Gen Relativ Gravit (2008) 40:1825–1830
DOI 10.1007/s10714-007-0579-z

RESEARCH ARTICLE

Some solutions of the Gauss–Bonnet gravity with scalar field in four dimensions

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Received: 5 September 2007 / Accepted: 10 December 2007 / Published online: 9 January 2008

Abstract We give all exact solutions of the Einstein–Gauss–Bonnet Field Equations coupled with a scalar field in four dimensions under certain assumptions. The main assumption we make in this work is to take the second covariant derivative of the coupling function proportional to the spacetime metric tensor. Although this assumption simplifies the field equations considerably, to obtain exact solutions we assume also that the spacetime metric is conformally flat. Then we obtain a class of exact solutions.

Keywords Gauss–Bonnet gravity · Conformally flat spacetimes · Exact solutions of field equations · Scalar fields · Gauss–Bonnet term

Recently there is an increasing interest in the Gauss–Bonnet theory with a scalar field to look for possible theoretical explanation to some cosmological problems such as acceleration of the universe [1]. Accelerated cosmological solutions were first suggested in [2,3] and also discussed in [4,5]. It is also expected that this theory or its modifications may have some contributions to some astrophysical phenomena. For this purpose, spherically symmetric solutions of this theory were first studied in [6,7]. It has been observed that the Post-Newtonian approximation does not give any new contribution in addition to the post-Newtonian parameters of the general relativity [8]. Black hole solutions in the framework of the GB gravity are investigated recently in [9] (see also [10,11]). There are also attempts to find exact solutions and to study the stability of the Gauss–Bonnet theory in various dimensions with actions containing higher derivative scalar field couplings [12–14].

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Since the Gauss–Bonnet term is a topological invariant in four dimensions it does not contribute to the Einstein field equations. On the other hand it contributes to the field equations if it couples to a spin-0 zero field. In this work we consider a four dimensional action containing the Einstein–Hilbert part, massless scalar field and the Gauss–Bonnet term coupled with the scalar field. The corresponding action is given by [8]

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} - \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) + f(\phi) GB \right] \tag{1}$$

where $\kappa^2 = 8\pi G$ ($c = \hbar = 1$) and

$$GB = R^2 - 4R^{\alpha\beta} R_{\alpha\beta} + R^{\alpha\beta\sigma\gamma} R_{\alpha\beta\sigma\gamma} \tag{2}$$

and f is an arbitrary function of the scalar filed ϕ (coupling function). Here V is potential term for the scalar field. The field equations are given by

$$\begin{aligned} R_{\mu\nu} = \kappa^2 \Bigg[& \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} V(\phi) \, g_{\mu\nu} + 2(\nabla_\mu \nabla_\nu f) R - g_{\mu\nu} (\nabla^\rho \nabla_\rho f) R \\ & - 4(\nabla^\rho \nabla_\mu f) R_{\nu\rho} - 4(\nabla^\rho \nabla_\nu f) R_{\mu\rho} + 4(\nabla^\rho \nabla_\rho f) R_{\mu\nu} \\ & + 2 g_{\mu\nu} (\nabla^\rho \nabla^\sigma f) R_{\rho\sigma} - 4(\nabla^\rho \nabla^\sigma f) R_{\mu\rho\nu\sigma} \Bigg] \end{aligned} \tag{3}$$

$$\nabla^\rho \nabla_\rho \phi - V'(\phi) + f' GB = 0 \tag{4}$$

Einstein field equations are usually solved under certain assumptions like spherical symmetry, plane symmetry and axial symmetry. In some cases we assume a form for the spacetime metric like conformally flat, Kerr–Schild and Gödel types. In each one we create a class of exact solutions of Einstein's field equations [15]. In this work our intention is open such a direction in GB theory and obtain exact solutions of this theory and its modifications under certain assumptions. To this end we now assume the spacetime geometry (M, g) is such that (*assumption 1*)

$$\nabla_\mu \nabla_\nu f = \Lambda_1 g_{\mu\nu} + \Lambda_2 \ell_\mu \ell_\nu \tag{5}$$

where Λ_1 and Λ_2 are scalar functions and ℓ_μ is a vector field. In the sequel we will assume that $\Lambda_2 = 0$ (*assumption 2*). Equation (5) restricts the space-time (M, g). Among these space-times admitting (5) we have conformally flat space-times (*assumption 3*).

$$g_{\mu\nu} = \psi^{-2} \eta_{\mu\nu} \tag{6}$$

where ψ is a scalar function. In such space-times the conformal tensor vanishes identically. Hence

$$GB = -2R^{\alpha\beta} R_{\alpha\beta} + \frac{2}{3} R^2 \tag{7}$$

Then the field equations (3) reduce to

$$(1 - 4\Lambda_1 \kappa^2)\, R_{\mu\nu} = \kappa^2 \left[\frac{1}{2}\partial_\mu \phi \partial_\nu \phi + \frac{1}{2} V(\phi)\, g_{\mu\nu}\right] \tag{8}$$

We have now the last assumption: All functions depend on $z = k_\mu x^\mu$ where k_μ is a constant vector, $\partial_\mu k_\nu = 0$. Then from (5) we get

$$f' = C\psi^{-2}, \quad \Lambda_1 = -Ck^2 \frac{\psi'}{\psi} \tag{9}$$

where C is an arbitrary constant and $k^2 = \eta^{\mu\nu} k_\mu k_\nu$. By using (8) and the Ricci tensor

$$R_{\mu\nu} = 2\frac{\psi_{,\mu\nu}}{\psi} + \left[\frac{1}{\psi}\eta^{\alpha\beta}\, \psi_{,\alpha\beta} - \frac{3}{\psi^2}\, \eta^{\alpha\beta}\, \psi_{,\alpha}\psi_{,\beta}\right] \eta_{\mu\nu}, \tag{10}$$

for the metric (6) we obtain the following equations

$$(1 - 4\Lambda_1 \kappa^2)\psi^{-1}\psi'' = \frac{\kappa^2}{4}(\phi')^2, \tag{11}$$

$$V = -\frac{2k^2}{\kappa^2}(1 - 4\Lambda_1 \kappa^2)[3(\psi')^2 - \psi\psi''], \tag{12}$$

$$k^2\psi^4(\psi^{-2}\phi')' - \dot{V} + \dot{f}\, GB = 0, \tag{13}$$

$$f' = C\psi^{-2}, \quad \Lambda_1 = -Ck^2 \frac{\psi'}{\psi} \tag{14}$$

where

$$GB = 72(k^2)^2\psi^4 \left[\left(\psi^{-1}\psi'\right)^2 - \psi^{-1}\psi''\right]\left(\psi^{-1}\psi'\right)^2 \tag{15}$$

and a dot over a letter denotes derivative with respect to the scalar field ϕ. Equations (11) and (13) give coupled ODEs for the functions ψ and ϕ. Letting $\psi'/\psi = u$ and $\phi' = v$ then these equations become

$$(1 + 4Ck^2\kappa^2 u)(u' + u^2) = \frac{\kappa^2}{4} v^2, \tag{16}$$

$$k^2\, \psi^2\, [(v' - 2uv)v - 27Ck^2\, u'u^2] = V' \tag{17}$$

where V is given by (from (12))

$$V = -\frac{2k^2}{\kappa^2}(1 + 4Ck^2\kappa^2 u)(2u^2 - u')\, \psi^2 \tag{18}$$

Inserting V from (18) into (17) [and using (16) in (17)] we obtain simply

$$3C(k^2)^2 u^2 u' = 0 \tag{19}$$

Hence we have the following solutions.

(A) $C = 0$**:** This corresponds to pure Einstein filed equations with a massless scalar field. The effect of the Gauss Bonnet term disappears. Solutions of these field equations have been given in [16]

(B) $k^2 = 0$**:** The vector field k_μ is null. Then the only field equation is

$$u' + u^2 = \frac{\kappa^2}{4} v^2 \tag{20}$$

and V becomes zero. There is a single equation for the two fields u and v. This means that, if one of the fields u or v is given then the other one is determined directly. The metric takes the form

$$ds^2 = \psi(p)^{-2} \, [2dpdq + dx^2 + dy^2] \tag{21}$$

where p and q are null coordinates and $k_\mu = \delta_\mu^p$ and the above equation (20) becomes

$$\psi_{pp} = \frac{\kappa^2}{4} (\phi')^2 \, \psi \tag{22}$$

and the Einstein tensor represents a null fluid with zero pressure.

$$G_{\mu\nu} = \frac{\kappa^2}{2} \, (\phi')^2 \, k_\mu \, k_\nu \tag{23}$$

Although the coupling function f is nonzero the effect of the GB term is absent in this type. Such a class of solutions belongs to class (A).

(C) $k^2 \neq 0$**:** The vector field k_μ is non-null. Then $u = m$ a real constant which leads to the following solution.

$$\psi = \psi_0 \, e^{m\,z}, \quad \phi = \phi_0 + \phi_1 \, z \tag{24}$$

where ψ_0 and ϕ_0 are arbitrary constants and

$$(1 + 4Ck^2\kappa^2 \, m)m^2 = \frac{\kappa^2}{4} \, \phi_1^2, \quad V = -k^2 \, \phi_1^2 \, \psi^2 \tag{25}$$

where $\phi_1 \neq 0$. The potential function V takes the form

$$V(\phi) = V_0 \, e^{\pm \frac{\phi}{\xi}}, \quad V_0 = -k^2 \, \phi_1^2 \, \psi_0^2 \, e^{\mp \frac{\phi_0}{\xi}} \tag{26}$$

where $\xi = 1 + 4Ck^2 \, \kappa^2 \, m$ and coupling function f takes the form

$$f = f_0 - f_1 \, e^{\mp \frac{\phi}{\xi}}, \quad f_1 = (C/\xi \, \psi_0^2) \, e^{\frac{\phi_0}{\xi}} \tag{27}$$

The solution we obtained here is free of singularities but not asymptotically flat. On the other hand, by using this solution it is possible to obtain an asymptotically flat cosmological solution.

This solution is well understood in a new coordinate chart $\{x^a, t\}$ where the line element takes the following form (after a scaling)

$$ds^2 = \frac{t^2}{t_0^2}\,\eta_{ab}\,dx^a dx^b + \epsilon dt^2 \tag{28}$$

where t_0 is a nonzero constant. If t is a spacelike coordinate then $\epsilon = 1$ and Latin indices take values $a = 0, 1, 2$. If t is a timelike coordinate then $\epsilon = -1$ and Latin indices take values $a = 1, 2, 3$. η_{ab} is the metric of the flat three dimensional geometry orthogonal to the u-direction. The Ricci tensor of the four dimensional metric

$$R_{tt} = 0, \quad R_{ta} = 0, \quad R_{ab} = -\frac{2\epsilon}{t_0^2}\,\eta_{ab} \tag{29}$$

Hence the solution takes the form

$$\phi' = \pm\frac{2\sqrt{\xi}}{t}, \quad V(\phi) = -\frac{4\epsilon\xi}{t^2} \tag{30}$$

where $\xi = 1 + 4\kappa^2 C$, $\Lambda_1 = C$ a constant, and $f = f_0 + \frac{\epsilon C}{2}t^2$, f_0 is an arbitrary constant. The curvature scalars are given by

$$R = \frac{6}{t^2}, \quad R_{\mu\nu}\,R^{\mu\nu} = \frac{12}{t^4} \tag{31}$$

and the Gauss–Bonnet scalar density $GB = 0$. It clear that $t = 0$ is the spacetime singularity. Letting $u_\alpha = \delta^t_\alpha$, the Einstein tensor becomes

$$G_{\alpha\beta} = \frac{2}{t^2}\,u_\alpha\,u_\beta + \frac{\epsilon}{t^2}\,g_{\alpha\beta} \tag{32}$$

This tensor has a physical meaning when $\epsilon = -1$ in which case the Gauss–Bonnet gravity produces a singular cosmological model. The Einstein tensor represents a perfect fluid with an energy density $\rho = 3/t^2$ and a negative pressure $p = -1/t^2$. Both of them are singular at $t = 0$.

We have found the most general solutions of the Gauss–Bonnet gravity coupled to a scalar field under the assumptions stated in the text. One solution (B) depends on a null coordinate whose Einstein tensor corresponds to the energy momentum tensor of a null fluid with zero pressure. The other solution (C) depends on variable t whose curvature invariants are all singular at $t = 0$. When t represents the time coordinate then GB gravity gives a cosmological model with a negative pressure. The solution is singular on the 3-surface $t = 0$.

We would like to conclude with a remark. The field equations (3) and (4) of the GB theory with a scalar field resemble to the field equations of the modified Gauss–Bonnet theory [1,17]. In the latter case the scalar field ϕ and the potential term $V(\phi)$ are absent in the action and the function $f = f(GB)$ depends on the GB term (2). We remark that the flat metric is the only solution of the modified Gauss–Bonnet filed equations under the assumptions made in the text. It seems that scalar field is crucial to obtain non-flat metrics. It is however interesting to search for the solutions of the modified GB field equations. For this purpose we are planning to relax our assumptions 2 and 3 in a forthcoming publication.

I would like to thank the referees for their constructive comments. This work is partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK) and Turkish Academy of Sciences (TUBA).

References

1. Nojiri, S., Odintsov, S.: Phys. Lett. **B631**, 1–6 (2005)
2. Nojiri, S., Odintsov, S.D., Sasaki, M.: Phys. Rev. **D71**, 123509 (2004)
3. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Zerbini, S.: Phys. Rev. **D73**, 084007 (2006)
4. Kolvisto, T., Mota, D.: Phys. Lett. **B644**, 104–108 (2007)
5. Kolvisto, T., Mota, D.: Phys. Rev. **D75**, 023518 (2007)
6. Boulware, D.G., Deser, S.: Phys. Rev. Lett. **55**, 2656 (1985)
7. Boulware, D.G., Deser, S.: Phys. Lett. **175B**, 409–412 (1986)
8. Sotiriou, T., Barausse, E.: Phys. Rev. **D75**, 084007 (2007)
9. Chen, C.-M., Galtsov, D.V., Orlov, D.G.: Extremal black goles in $D = 4$ Gauss–Bonnet gravity (hep-th/0701004)
10. Mignemi, S., Stewart, N.R.: Phys. Rev. **D47**, 5259 (1993)
11. Kanti, P., Mavromatos, N.E., Rizos, J., Tamvakis, K., Winstanley, E.: Phys. Rev. **D54**, 5049 (1996)
12. Binetruy, P., Charmousis, C., Davis, S., Dufaus, J.: Phys. Lett. **B544**, 183–191 (2002)
13. Jakobek, A., Meissner, K.A., Olechowski, M.: Nucl. Phys. **B645**, 217–236 (2002)
14. Davis, S.C.: Gravity on a dilatonic Gauss–Bonnet brane world. In: Proceedings for the XXXIX the Rencontres de Moriond on Exploring the Universe (2005)
15. Stephani, H., Cramer, D., Maccallum, M., Hoenselaers, C., Herlt, E.: Exact Solutions of Einstein's Field Equations, 2nd edn. University Press, Cambridge (2003)
16. Gürses, M.: Phys. Rev. **D15**, 2731 (1977)
17. Nojiri, S., Odintsov, S.: Introduction to modified gravity and gravitational alternative for dark energy, hep-th/0601213