.178	1.808	1.117	0.700	1.796
8	1.970	1.954	1.963	1.980	2.009	2.054	2.112	2.172	2.221	2.255	2.069
16	1.970	1.949	1.954	1.963	1.980	2.009	2.054	2.112	2.172	2.221	2.038
32	1.970	1.895	1.828	1.712	1.550	1.461	1.538	1.655	1.807	1.965	1.738
64	1.970	2.046	2.122	2.277	2.596	3.260	4.662	5.785	5.266	4.584	3.457
128	1.970	2.008	2.046	2.122	2.277	2.596	3.260	4.662	7.198	10.849	3.899
256	1.970	2.008	2.046	2.071	1.927	1.661	1.210	0.579	1.311	0.689	1.547
512	1.970	2.008	2.046	2.071	1.927	1.661	1.210	0.579	1.311	0.689	1.547
Avg $\mathscr{IR}$	1.970	1.930	1.909	1.881	1.834	1.829	1.927	2.013	2.302	2.448	2.004
Range 3: $\frac{K_w h_r}{K_r h_w} = 2.0301 > 2, \frac{K_w h'}{K_r h_w} = 0.0201 < 2, K_r = 160, K_w = 321.6, h_r = 10.1, h_w = 10, h' = 0.1, D = 10$											
		R = 2.5										
2	13.147	11.435	10.132	8.278	6.115	4.108	2.603	1.645	1.098	0.804	5.937
4	13.147	12.222	11.429	10.126	8.273	6.111	4.105	2.601	1.644	1.098	7.076
8	13.147	12.661	12.220	11.427	10.124	8.271	6.110	4.105	2.601	1.644	8.231
16	13.147	12.845	12.598	12.159	11.370	10.074	8.231	6.081	4.086	2.590	9.318
32	13.147	12.879	12.658	12.429	11.996	11.219	9.941	8.123	6.003	4.036	10.243
64	13.147	12.984	12.944	12.992	13.084	13.255	13.557	14.038	14.690	15.310	13.600
128	13.147	13.037	13.089	13.280	13.652	14.363	15.667	17.884	21.055	18.442	15.362
256	13.147	13.092	13.037	12.931	12.727	12.354	11.734	10.903	10.380	11.502	12.181
512	13.147	13.092	13.037	12.931	12.727	12.354	11.734	10.903	10.380	11.502	12.181
Avg $\mathscr{IR}$	13.147	12.438	11.942	11.267	10.418	9.471	8.533	7.738	7.273	6.737	9.896

In fact, the threshold P values mentioned above represent cases in which Problems I and II reduce to Goyal's Problem. That is, once these threshold values are reached, the truck capacity is sufficiently large so that $\mathscr{IR}$ does not depend on the capacity at all. In order to illustrate how this observation enables us to use our analytical results on Goyal's Problem, let us consider a specific instance of Problem II in Table 7, i.e., the problem instance where $R = 2.5$ and the other model parameters are in Range 3 so that the corresponding $\mathscr{IR}$ value remains constant at 13.203% over $P \geqslant 256$. As we have noted at the end of Section 4.2, when the truck capacity is sufficiently large, we can use Proposition 4 to compute $\mathscr{IR}$ by adding the per truck cost R to fixed replenishment costs, i.e., by substituting $K_w = 312.6 + 2.5 = 324.1$ and $K_r = 160 + 2.5 = 162.5$ in Expression (14). It follows from Expressions (9) and (11) that $n_d^* = 2$ and $n_c^* = 1$, and hence, Expression (14) leads to $\mathscr{IR} = 13.203\%$, which is the same as our experimental result in Table 7.

For Problem I, over Range 1, considering a fixed value of P such that $P \leqslant 16$, we observe that as R increases, $\mathscr{IR}$ decreases (see Table 6.) However, this is no longer true for fixed values of P such that $P \geqslant 32$. In fact, there exist threshold values of P up to which, as R increases, $\mathscr{IR}$ decreases, not only for Range 1 but also for Ranges 2 and 3 for both Problems I and II. For Problem 2, over Range 1, considering

Table 7
The impact of P and R on $\mathscr{IR}$: Problem II

P	$R = 0$	$R = 2.5$	$R = 5$	$R = 10$	$R = 20$	$R = 40$	$R = 80$	$R = 160$	$R = 320$	$R = 640$	Avg $\mathscr{IR}$
Range 1: $1 \leqslant \frac{K_w h_r}{K_r h_w} = 1.0121 \leqslant 2, K_r = 160, K_w = 161.6, h_r = 10.1, h_w = 10, h' = 0.1, D = 10$											
2	5.798	3.988	3.085	2.123	1.308	0.740	0.396	0.205	0.104	0.053	1.780
4	5.798	4.672	3.988	3.085	2.123	1.308	0.740	0.396	0.205	0.104	2.242
8	5.798	5.110	4.672	3.988	3.085	2.123	1.308	0.740	0.396	0.205	2.743
16	5.798	5.361	5.110	4.672	3.988	3.085	2.123	1.308	0.740	0.396	3.258
32	5.798	5.097	4.461	3.348	1.317	0.357	0.274	0.187	0.114	0.064	2.102
64	5.798	5.795	5.793	5.789	12.348	11.361	9.807	7.701	5.387	3.365	7.314
128	5.798	5.796	5.795	5.793	5.789	5.782	5.499	3.481	0.000	0.000	4.373
256	5.798	5.796	5.795	5.793	5.789	5.782	5.771	5.758	5.745	5.735	5.776
512	5.798	5.796	5.795	5.793	5.789	5.782	5.771	5.758	5.745	5.735	5.776
Avg $\mathscr{IR}$	5.798	5.050	4.662	4.169	4.228	3.672	3.189	2.564	1.849	1.568	3.675
Range 2: $\frac{K_w h_r}{K_r h_w} = 5.8491 > 2, \frac{K_w h'}{K_r h_w} = 3.8391 \geqslant 2, K_r = 160, K_w = 321.6, h_r = 29.1, h_w = 10, h' = 19.1, D = 10$											
	R = 0										
2	1.970	1.825	1.548	1.187	0.810	0.495	0.278	0.149	0.077	0.039	0.838
4	1.970	1.368	1.245	1.055	0.808	0.551	0.336	0.189	0.101	0.052	0.768
8	1.970	1.440	1.368	1.245	1.055	0.808	0.551	0.336	0.189	0.101	0.906
16	1.970	1.294	1.068	0.846	0.770	0.652	0.500	0.341	0.208	0.117	0.777
32	1.970	1.747	1.968	1.967	1.965	1.551	0.000	0.000	0.000	0.000	1.117
64	1.970	1.969	1.969	1.968	1.967	1.965	1.551	0.000	0.000	0.000	1.336
128	1.970	1.932	1.896	1.828	1.704	1.500	1.095	6.977	8.375	5.540	3.282
256	1.970	1.932	1.896	1.828	1.704	1.500	1.209	0.876	0.583	0.383	1.388
512	1.970	1.932	1.896	1.828	1.704	1.500	1.209	0.876	0.583	0.383	1.388
Avg $\mathscr{IR}$	1.970	1.678	1.588	1.445	1.292	1.076	0.686	0.981	1.015	0.663	1.239
Range 3: $\frac{K_w h_r}{K_r h_w} = 2.0301 > 2, \frac{K_w h'}{K_r h_w} = 0.0201 < 2, K_r = 160, K_w = 321.6, h_r = 10.1, h_w = 10, h' = 0.1, D = 10$											
		R = 2.5									
2	13.147	10.025	8.102	5.855	3.766	2.198	1.199	0.628	0.322	0.163	4.541
4	13.147	11.369	10.019	8.097	5.851	3.764	2.197	1.198	0.628	0.322	5.659
8	13.147	12.187	11.367	10.017	8.095	5.850	3.763	2.196	1.198	0.628	6.845
16	13.147	12.581	12.127	11.310	9.967	8.055	5.821	3.744	2.185	1.192	8.013
32	13.147	12.301	11.402	9.656	6.333	4.488	3.586	2.558	1.626	0.940	6.604
64	13.147	12.925	12.636	12.066	13.938	13.019	11.502	9.328	6.770	4.371	10.970
128	13.147	13.072	12.925	12.636	12.066	10.960	8.827	4.703	0.000	0.000	8.834
256	13.147	13.203	13.248	13.039	12.654	11.990	10.978	9.688	8.375	7.268	11.359
512	13.147	13.203	13.248	13.039	12.654	11.990	10.978	9.688	8.375	7.313	11.364
Avg $\mathscr{IR}$	13.147	11.897	11.093	9.948	8.752	7.351	5.948	4.405	2.964	2.228	7.773

a fixed value of R such that $R \leqslant 10$, we observe that as P increases, $\mathscr{IR}$ increases, i.e., does not decrease (see Table 7.) However, this is no longer true for fixed values of R such that $R \geqslant 20$. In fact, there exist threshold values of R up to which, as P increases, $\mathscr{IR}$ increases, not only for Range 1 but also for Ranges 2 and 3 for both Problems I and II.

We note that some of the most significant $\mathscr{IR}$ values in Tables 6 and 7 are observed after the threshold P and R values[2] mentioned above are exceeded. Finally, we also note that the impact of P and R on $\mathscr{IR}$ is difficult to characterize in general, especially once these threshold values are exceeded.

6. Conclusions and future research

Our results demonstrate that significant cost savings can be achieved through coordination; however, these savings are not guaranteed in general, i.e., for all parameter settings, and, hence, for all practical purposes. These results provide simple practical rules characterizing the cost improvement rates for different parameter settings for problems with and without explicit transportation considerations, and such rules are useful for managers to use as practical guidelines in preliminary analysis. Overall, we conclude that although the potential maximum savings are more significant under explicit transportation considerations, i.e., for Problems I and II, it is difficult to predict the actual savings without a careful investigation. Hence, for these problems, there is a need to use the technical development in Section 4 to compare the centralized and decentralized solutions for the parameter set of interest. Also, the decentralized analysis provided in this paper for the models with transportation considerations is important for the following additional reason. Classical methods proposed in the literature for achieving channel coordination assume that it is always to the vendor's advantage to influence the ordering behavior of the buyer in such a way that he/she orders more. For this reason, coordination mechanisms such as quantity discounts, rebate policies, buyback policies, and fixed payments aim to increase the order quantity of the buyer. However, when the vendor's profit function is not an increasing function of the buyer's order quantity, a larger order from the buyer can actually be disadvantageous to the vendor. One such practical case is when the parties incur stepwise transportation costs as in this paper. Careful investigation of other practical settings where similar nontraditional results apply and significant cost savings are achievable through coordination remains an area for future research.

It is also important to reiterate that our analytical and numerical results for Goyal's problem indicate that the value of coordination depends only on two important ratios that can be expressed in terms of the critical cost parameters whereas it does not depend on the demand rate. However, these results do not hold under general replenishment costs. In fact, when the demand is stochastic, the value of coordination would depend on the demand process even under simpler replenishment cost structures. A careful investigation of the value of coordination under stochastic demand with or without stepwise transportation costs remains another area for future investigation. Additional important avenues for future research include quantifying the value of coordination under information asymmetry considerations and in multi-buyer and/or multi-vendor settings.

Appendix A

Proof of Proposition 1. Suppose that $Q_c^* \leqslant Q_d^*$. Then, using Expressions (10) and (12), it is easy to show that

$$K_r h_r + \frac{K_w h_r}{n_c^*} \leqslant n_c^* K_r h_w + K_r h'.$$

Substituting $h' = h_r - h_w$ in the above inequality, and rearranging the terms we have

$$n_c^*(n_c^* - 1) \geqslant \frac{K_w h_r}{K_r h_w}.$$

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[2] These threshold values are indicated by bold characters for each of the regions illustrated in Tables 6 and 7 for Problems I and II.

Since $h_r > h'$, the above inequality implies that $n_c^*(n_c^* - 1) > \frac{K_w h'}{K_r h_w}$. This contradicts Expression (11), and, hence, $Q_c^* > Q_d^*$. $\square$

Proof of Proposition 2. Suppose that $n_d^* < n_c^*$. Then, since n_d^* and n_c^* are both positive integers, $(n_c^* - n_d^*) \geqslant 1$. Multiplying both sides of this inequality by $n_c^* + n_d^*$, we obtain $(n_c^*)^2 - (n_d^*)^2 \geqslant n_c^* + n_d^*$ which, in turn, implies that $n_c^*(n_c^* - 1) \geqslant n_d^*(n_d^* + 1)$. Using Expression (9), we also have

$$n_d^*(n_d^* + 1)\frac{K_r h_w}{K_w} \geqslant h_r,$$

and, hence,

$$n_c^*(n_c^* - 1)\frac{K_r h_w}{K_w} \geqslant n_d^*(n_d^* + 1)\frac{K_r h_w}{K_w} \geqslant h_r.$$

Noting that $h_r > h'$, by definition, the above inequality implies that

$$n_c^*(n_c^* - 1) > \frac{K_w h'}{K_r h_w},$$

which, in turn, contradicts Expression (11). Therefore, $n_d^* \geqslant n_c^*$. $\square$

Proof of Proposition 3. Under the coordinated solution, the buyer's cost is $G_r(Q)$ for $Q < Q_c^*$ and $G_r(Q) - \Delta \times D$ for $Q \geqslant Q_c^*$ where $G_r(\cdot)$ is given by Expression (6). Since $Q_c^* > Q_d^*$ and Q_d^* is the minimizer of $G_r(Q)$, we have $G_r(Q_d^*) < G_r(Q)$, $\forall Q < Q_c^*$ and $Q \neq Q_d^*$. For $Q \geqslant Q_c^*$, the cost function $G_r(Q) - \Delta \times D$ is increasing in Q, and, therefore, $G_r(Q_c^*) < G_r(Q)$, $\forall Q > Q_c^*$. At order quantity Q_c^*, the buyer's cost is given by $G_r(Q_c^*) - \Delta \times D = G_r(Q_d^*)$ so that the buyer stays in a no-worse situation under the coordinated solution Also, under the coordinated solution, the vendor's profit is

$$(c - \Delta)D - G_w(Q_c^*, n_c^*) = cD - (G_r(Q_c^*) - G_r(Q_d^*)) - G_w(Q_c^*, n_c^*),$$

where $G_w(\cdot, \cdot)$ is given by Expression (7). Since (Q_c^*, n_c^*) is the minimizer of $G_w(Q, n) + G_r(Q_r)$, we have $G_r(Q_c^*) - G_r(Q_d^*) < G_w(Q_d^*, n_d^*) - G_w(Q_c^*, n_c^*)$, and it follows that

$$(c - \Delta)D - G_w(Q_c^*, n_c^*) > cD - \left(G_w(Q_d^*, n_d^*) - G_w(Q_c^*, n_c^*)\right) - G_w(Q_c^*, n_c^*) > cD - G_w(Q_d^*, n_d^*).$$

Consequently, the vendor's profit under the coordinated solution, given by $(c - \Delta)D - G_w(Q_c^*, n_c^*)$, is improved relative to his/her profit in the decentralized setting, i.e., $cD - G_w(Q_d^*, n_d^*)$. Recall that we concentrate on the case where the decentralized transactional setting makes economical sense for the vendor, i.e., $cD - G_w(Q_d^*, n_d^*) > 0$. Then, $(c - \Delta)D - G_w(Q_c^*, n_c^*) > 0$ so that $c > \Delta$. $\square$

Proof of Proposition 4. Utilizing Expressions (10) and (12) in Expression (13), and performing algebraic manipulations result in Expression (14). $\square$

Proof of Corollary 1. It follows from Expression (9) that if

$$0 \leqslant \frac{K_w h_r}{K_r h_w} \leqslant 2, \tag{18}$$

then $n_d^* = 1$. If $n_d^* = 1$, then Proposition 2 implies that $n_c^* = 1$. The result follows from substituting $n_d^* = 1$ and $n_c^* = 1$ in Expression (14). $\square$

Proof of Lemma 1. Clearly, $f[(x, y)]$ is a relation from the set of possible $\left(\frac{K_w}{K_r}, \frac{h_w}{h_r}\right)$ pairs to the set of $(\frac{K_w h_r}{K_r h_w}, \frac{K_w h'}{K_r h_w})$ pairs. The uniqueness of the output of this relation is based on the fact that $f_1[(x, y)]$ and $f_2[(x, y)]$ are real-valued functions. The same argument can be extended for the second part of the lemma. $\square$

Proof of Corollary 2. The corollary is a direct result of Expression (14), Lemma 1, and Inequalities (9) and (11). $\square$

Proof of Proposition 5. First, we consider the case $0 < \frac{K_w h_r}{K_r h_w} \leqslant 2$. It follows from Corollary 1 that

$$\mathscr{IR} = \left(1 - \frac{2\sqrt{1 + \frac{K_w}{K_r}}}{2 + \frac{K_w}{K_r}} \right) \times 100\%.$$

Letting

$$x = \frac{K_w}{K_r} \quad \text{and} \quad f(x) = \frac{2\sqrt{1 + x}}{2 + x}$$

for the first part of the proposition it is sufficient to show that

$$f(x) > \frac{\sqrt{3}}{2} \quad \text{when } 0 < x = \frac{K_w}{K_r} \leqslant 2,$$

and that

$$\frac{\sqrt{3}}{2} < f(x) < \frac{2\sqrt{2}}{3} \quad \text{when we additionally have } K_w/K_r > 1.$$

Letting $f'(x)$ denote the first derivative of $f(x)$, we have

$$f'(x) = \frac{-x}{\sqrt{1 + x}(x + 2)^2}.$$

Observe that $f'(x) < 0$ for $x > 0$ so that $f(x)$ is decreasing in x. Now, recall that $\frac{h_r}{h_w} > 1$, and we consider the case $\frac{K_w h_r}{K_r h_w} \leqslant 2$. It follows that we are interested in $f(x)$ where $x = \frac{K_w}{K_r} < 2$. Using the fact that $f(x)$ is decreasing in x, this implies $f(x) > \lim_{x \to 2} f(x) = \frac{\sqrt{3}}{2}$, $\forall x$ such that $0 < x \leqslant 2$. Similarly, when $K_w/K_r > 1$, we can easily show that $f(x) < \lim_{x \to 1} f(x) = \frac{2\sqrt{2}}{3}$.

Next, we consider the case $\frac{K_w h_r}{K_r h_w} > 2$, and, similar to the proof of the first part of the proposition, we show that

$$\frac{G_r(Q_c^*) + G_w(Q_c^*, n_c^*)}{G_r(Q_d^*) + G_w(Q_d^*, n_d^*)} \geqslant \frac{2}{3}.$$

Simultaneously, we also extend the proof to consider the case where we additionally have $K_w/K_r > 1$ in which case it suffices to show

$$\frac{G_r(Q_c^*) + G_w(Q_c^*, n_c^*)}{G_r(Q_d^*) + G_w(Q_d^*, n_d^*)} > \frac{2\sqrt{3}}{5},$$

where $G_w(\cdot, \cdot)$ and $G_r(\cdot)$ are given by Expressions (7) and (6), respectively.

Since, by definition, $G_r(Q_d^*) \leqslant G_r(Q_c^*)$ and $G_w(Q_c^*, n_c^*) \leqslant G_w(Q_d^*, n_d^*)$, we can write

$$\frac{G_r(Q_c^*) + G_w(Q_c^*, n_c^*)}{G_r(Q_d^*) + G_w(Q_d^*, n_d^*)} \geqslant \frac{G_r(Q_d^*) + G_w(Q_c^*, n_c^*)}{G_r(Q_d^*) + G_w(Q_d^*, n_d^*)} \geqslant \frac{G_w(Q_c^*, n_c^*)}{G_w(Q_d^*, n_d^*)}. \tag{19}$$

For a fixed value of n, it is easy to show that $G_w(Q_r, n)$ is minimized at

$$Q_r^*(n) = \sqrt{\frac{2K_w D}{n(n - 1)h_w}}.$$

Therefore,

$$G_w\left(\sqrt{\frac{2K_w D}{n_c^*(n_c^* - 1)h_w}}, n_c^* \right) \leqslant G_w(Q_c^*, n_c^*), \tag{20}$$

and combining Inequalities (19) and (20) leads to

$$\frac{G_{\mathrm{r}}(Q_{\mathrm{c}}^*) + G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*)}{G_{\mathrm{r}}(Q_{\mathrm{d}}^*) + G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*)} \geqslant \frac{G_{\mathrm{w}}\left(\sqrt{\frac{2K_{\mathrm{w}}D}{n_{\mathrm{c}}^*(n_{\mathrm{c}}^*-1)h_{\mathrm{w}}}}, n_{\mathrm{c}}^*\right)}{G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*)}.$$

Substituting in $Q_{\mathrm{d}}^* = \sqrt{2K_{\mathrm{r}}D/h_{\mathrm{r}}}$ in the above expression and rearranging its terms, we have

$$\frac{G_{\mathrm{w}}\left(\sqrt{\frac{2K_{\mathrm{w}}D}{n_{\mathrm{c}}^*(n_{\mathrm{c}}^*-1)h_{\mathrm{w}}}}, n_{\mathrm{c}}^*\right)}{G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*)} = \frac{\sqrt{\frac{2K_{\mathrm{w}}h_{\mathrm{w}}(n_{\mathrm{c}}^*-1)}{n_{\mathrm{c}}^*}}}{\frac{K_{\mathrm{w}}}{n_{\mathrm{d}}^*}\sqrt{\frac{h_{\mathrm{r}}}{2K_{\mathrm{r}}}} + \frac{h_{\mathrm{w}}(n_{\mathrm{d}}^*-1)}{2}\sqrt{\frac{2K_{\mathrm{r}}}{h_{\mathrm{r}}}}} \geqslant \frac{2\sqrt{\frac{(n_{\mathrm{c}}^*-1)}{n_{\mathrm{c}}^*}}}{\frac{\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}}}{n_{\mathrm{d}}^*} + \frac{n_{\mathrm{d}}^*}{\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}}}}.$$

Since we now consider the case $\frac{K_{\mathrm{w}}h'}{K_{\mathrm{r}}h_{\mathrm{w}}} \geqslant 2$, Expression (11) implies that $n_{\mathrm{c}}^* \geqslant 2$. Consequently

$$\sqrt{\frac{n_{\mathrm{c}}^*-1}{n_{\mathrm{c}}^*}} \geqslant \frac{1}{\sqrt{2}},$$

and we can write

$$\frac{G_{\mathrm{w}}\left(\sqrt{\frac{2K_{\mathrm{w}}D}{n_{\mathrm{c}}^*(n_{\mathrm{c}}^*-1)h_{\mathrm{w}}}}, n_{\mathrm{c}}^*\right)}{G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*)} \geqslant \frac{\sqrt{2}}{\frac{\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}}}{n_{\mathrm{d}}^*} + \frac{n_{\mathrm{d}}^*}{\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}}}}. \tag{21}$$

In order to complete this part of the proof, we analyze the following two cases:

Case 1: $n_{\mathrm{d}}^* \leqslant \sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}} \leqslant \sqrt{n_{\mathrm{d}}^*(n_{\mathrm{d}}^*+1)}$. In this case,

$$\frac{\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}}}{n_{\mathrm{d}}^*} + \frac{n_{\mathrm{d}}^*}{\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}}} \tag{22}$$

reaches its maximum value at

$$\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}} = \sqrt{n_{\mathrm{d}}^*(n_{\mathrm{d}}^*+1)}.$$

As a result,

$$\frac{\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}}}{n_{\mathrm{d}}^*} + \frac{n_{\mathrm{d}}^*}{\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}}} \leqslant \frac{2n_{\mathrm{d}}^*+1}{\sqrt{n_{\mathrm{d}}^*(n_{\mathrm{d}}^*+1)}}. \tag{23}$$

Using Inequalities (21) and (23), we conclude that

$$\frac{G_{\mathrm{w}}\left(\sqrt{\frac{2K_{\mathrm{w}}D}{n_{\mathrm{c}}^*(n_{\mathrm{c}}^*-1)h_{\mathrm{w}}}}, n_{\mathrm{c}}^*\right)}{G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*)} \geqslant \frac{\sqrt{2n_{\mathrm{d}}^*(n_{\mathrm{d}}^*+1)}}{2n_{\mathrm{d}}^*+1}.$$

Since we consider the case $\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}} > 2$, we know that $n_{\mathrm{d}}^* \geqslant 2$, and, hence,

$$\frac{G_{\mathrm{r}}(Q_{\mathrm{c}}^*) + G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*)}{G_{\mathrm{r}}(Q_{\mathrm{d}}^*) + G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*)} \geqslant \frac{G_{\mathrm{w}}\left(\sqrt{\frac{2K_{\mathrm{w}}D}{n_{\mathrm{c}}^*(n_{\mathrm{c}}^*-1)h_{\mathrm{w}}}}, n_{\mathrm{c}}^*\right)}{G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*)} \geqslant \frac{2\sqrt{3}}{5}.$$

Case 2: $\sqrt{n_{\mathrm{d}}^*(n_{\mathrm{d}}^*-1)} \leqslant \sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}} < n_{\mathrm{d}}^*$. In this case, Expression (22) reaches its maximum at

$$\sqrt{\frac{K_{\mathrm{w}}h_{\mathrm{r}}}{K_{\mathrm{r}}h_{\mathrm{w}}}} = \sqrt{n_{\mathrm{d}}^*(n_{\mathrm{d}}^*-1)},$$

and, hence,

$$\frac{\sqrt{\frac{K_w h_r}{K_r h_w}}}{n_d^*} + \frac{n_d^*}{\sqrt{\frac{K_w h_r}{K_r h_w}}} \leqslant \frac{2n_d^* - 1}{\sqrt{n_d^*(n_d^* - 1)}}. \tag{24}$$

In order to complete the proof, we analyze Case 2 considering two possibilities. Namely, $n_d^* \geqslant 3$ and $n_d^* = 2$.
Case 2.1: $n_d^* \geqslant 3$. Considering $n_d^* \geqslant 3$, the right hand side of Inequality (24) reaches its maximum at $n_d^* = 3$, and combining Inequalities (21) and (24) leads to

$$\frac{G_w\left(\sqrt{\frac{2K_w D}{n_c^*(n_c^* - 1)h_w}}, n_c^*\right)}{G_w(Q_d^*, n_d^*)} \geqslant \frac{\sqrt{2n_d^*(n_d^* - 1)}}{2n_d^* - 1} \geqslant \frac{2\sqrt{3}}{5}.$$

Case 2.2: $n_d^* = 2$.

If we do not have the constraint that $K_w/K_r > 1$, under the general assumptions of Case 2, the proof of Case 2.2 is similar to that of Case 2.1, so using Inequalities (21) and (24) results in

$$\frac{G_w\left(\sqrt{\frac{2K_w D}{n_c^*(n_c^* - 1)h_w}}, n_c^*\right)}{G_w(Q_d^*, n_d^*)} \geqslant \frac{\sqrt{2n_d^*(n_d^* - 1)}}{2n_d^* - 1} = \frac{2}{3}.$$

If we additionally have $K_w/K_r > 1$, it follows from $\frac{K_w h'}{K_r h_w} \geqslant 2$ that $\frac{K_w h_r}{K_r h_w} > 3$. Recalling the original assumptions of Case 2 and using $n_d^* = 2$, we have $\sqrt{3} < \sqrt{\frac{K_w h_r}{K_r h_w}} < 2$. Then, utilizing Inequality (21), it suffices to analyze Expression (22). Observe that, within the parameter range of interest, this ratio reaches its maximum at $\sqrt{\frac{K_w h_r}{K_r h_w}} = \sqrt{3}$ so that we can write

$$\left(\frac{\sqrt{\frac{K_w h_r}{K_r h_w}}}{n_d^*} + \frac{n_d^*}{\sqrt{\frac{K_w h_r}{K_r h_w}}}\right) < \frac{7}{2\sqrt{3}}.$$

As a result, it follows from Inequality (21) that

$$\frac{G_w\left(\sqrt{\frac{2K_w D}{n_c^*(n_c^* - 1)h_w}}, n_c^*\right)}{G_w(Q_d^*, n_d^*)} > \frac{2\sqrt{6}}{7}.$$

Combining our results for Case 2.1 and Case 2.2, we conclude that if $\frac{K_w h_r}{K_r h_w} > 2$ and $\frac{K_w h'}{K_r h_w} \geqslant 2$, then

$$\frac{G_r(Q_c^*) + G_w(Q_c^*, n_c^*)}{G_r(Q_d^*) + G_w(Q_d^*, n_d^*)} \geqslant \frac{2}{3},$$

so that $\mathscr{IR} \leqslant \left(\frac{1}{3}\right) \times 100\%$. Also, if we additionally have $K_w/K_r > 1$, then

$$\frac{G_r(Q_c^*) + G_w(Q_c^*, n_c^*)}{G_r(Q_d^*) + G_w(Q_d^*, n_d^*)} > \frac{2\sqrt{3}}{5},$$

and hence, $\mathscr{IR} < \left(1 - \frac{2\sqrt{3}}{5}\right) \times 100\%$. This completes the proof for the second part of the proposition.

Finally, the following example proves that when $\frac{K_w h_r}{K_r h_w} > 2$ and $\frac{K_w h'}{K_r h_w} < 2$, $\mathscr{IR}$ can be very high. Let $K_w = 10^k$, $K_r = 1$, $h_r = 1$, $h_w = (1 - 10^{-k})$ where k is a very large integer. Then, we have $\frac{K_w h_r}{K_r h_w} \cong 10^k + 1 + \frac{1}{10^k - 1}$ and $\frac{K_w h'}{K_r h_w} \cong 1 + \frac{10^{-k}}{1 - 10^{-k}}$. Therefore, $n_d^* = 10^k + 1$ and $n_c^* = 1$. For general demand rate, we have $Q_d^* = \sqrt{2D}$ and $Q_c^* = \sqrt{2D}10^{k/2}$. Now consider the ratio of decentralized total costs over the centralized total costs. It follows that

$$\frac{G_r(Q_d^*) + G_w(Q_d^*, n_d^*)}{G_r(Q_c^*) + G_w(Q_c^*, n_c^*)} = \frac{1 + 10^k + \frac{10^k}{10^k + 1}}{2 \times 10^{k/2} + \frac{1}{10^{k/2}}} \xrightarrow{k \to \infty} \infty.$$

Hence, using Expression (13), it is easy to see that the improvement rate in this case is almost 100%. $\quad\square$

Proof of Proposition 6. Recalling Expression (15), observe that

$$\psi(n) \geqslant \phi(n) = \frac{KD}{nQ} + \frac{(nQ/P)RD}{nQ} + \frac{h(n-1)Q}{2}, \quad \forall n \geqslant 1. \tag{25}$$

Treating n as a continuous variable, it is straightforward to show that $\phi(n)$ is a strictly convex function of n with a minimizer, denoted by n_{o}, where $n_{\mathrm{o}} = \sqrt{2KD/h}/Q$. By definition, $\psi(1) = KD/Q + \lceil Q/P \rceil RD/Q \geqslant \psi(n^*)$. Hence, noting that $\lceil Q/P \rceil < Q/P + 1$, we can write $(K + R)D/Q + RD/P > \psi(1) \geqslant \psi(n^*)$. Letting $A = (K + R)D/Q + RD/P$, and using Expression (25) in the above inequality we have $A > \psi(n^*) > \phi(n^*)$. Since $A > \psi(n^*) > \phi(n^*)$ and $\phi(n)$ is a strictly convex function of n, $\phi(n) - A$ has two roots leading to $n_{\min}$ and $n_{\max}$. $\quad\square$

Proof of Proposition 7. Since the buyer's cost function in Problem I (given by Expression (6)) has the same structure as in Goyal's Problem, the pricing mechanism described in this proposition is similar to the one in Proposition 5, so the proof is also similar to the proof of Proposition 5. However, we need to take into account the additional case where $Q_{\mathrm{d}}^* > Q_{\mathrm{c}}^*$. First, we show that, under the coordinated solution, the buyer stays in a no-worse situation as far as his/her cost is concerned.

If $Q_{\mathrm{d}}^* < Q_{\mathrm{c}}^*$ then the buyer's cost is given by $G_{\mathrm{r}}(Q)$ for $Q < Q_{\mathrm{c}}^*$ and by $G_{\mathrm{r}}(Q) - \varDelta \times D$ for $Q \geqslant Q_{\mathrm{c}}^*$, where $G_{\mathrm{r}}(\cdot)$ is given by Expression (6). Since $Q_{\mathrm{c}}^* > Q_{\mathrm{d}}^*$ and Q_{d}^* is the minimizer of $G_{\mathrm{r}}(Q)$, we have $G_{\mathrm{r}}(Q_{\mathrm{d}}^*) < G_{\mathrm{r}}(Q)$ for $Q < Q_{\mathrm{c}}^*$ and $Q \neq Q_{\mathrm{d}}^*$. For $Q \geqslant Q_{\mathrm{c}}^*$, the cost function $G_{\mathrm{r}}(Q) - \varDelta \times D$ is increasing in Q, and, therefore, $G_{\mathrm{r}}(Q_{\mathrm{c}}^*) < G_{\mathrm{r}}(Q)$ for $Q > Q_{\mathrm{c}}^*$. At order quantity Q_{c}^*, the buyer's cost is given by $G_{\mathrm{r}}(Q_{\mathrm{c}}^*) - \varDelta \times D = G_{\mathrm{r}}(Q_{\mathrm{d}}^*)$, and, as a result, the buyer stays in a no-worse situation by ordering Q_{c}^* units.

Similarly, if $Q_{\mathrm{d}}^* > Q_{\mathrm{c}}^*$ then the buyer's cost is given by $G_{\mathrm{r}}(Q)$ for $Q > Q_{\mathrm{c}}^*$ and by $G_{\mathrm{r}}(Q) - \varDelta \times D$ for $Q \leqslant Q_{\mathrm{c}}^*$. For $Q \leqslant Q_{\mathrm{c}}^*$, the cost function $G_{\mathrm{r}}(Q) - \varDelta \times D$ is decreasing in Q, and, therefore, $G_{\mathrm{r}}(Q_{\mathrm{c}}^*) < G_{\mathrm{r}}(Q)$ for $Q < Q_{\mathrm{c}}^*$. For $Q > Q_{\mathrm{c}}^*$, we have $G_{\mathrm{r}}(Q_{\mathrm{d}}^*) < G_{\mathrm{r}}(Q)$ where $Q \neq Q_{\mathrm{d}}^*$. Consequently, Q_{c}^* minimizes the buyer's cost under the coordinated solution, and, at this order quantity, the buyer's cost is given by $G_{\mathrm{r}}(Q_{\mathrm{c}}^*) - \varDelta \times D = G_{\mathrm{r}}(Q_{\mathrm{d}}^*)$.

In both cases, i.e., when $Q_{\mathrm{d}}^* < Q_{\mathrm{c}}^*$ or $Q_{\mathrm{d}}^* > Q_{\mathrm{c}}^*$, under the coordinated solution, the vendor's profit is

$$(c - \varDelta)D - G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*) = cD - \big(G_{\mathrm{r}}(Q_{\mathrm{c}}^*) - G_{\mathrm{r}}(Q_{\mathrm{d}}^*)\big) - G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*),$$

where $G_{\mathrm{r}}(\cdot)$ and $G_{\mathrm{w}}(\cdot, \cdot)$ are given by Expressions (6) and (16), respectively.

Since $(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*)$ is the minimizer of $G_{\mathrm{w}}(Q, n) + G_{\mathrm{r}}(Q_{\mathrm{r}})$, we have

$$G_{\mathrm{r}}(Q_{\mathrm{c}}^*) - G_{\mathrm{r}}(Q_{\mathrm{d}}^*) < G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*) - G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*),$$

and it follows that

$$(c - \varDelta)D - G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*) > cD - (G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*) - G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*)) - G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*) > cD - G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*).$$

Consequently, the vendor's profit under the coordinated solution, i.e., $(c - \varDelta)D - G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*)$, is improved relative to his/her profit in the decentralized setting, i.e., $cD - G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*)$. Since we concentrate on the case where the decentralized transactional setting makes economical sense for the vendor, i.e., $cD - G_{\mathrm{w}}(Q_{\mathrm{d}}^*, n_{\mathrm{d}}^*) > 0$, we also have $(c - \varDelta)D - G_{\mathrm{w}}(Q_{\mathrm{c}}^*, n_{\mathrm{c}}^*) > 0$ so $c > \varDelta$. $\quad\square$

Proof of Proposition 8. First, we show that, under the coordinated solution, Q_{c}^* minimizes the buyer's cost function in such a way that by ordering this quantity his/her cost does not exceed $G_{\mathrm{r}}(Q_{\mathrm{d}}^*)$, where $G_{\mathrm{r}}(\cdot)$ is given by Expression (17), leaving the buyer in a no-worse situation relative to the decentralized setting.

- $Q_{\mathrm{d}}^* < Q_{\mathrm{c}}^*$:
  - If $Q_{\mathrm{d}}^* < Q_{\mathrm{c}}^*$ and $Q_{\mathrm{c}}^* \geqslant Q_{l_2}$, under the coordinated solution, the buyer's cost is given by $G_{\mathrm{r}}(Q)$ for $Q < Q_{\mathrm{c}}^*$ and by $G_{\mathrm{r}}(Q) - G_{\mathrm{r}}(Q_{\mathrm{c}}^*) + G_{\mathrm{r}}(Q_{\mathrm{d}}^*)$ for $Q \geqslant Q_{\mathrm{c}}^*$. Since $Q_{\mathrm{d}}^* < Q_{\mathrm{c}}^*$ and Q_{d}^* is the minimizer of $G_{\mathrm{r}}(Q)$, we have $G_{\mathrm{r}}(Q_{\mathrm{d}}^*) < G_{\mathrm{r}}(Q)$ for $Q < Q_{\mathrm{c}}^*$ and $Q \neq Q_{\mathrm{d}}^*$. Let us examine the region $Q \geqslant Q_{\mathrm{c}}^*$ in two parts; namely, $Q_{\mathrm{c}}^* \leqslant Q \leqslant l_2 P$ and $Q > l_2 P$.
    $Q_{\mathrm{c}}^* \leqslant Q \leqslant l_2 P$: Since Q_{l_2} is the economic order quantity when l_2 trucks are used and $Q_{\mathrm{c}}^* \geqslant Q_{l_2}$, we have $G_{\mathrm{r}}(Q_{\mathrm{c}}^*) \leqslant G_{\mathrm{r}}(Q)$ for $Q_{\mathrm{c}}^* \leqslant Q \leqslant l_2 P$. Subtracting $G_{\mathrm{r}}(Q_{\mathrm{c}}^*) - G_{\mathrm{r}}(Q_{\mathrm{d}}^*)$ from both sides of this inequality results in $G_{\mathrm{r}}(Q_{\mathrm{d}}^*) \leqslant G_{\mathrm{r}}(Q) - G_{\mathrm{r}}(Q_{\mathrm{c}}^*) + G_{\mathrm{r}}(Q_{\mathrm{d}}^*)$. Note that the right hand side of this final inequality is the buyer's cost under the coordinated solution for $Q \geqslant Q_{\mathrm{c}}^*$, and, $G_{\mathrm{r}}(Q_{\mathrm{d}}^*)$ is the buyer's cost when $Q = Q_{\mathrm{c}}^*$.

$Q > l_2P$: Using Property 3 in [13], we know that $G_r(Q) > G_r(l_2P)$ for $Q > l_2P$. Since $G_r(l_2P) \geqslant G_r(Q_c^*)$, it follows that $G_r(Q) > G_r(Q_c^*)$ for $Q > l_2P$. Again, subtracting $G_r(Q_c^*) - G_r(Q_d^*)$ from both sides of this inequality results in $G_r(Q_d^*) < G_r(Q) - G_r(Q_c^*) + G_r(Q_d^*)$.

– Considering the case $Q_d^* < Q_c^*$ and $Q_c^* < Q_{l_2}$, we analyze the buyer's cost function under the coordinated solution over three regions; namely, $Q \leqslant l_1P$, $l_1P < Q \leqslant Q_c^*$, and $Q > Q_c^*$. The buyer's cost is given by $G_r(Q)$ for $Q \leqslant l_1P$ and $Q > Q_c^*$, and it is given by $G_r(Q) - G_r(Q_c^*) + G_r(Q_d^*)$ for $l_1P < Q \leqslant Q_c^*$. Since Q_d^* is the minimizer of $G_r(Q)$, we have $G_r(Q_d^*) < G_r(Q)$ for $Q \neq Q_d^*$ over $Q \leqslant l_1P$ and $Q > Q_c^*$. Now, let us consider those Q such that $l_1P < Q \leqslant Q_c^*$. Since $Q_c^* < Q_{l_2}$ and Q_{l_2} is the economic order quantity when l_2 trucks are used, $G_r(Q)$ is decreasing over $l_1P < Q \leqslant Q_c^*$, and, hence, $G_r(Q) - G_r(Q_c^*) + G_r(Q_d^*)$ is decreasing. This implies that the cost at $Q = Q_c^*$, given by $G_r(Q_d^*)$, is less than $G_r(Q) - G_r(Q_c^*) + G_r(Q_d^*)$ over $l_1P < Q < Q_c^*$. It follows that Q_c^* is the minimizer over $Q_c^* < Q_{l_2}$.

• $Q_d^* > Q_c^*$: It is easy to show that $G_r(Q)$ is decreasing in Q over $l_1P < Q \leqslant Q_c^*$ (see [7] where some specific properties of the cost function in Expression (17) are examined). The remainder of the proof builds on this result and is similar to the previous case, and, hence, the details are omitted here.

In all cases of the proposition, the vendor's average annual profit is improved relative to the decentralized setting. This is because, Q_c^* is the minimizer of $G_w(Q_r, n) + G_r(Q)$ and $G_r(Q_d^*) \leqslant G_r(Q_c^*)$ where $G_w(\cdot, \cdot)$ and $G_r(\cdot)$ are given by Expressions (16) and (17), respectively. It follows that

$$0 \leqslant G_r(Q_c^*) - G_r(Q_d^*) < G_w(Q_d^*, n_d^*) - G_w(Q_c^*, n_c^*),$$

and, therefore, $G_w(Q_d^*, n_d^*) > G_w(Q_c^*, n_c^*)$. □

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