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On Some Algebraic Properties of Semi-Discrete Hyperbolic Type Equations

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Abstract

Nonlinear semi-discrete equations of the form $t_x(n+1) = f(t(n), t(n+1), t_x(n))$ are studied. An adequate algebraic formulation of the Darboux integrability is discussed and an attempt to adopt this notion to the classification of Darboux integrable chains has been undertaken.

Key Words: Darboux integrability; Characteristic Lie Algebra; First Integrals; Integrability test.

1. Introduction

The notion of integrability has various of meanings. Different approaches and methods are applied to classify different types of integrable equations (see [1], [8]-[11], [14], [15], [17] and [20]).

Investigation of the class of hyperbolic type differential equations of the form

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{1}$$

has a very long history. Various approaches have been developed to look for particular and general solutions of these kind equations. In the literature one can find several definitions of integrability. According to one given by G. Darboux (see [5], [7]),

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equation (1) is called integrable if there exist functions $F(x, y, u, u_x, u_{xx}, ..., D_x^m u)$ and $G(x, y, u, u_y, u_{yy}, ..., D_y^n u)$ such that arbitrary solution of (1) satisfies $D_y F = 0$ and $D_x G = 0$, where D_x and D_y are operators of differentiation with respect to x and y. Functions F and G are called y- and x-integrals of equation (1), respectively.

An effective criterion of Darboux integrability has been proposed by G. Darboux himself. Equation (1) is integrable if and only if the Laplace sequence of the linearized equation terminates at both ends. The reader may find the definition of the Laplace sequence and the proof of the criterion in [3], [19]. A complete list of the Darboux integrable equations of the form (1) is given in [21].

1.1. Characteristic Lie algebras. Continuous Case.

An alternative method of investigation and classification of the Darboux integrable equations has been developed by A. B. Shabat in [18], based on the notion of characteristic Lie algebra. Let us give a brief explanation of this notion. Define two vector fields as

$$T_1 = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + f \frac{\partial}{\partial u_x} + D_x(f) \frac{\partial}{\partial u_{xx}} + ..., \qquad T_2 = \frac{\partial}{\partial u_y}.$$

Denote by L_y the Lie algebra generated by T_1 and T_2. Any vector field T from L_y satisfies $TF = 0$. Algebra L_y is called the characteristic Lie algebra of equation (1) in the direction of y. Characteristic Lie algebra in the x-direction is defined in a similar way. By virtue of the famous Jacobi theorem, equation (1) is Darboux integrable if and only if both of its characteristic Lie algebras are of finite dimension. In [16] and [18] the characteristic Lie algebras for the systems of nonlinear hyperbolic equations and their applications are studied.

1.2. Characteristic Lie Algebras. Semi-Discrete Case.

In this paper we study semi-discrete chains of the form

$$t_{1x} = f(t, t_1, t_x) \tag{2}$$

from the Darboux integrability point of view. The unknown $t = t(n, x)$ is a function of two independent variables: one discrete n and one continuous x. It is assumed that $\frac{\partial f}{\partial t_x} \neq 0$. Subindex means shift or derivative, for instance, $t_1 = t(n + 1, x)$ and $t_x = \frac{\partial}{\partial x} t(n, x)$. Below we use D to denote the shift operator and D_x to denote the x-

derivative: $Dh(n,x)=h(n+1,x)$ and $D_x h(n,x)=\frac{\partial}{\partial x}h(n,x)$. For the iterated shifts we use the subindex: $D^j h = h_j$.

The characteristic Lie algebra has proved to be an effective tool for classifying nonlinear hyperbolic partial differential equations. This concept can be extended to discrete versions of partial differential equations (see [12]). Discrete models have become rather popular in the last decade because of their applications in physics and biology (see survey [22]). The problem of classification of the discrete Darboux integrable equations is very important and, to our knowledge, still open.

In accordance with the continuous case, function $I = I(x,n,t,t_x,t_{xx},...D_x^m t)$ is called an n-integral of the chain (2) if it satisfies the equation $(D-1)I=0$. In other words, n-integral should still be unchanged under the action of the shift operator $DI = I$, (see also [2]). One can write it in an enlarged form:

$$I(x,n+1,t_1,f,f_x,f_{xx},...) = I(x,n,t,t_x,t_{xx},...). \tag{3}$$

Notice that it is a functional equation, the unknown is taken at two different "points". This circumstance causes the main difficulty in studying discrete chains. Problems of this kind appear when the symmetry approach is applied to discrete equations (see [4], [6]). However, the concept of the Lie algebra of characteristic vector fields can serve as a basis for chains' investigation.

Introduce vector fields in the following way. Concentrate on the main equation (3). The left hand side of (3) contains the variable t_1, while the right hand side does not. Hence the total derivative of the function DI with respect to t_1 should vanish. In other words, the n-integral is in the kernel of the operator $Y_1 := D^{-1}\frac{\partial}{\partial t_1}D$. Similarly one can check that I is in the kernel of the operator $Y_2 := D^{-2}\frac{\partial}{\partial t_1}D^2$. Really, the right hand side of the equation $D^2 I = I$, which immediately follows from (3), does not depend on t_1, therefore the derivative of the function $D^2 I$ with respect to t_1 vanishes. Proceeding this way one can easily prove that for any $j \geq 1$ the operator $Y_j = D^{-j}\frac{\partial}{\partial t_1}D^j$ solves the equation $Y_j I = 0$.

Rewrite the original equation (2) in the form

$$t_{-1x} = g(t,t_{-1},t_x). \tag{4}$$

This can be done because of the condition $\frac{\partial f}{\partial t_x} \neq 0$ assumed above. In the enlarged form the equation $D^{-1}I = I$ looks like

$$I(x,n-1,t_{-1},g,g_x,g_{xx},...) = I(x,n,t,t_x,t_{xx},...). \tag{5}$$

The right side of equation (5) does not depend on t_{-1} so the total derivative of $D^{-1}I$ with respect to t_{-1} is zero, i.e. the operator $Y_{-1} := D\frac{\partial}{\partial t_{-1}}D^{-1}$ solves the equation $Y_{-1}I = 0$. Moreover, the operators $Y_{-j} = D^j\frac{\partial}{\partial t_{-1}}D^{-j}$, $j \geq 1$, also satisfy similar conditions $Y_{-j}I = 0$.

Summarizing the reasonings above one can conclude that the n-integral is annulated by any operator from the Lie algebra $\tilde{L}_n$ generated by the set of operators $\mathcal{Y} = \{..., Y_{-2}, Y_{-1}, Y_{-0}, Y_0, Y_1, Y_2, ..., \}$, where $Y_0 = \frac{\partial}{\partial t_1}$ and $Y_{-0} = \frac{\partial}{\partial t_{-1}}$.

The algebra $\tilde{L}_n$ consists of the operators from the set $\mathcal{Y}$, all possible commutators and linear combinations with coefficients depending on the variables n and x. Evidently equation (2) admits a nontrivial n-integral only if the dimension of the algebra $\tilde{L}_n$ is finite. However the converse is not true: $dim\tilde{L}_n < \infty$ does not imply the existence of n-integrals. By this reason we introduce another Lie algebra, called the characteristic Lie algebra L_n of equation (2) in the direction of n. First we define in addition to the operators $Y_1, Y_2, ...$ differential operators $X_j = \frac{\partial}{\partial_{t_{-j}}}$ for $j \geq 1$.

The following theorem (see [13]) allows us to define this characteristic Lie algebra.

Theorem 1.1 *Equation (2) admits a nontrivial n-integral if and only if the following two conditions hold:*
1) Linear space spanned by the operators $\{Y_j\}_1^\infty$ is of finite dimension. Denote this dimension by N.
2) Lie algebra L_n generated by the operators $Y_1, Y_2, ..., Y_N, X_1, X_2, ..., X_N$ is of finite dimension. We call L_n the characteristic Lie algebra of (2) in the direction of n.

Note that elements of the algebra L_n are operators acting on locally analytical functions of a finite number of the dynamical variables: t, $t_{\pm 1}$, $t_{\pm 2}$, $\cdots$, t_x, t_{xx}, $\cdots$.

Remark 1.2 *If dimension of the linear space L_Y generated by $\{Y_j\}_1^\infty$ is N then the set $\{Y_j\}_1^N$ constitutes a basis in L_Y.*

The x-integral and the characteristic Lie algebra in the x-direction of equation (2) are defined in a similar way to the continuous case. We call a function $F = F(x, n, t, t_{\pm 1}, t_{\pm 2}, ...)$ depending on a finite number of shifts an x-integral of the chain (2), if the following condition is valid $D_xF = 0$, i.e. $K_0F = 0$, where

$$K_0 = \frac{\partial}{\partial x} + t_x\frac{\partial}{\partial t} + f\frac{\partial}{\partial t_1} + g\frac{\partial}{\partial t_{-1}} + f_1\frac{\partial}{\partial t_2} + g_{-1}\frac{\partial}{\partial t_{-2}} + \cdots . \quad (6)$$

Vector fields K_0 and

$$X = \frac{\partial}{\partial t_x}, \tag{7}$$

as well as any vector field from the Lie algebra generated by K_0 and X, annulate F. This algebra is called the characteristic Lie algebra L_x of the chain (2) in the x-direction. The following result is essential; its proof can be found in [18].

Theorem 1.3 *Equation (2) admits a nontrivial x-integral if and only if its Lie algebra L_x is of finite dimension.*

The article is organized as follows. In Section 2 we study the algebra L_n introduced in Theorem 1.1. Section 3 is devoted to properties of the Lie algebra L_x. These algebras L_n and L_x can be used as a new classifying tool for equations on a lattice. From this viewpoint the system of equations (26) is of special importance. Actually, the consistency condition of this overdetermined system of "ordinary" difference equations provides necessary conditions of the Darboux integrability of the original equation (2). As an illustration of efficiency of our approach in the last Section 4 we study in details equation (2) admitting characteristic Lie algebras L_n and L_x of minimal possible dimensions equal 2 and 3 respectively. It is proved that in this case the equation (2) can be reduced to $t_{1x} = t_x + t_1 - t$.

2. Characteristic Lie Algebra L_n

The proof of the first two lemmas can be found in [13].

Lemma 2.1 *If for some integer N the operator Y_{N+1} is a linear combination of the operators Y_i with $i \leq N$: $Y_{N+1} = \alpha_1 Y_1 + \alpha_2 Y_2 + ... + \alpha_N Y_N$, then for any integer $j > N$, we have a similar expression $Y_j = \beta_1 Y_1 + \beta_2 Y_2 + ... + \beta_N Y_N$.*

Lemma 2.2 *The following commutativity relations take place: $[Y_0, Y_{-0}] = 0$, $[Y_0, Y_1] = 0$ and $[Y_{-0}, Y_{-1}] = 0$.*

Note that by direct computations

$$\begin{aligned} Y_1 H &= D^{-1} \frac{d}{dt_1} D H(t, t_x, t_{xx}, ...) \\ &= \Big\{ \frac{\partial}{\partial t} + D^{-1} \Big(\frac{\partial f}{\partial t_1} \Big) \frac{\partial}{\partial t_x} + D^{-1} \Big(\frac{\partial f_x}{\partial t_1} \Big) \frac{\partial}{\partial t_{xx}} + ... \Big\} H(t, t_x, t_{xx}, ...) \end{aligned}$$

one gets

$$Y_1 = \frac{\partial}{\partial t} + D^{-1}\Big(\frac{\partial f}{\partial t_1}\Big)\frac{\partial}{\partial t_x} + D^{-1}\Big(\frac{\partial f_x}{\partial t_1}\Big)\frac{\partial}{\partial t_{xx}} + D^{-1}\Big(\frac{\partial f_{xx}}{\partial t_1}\Big)\frac{\partial}{\partial t_{xxx}} + \dots . \tag{8}$$

Now notice that all of the functions f, f_x, f_{xx}, ... depend on the variables t_1, t, t_x, t_{xx}, ... and do not depend on t_2 hence the coefficients of the vector field Y_1 do not depend on t_1 and therefore the operators Y_1 and Y_0 commute. In a similar way, by using the explicit coordinate representation, we have $Y_{-1} = \frac{\partial}{\partial t} + D\Big(\frac{\partial g}{\partial t_{-1}}\Big)\frac{\partial}{\partial t_x} + D\Big(\frac{\partial g_x}{\partial t_{-1}}\Big)\frac{\partial}{\partial t_{xx}} + \dots$, where g is defined by (4).

The following statement turned out to be very useful for studying the characteristic Lie algebra L_n.

Lemma 2.3 *(1) Suppose that the vector field*

$$Y = \alpha(0)\frac{\partial}{\partial t} + \alpha(1)\frac{\partial}{\partial t_x} + \alpha(2)\frac{\partial}{\partial t_{xx}} + \dots,$$

where $\alpha_x(0) = 0$, solves the equation $[D_x, Y] = 0$, then $Y = \alpha(0)\frac{\partial}{\partial t}$.
(2) Suppose that the vector field

$$Y = \alpha(1)\frac{\partial}{\partial t_x} + \alpha(2)\frac{\partial}{\partial t_{xx}} + \alpha(3)\frac{\partial}{\partial t_{xxx}} + \dots$$

solves the equation $[D_x, Y] = hY$, where h is a function of variables t, t_x, t_{xx}, ..., $t_{\pm 1}$, $t_{\pm 2}$, ..., then $Y = 0$.

The proof of Lemma 2.3 can be easily derived from the formula

$$\begin{aligned}[D_x, Y] &= -(\alpha(0)f_t + \alpha(1)f_{t_x})\frac{\partial}{\partial t_1} + (\alpha_x(0) - \alpha(1))\frac{\partial}{\partial t} \\ &+ (\alpha_x(1) - \alpha(2))\frac{\partial}{\partial t_x} + (\alpha_x(2) - \alpha(3))\frac{\partial}{\partial t_{xx}} + \dots . \end{aligned} \tag{9}$$

In formula (8) we have already given an enlarged coordinate form of the operator Y_1. One can check that the operator Y_2 is a vector field of the form

$$Y_2 = D^{-1}(Y_1(f))\frac{\partial}{\partial t_x} + D^{-1}(Y_1(f_x))\frac{\partial}{\partial t_{xx}} + D^{-1}(Y_1(f_{xx}))\frac{\partial}{\partial t_{xxx}} + \dots . \tag{10}$$

It immediately follows from the equation $Y_2 = D^{-1}Y_1D$ and the coordinate representation (8). By induction one can prove similar formulas for arbitrary Y_{j+1}, $j \geq 1$:

$$Y_{j+1} = D^{-1}(Y_j(f))\frac{\partial}{\partial t_x} + D^{-1}(Y_j(f_x))\frac{\partial}{\partial t_{xx}} + D^{-1}(Y_j(f_{xx}))\frac{\partial}{\partial t_{xxx}} + \dots . \tag{11}$$

Lemma 2.4 *For any $n \geq 0$, we have*

$$[D_x, Y_n] = -\sum_{j=0}^{n} D^{-j}(Y_{n-j}(f))Y_j \, . \tag{12}$$

In particular,

$$[D_x, Y_0] = -Y_0(f)Y_0 \quad , \quad [D_x, Y_1] = -Y_1(f)Y_0 - D^{-1}(Y_0(f))Y_1. \tag{13}$$

Proof. We have,

$$\begin{aligned}
[D_x, Y_0]H(t, t_1, t_x, t_{xx}, ...) &= D_xH_{t_1} - Y_0D_xH \\
&= (H_{tt_1}t_x + H_{t_1t_1}t_{1x} + ...) - \frac{\partial}{\partial_{t_1}}(H_tt_x + H_{t_1}t_{1x} + ...) \\
&= -H_{t_1}f_{t_1} = -Y_0(f)Y_0H,
\end{aligned}$$

i.e. the first equation of (13) holds. By (8), (9) and $[D_x, Y_0] = -Y_0(f)Y_0$,

$$\begin{aligned}
&[D_x, Y_1] = -Y_1(f)\frac{\partial}{\partial t_1} - D^{-1}(Y_0(f))\frac{\partial}{\partial t} + D^{-1}[D_x, Y_0]f\frac{\partial}{\partial t_x} + D^{-1}[D_x, Y_0]f_x\frac{\partial}{\partial t_{xx}} + \dots \\
&= -Y_1(f)Y_0 - D^{-1}(Y_0(f))\frac{\partial}{\partial t} - D^{-1}(Y_0(f)Y_0(f))\frac{\partial}{\partial t_x} - D^{-1}(Y_0(f)Y_0(f_x))\frac{\partial}{\partial t_{xx}} - \dots \\
&= -Y_1(f)Y_0 - D^{-1}(Y_0(f))Y_1.
\end{aligned}$$

By Mathematical Induction we have the equation (12). □

Lemma 2.5 *Lie algebra generated by the operators $Y_1, Y_2, Y_3, ...$ is commutative.*

Proof. By Lemma 2.2, $[Y_1, Y_0] = 0$. The reason for this equality is that the coefficients of the vector field Y_1 do not depend on the variable t_1. They might depend only on t_{-1}, t, t_x, t_{xx}, t_{xxx}, ... The coefficients of the vector field Y_2 being of the form $D^{-1}(Y_1(D_x^jf))$ (see (10)) also do not depend on the variable t_1. They might depend only on t_{-2}, t_{-1},

t, t_x, t_{xx}, t_{xxx}, ... Therefore, we have $[Y_2, Y_0] = 0$. Continuing this reasoning we see that for any $n \geq 1$ the commutativity relation $[Y_n, Y_0] = 0$ takes place. Consider now the commutator $[Y_n, Y_{n+m}]$, $n \geq 1$, $m \geq 1$. We have,

$$[Y_n, Y_{n+m}] = [D^{-n}Y_0D^n, D^{-(n+m)}Y_0D^{n+m}] = D^{-n}[Y_0, Y_m]D^n = 0,$$

that finishes the proof of Lemma 2.5. □

Lemma 2.6 *If the operator $Y_2 = 0$ then $[X_1, Y_1] = 0$.*

Proof. By (10), $Y_2 = 0$ implies that $Y_1(f) = 0$. Due to (8), $Y_1(f) = 0$ means that $f_t + D^{-1}(f_{t_1})f_{t_x} = 0$ and, therefore, $D^{-1}(f_{t_1})$ does not depend on t_{-1}. Together with Lemma 2.4 and the fact that $[D_x, X_1] = 0$, it allows us to conclude that $[D_x, [X_1, Y_1]] = -[X_1, D^{-1}(f_{t_1})Y_1] = -D^{-1}(f_{t_1})[X_1, Y_1]$ i.e. $[D_x, [X_1, Y_1]] = -D^{-1}(f_{t_1})[X_1, Y_1]$. By Lemma 2.4, part (2), it follows that $[X_1, Y_1] = 0$. □

Lemma 2.7 *The operator $Y_2 = 0$ if and only if we have*

$$f_t + D^{-1}(f_{t_1})f_{t_x} = 0. \tag{14}$$

Proof. Assume $Y_2 = 0$. By (10), $Y_1(f) = 0$. Due to (8) equality $Y_1(f) = 0$ is another way of writing (14).
Conversely, assume (14) holds, i.e. $Y_1(f) = 0$. It follows from (10) that $Y_2(f) = 0$. Due to Lemma 2.4, we have $[D_x, Y_2] = -D^{-2}(Y_0(f))Y_2$ that implies, by Lemma 2.3, part (2), that $Y_2 = 0$. □

Corollary 2.8 *The dimension of Lie algebra L_n associated with n-integral is equal to* 2 *if and only if (14) holds, or the same $Y_2 = 0$.*

Proof. By Theorem 1.1, the dimension of L_n is 2 if and only if $Y_2 = \lambda_1 X_1 + \mu_1 Y_1$ and $[X_1, Y_1] = \lambda_2 X_1 + \mu_2 Y_1$ for some λ_i, μ_i, $i = 1, 2$.

Assume the dimension of L_n is 2. Then $Y_2 = \lambda_1 X_1 + \mu_1 Y_1$. Since among X_1, Y_1, Y_2 differentiation by t_{-1} is used only in X_1, differentiation by t is used only in Y_1, then $\lambda_1 = \mu_1 = 0$. Therefore, $Y_2 = 0$, or the same, by Lemma 2.7, (14) holds.
Conversely, assume (14) holds, that is $Y_2 = 0$. By Lemma 2.6, $[X_1, Y_1] = 0$. Since Y_2 and $[X_1, Y_1]$ are trivial linear combinations of X_1 and Y_1 then the dimension of L_n is 2. □

3. Characteristic Lie Algebra L_x

Denote by

$$K_1=[X,K_0],\quad K_2=[X,K_1],\quad \ldots,\quad K_{n+1}=[X,K_n],\quad n\geq 1\,, \tag{15}$$

where X and K_0 are defined by (7) and (6).
It is easy to see that

$$K_1=\frac{\partial}{\partial t}+X(f)\frac{\partial}{\partial t_1}+X(g)\frac{\partial}{\partial t_{-1}}+X(f_1)\frac{\partial}{\partial t_2}+X(g_{-1})\frac{\partial}{\partial t_{-2}}+\ldots\,, \tag{16}$$

$$K_n=\sum_{j=1}^{\infty}\left\{X^n(f_{j-1})\frac{\partial}{\partial t_j}+X^n(g_{-j+1})\frac{\partial}{\partial t_{-j}}\right\},\quad n\geq 2, \tag{17}$$

where $f_0:=f$ and $g_0:=g$.

Lemma 3.1 *We have,*

$$DXD^{-1}=\frac{1}{f_{t_x}}X,\qquad DK_0D^{-1}=K_0-\frac{t_xf_t+ff_{t_1}}{f_{t_x}}X, \tag{18}$$

$$DK_1D^{-1}=\frac{1}{f_{t_x}}K_1-\frac{f_t+f_{t_x}f_{t_1}}{f_{t_x}^2}X,\qquad DK_2D^{-1}=\frac{1}{f_{t_x}^2}K_2-\frac{f_{t_xt_x}}{f_{t_x}^3}K_1+\frac{f_{t_xt_x}f_t}{f_{t_x}^4}X, \tag{19}$$

$$DK_3D^{-1}=\frac{1}{f_{t_x}^3}K_3-3\frac{f_{t_xt_x}}{f_{t_x}^4}K_2+\left(3\frac{f_{t_xt_x}^2}{f_{t_x}^5}-\frac{f_{t_xt_xt_x}}{f_{t_x}^4}\right)K_1-\frac{f_t}{f_{t_x}}\left(3\frac{f_{t_xt_x}^2}{f_{t_x}^5}-\frac{f_{t_xt_xt_x}}{f_{t_x}^4}\right)X. \tag{20}$$

Proof. By simple calculations we find the equations (18), (19) and (20). □

Lemma 3.2 *For any $n\geq 1$ we have,*

$$DK_nD^{-1}=a_n^{(n)}K_n+a_{n-1}^{(n)}K_{n-1}+a_{n-2}^{(n)}K_{n-2}+\ldots+a_1^{(n)}K_1+b^{(n)}X, \tag{21}$$

where coefficients $b^{(n)}$ and $a_k^{(n)}$ are functions that depend only on variables t, t_1 and t_x for all k, $1\leq k\leq n$. Moreover,

$$a_n^{(n)}=\frac{1}{f_{t_x}^n},\quad n\geq 1,\qquad a_{n-1}^{(n)}=-\frac{n(n-1)}{2}\frac{f_{t_xt_x}}{f_{t_x}^{n+1}},\quad n\geq 2,$$

$$b^{(n)}\quad=\quad-\frac{f_t}{f_{t_x}}a_1^{(n)},\quad n\geq 2, \tag{22}$$

$$a^{(n)}_{n-2} = \frac{(n-2)(n^2-1)n}{4}\frac{f^2_{t_xt_x}}{2f^{n+2}_{t_x}} - \frac{(n-2)(n-1)n}{3}\frac{f_{t_xt_xt_x}}{2f^{n+1}_{t_x}}, \quad n \geq 3\,. \tag{23}$$

Proof. It is easy to prove the Lemma by using Mathematical Induction. □

Lemma 3.3 *Suppose that the vector field*

$$K = \sum_{j=1}^{\infty}\left\{\alpha(k)\frac{\partial}{\partial t_k} + \alpha(-k)\frac{\partial}{\partial t_{-k}}\right\}$$

solves the equation $DKD^{-1} = hK$, *where* h *is a function of variables* t, $t_{\pm 1}$, $t_{\pm 2}$, $\ldots$, t_x, t_{xx}, $\ldots$, *then* $K = 0$.

The proof of Lemma 3.3 can be easily derived from the following formula

$$\begin{aligned} DKD^{-1} &= -\frac{f_t}{f_{t_x}}D(\alpha(-1))X + D(\alpha(-1))\frac{\partial}{\partial t} + D(\alpha(-2))\frac{\partial}{\partial t_{-1}} \\ &+ \sum_{j=2}^{\infty}\left\{D(\alpha(j-1))\frac{\partial}{\partial t_j} + D(\alpha(-j-1))\frac{\partial}{\partial t_{-j}}\right\}. \end{aligned} \tag{24}$$

Consider the linear space L^* generated by X and K_n, $n \geq 0$. It is a subset in the finite dimensional Lie algebra L_x. Therefore, there exists a natural number N such that

$$K_{N+1} = \mu X + \lambda_0 K_0 + \lambda_1 K_1 + \ldots + \lambda_N K_N, \tag{25}$$

where X, K_n, $0 \leq n \leq N$ are linearly independent. It can be proved that the coefficients μ, λ_i, $0 \leq i \leq N$, are functions depending on a finite number of the dynamical variables. Since $\mu = \lambda_0 = \lambda_1 = 0$, then the equality above should be studied only if $N \geq 2$, or the same, if the dimension of L_x is 4 or more. The case of when the dimension of L_x is equal to 3 must be considered separately.
Assume $N \geq 2$. Then

$$\begin{aligned} DK_{N+1}D^{-1} &= D(\lambda_2)DK_2D^{-1} + D(\lambda_3)DK_3D^{-1} + \ldots + D(\lambda_{N-1})DK_{N-1}D^{-1} \\ &+ D(\lambda_N)DK_ND^{-1}. \end{aligned}$$

Rewriting DK_kD^{-1} in the last equation for each k, $2 \leq k \leq N+1$, using formulas (21), and K_{N+1} as a linear combination (25), allows us to compare coefficients before K_k,

$2 \leq k \leq N$ and obtain the following system of equations:

$$\begin{array}{l} a_{N+1}^{(N+1)}\lambda_N + a_N^{(N+1)} = D(\lambda_N)a_N^{(N)} \\ a_{N+1}^{(N+1)}\lambda_{N-1} + a_{N-1}^{(N+1)} = D(\lambda_{N-1})a_{N-1}^{(N-1)} + D(\lambda_N)a_{N-1}^{(N)} \\ \dots \\ a_{N+1}^{(N+1)}\lambda_k + a_k^{(N+1)} = D(\lambda_k)a_k^{(k)} + D(\lambda_{k+1})a_k^{(k+1)} + \dots + D(\lambda_N)a_k^{(N)}, \end{array} \tag{26}$$

for $2 \leq k \leq N$. Using the fact that coefficients λ_k, $2 \leq k \leq N$, depend on a finite number of arguments, it is easy to see that all of them are functions of only variables t and t_x.

Lemma 3.4 $K_2 = 0$ *if and only if* $f_{t_x t_x} = 0$.

Proof. Assume $K_2 = 0$. By representation (17) we have $X^2(f) = 0$, that is $f_{t_x t_x} = 0$. Conversely, assume that $f_{t_x t_x} = 0$. By (19) we have $DK_2D^{-1} = \frac{1}{f_{t_x}^2}K_2$ that implies, by Lemma 3.3, that $K_2 = 0$. □

Introduce

$$Z_2 = [K_0, K_1]. \tag{27}$$

Lemma 3.5 *We have,*

$$DZ_2D^{-1} = \frac{1}{f_{t_x}}Z_2 - \frac{t_x f_t + f f_{t_1}}{f_{t_x}^2}K_2 + CK_1 - \frac{f_t}{f_{t_x}}CX, \tag{28}$$

where $C = -\frac{t_x f_{t_x t}}{f_{t_x}^2} - \frac{f f_{t_x t_1}}{f_{t_x}^2} + \frac{f_t}{f_{t_x}^2} + \frac{f_{t_1}}{f_{t_x}} + \frac{t_x f_t f_{t_x t_x}}{f_{t_x}^3} + \frac{f f_{t_1} f_{t_x t_x}}{f_{t_x}^3}$.

Proof. Using formulas (18) and (19) for DK_0D^{-1}, DK_1D^{-1} and definition (27), we can easily get the desired results. □

Lemma 3.6 *The dimension of the Lie algebra L_x generated by X and K_0 is equal to 3 if and only if*

$$f_{t_x t_x} = 0 \tag{29}$$

and

$$-\frac{t_x f_{t_x t}}{f_{t_x}^2} - \frac{f f_{t_x t_1}}{f_{t_x}^2} + \frac{f_t}{f_{t_x}^2} + \frac{f_{t_1}}{f_{t_x}} = 0\,. \tag{30}$$

Proof. Assume the dimension of the Lie algebra L_x generated by X and K_0 is equal to 3. It means that the algebra consists of X, K_0 and K_1 only, and $K_2 = \lambda_1 X + \lambda_2 K_0 + \lambda_3 K_1$, $Z_2 = \mu_1 X + \mu_2 K_0 + \mu_3 K_1$ for some functions λ_i and μ_i. Since among X, K_0, K_1, K_2 and Z_2 we have differentiation by t_x only in X, differentiation by x only in K_0, then $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. Therefore, $K_2 = \lambda_3 K_1$ and $Z_2 = \mu_3 K_1$. Also, among K_1, K_2 and Z_2 we have differentiation by t only in K_1 then $\lambda_3 = \mu_3 = 0$. We have proved that if the dimension of the Lie algebra L_x is 3 then $K_2 = 0$ and $Z_2 = 0$. By Lemma 3.4, condition (29) is satisfied. It follows from (28) that

$$0 = DZ_2D^{-1} = \frac{1}{f_{t_x}} Z_2 - \frac{t_x f_t + f f_{t_1}}{f_{t_x}^2} K_2 + CK_1 - \frac{f_t}{f_{t_x}} CX = CK_1 - \frac{f_t}{f_{t_x}} CX.$$

Since X and K_1 are linearly independent then equality $CK_1 - \frac{f_t}{f_{t_x}} CX = 0$ implies $C = 0$. Equality (30) follows from (29) and $C = 0$.
Conversely, assume that properties (29) and (30) are satisfied. To prove that the dimension of the Lie algebra L_x is equal to 3 it is enough to show that $K_2 = 0$ and $Z_2 = 0$. It follows from (29) and Lemma 3.4 that $K_2 = 0$. From formula (28) for DZ_2D^{-1}, property (30) and knowing that $K_2 = 0$ we have that $DZ_2D^{-1} = \frac{1}{f_{t_x}} Z_2$ that implies, by Lemma 3.3, that $Z_2 = 0$. □

4. Equations with Characteristic Algebras of the Minimal Possible Dimensions.

Corollary 4.1 *If Lie algebras for $n-$ and $x-$ integrals have dimensions 2 and 3 respectively, then equation $t_{1x} = f(t, t_1, t_x)$ can be reduced to $t_{1x} = t_x + t_1 - t$.*

Proof. By Lemma 3.6 and Corollary 2.8, the dimensions of n- and x-Lie algebras are 2 and 3 correspondingly mean equations (14), (29), and (30) are satisfied. It follows from property (29) that $f(t, t_1, t_x) = G(t, t_1)t_x + H(t, t_1)$ for some functions $G(t, t_1)$ and $H(t, t_1)$. By (14), $G_t t_x + H_t + \{D^{-1}(G_{t_1} t_x + H_{t_1})\}G = 0$, that is

$$D^{-1}(G_{t_1} t_x + H_{t_1}) = -\frac{G_t}{G} t_x - \frac{H_t}{G}\,. \tag{31}$$

Note that $t_{1x} = Gt_x + H$ implies $t_x = D^{-1}(G)t_{-1x} + D^{-1}(H)$ and, therefore, $t_{-1x} =$

$\frac{1}{D^{-1}(G)}t_x - \frac{D^{-1}(H)}{D^{-1}(G)}$. We continue with (31) and obtain the equality

$$D^{-1}\left(\frac{G_{t_1}}{G}\right)t_x - D^{-1}\left(\frac{G_{t_1}H}{G}\right) + D^{-1}(H_{t_1}) = -\frac{G_t}{G}t_x - \frac{H_t}{G}$$

which gives rise to the two equations

$$D^{-1}\left(\frac{G_{t_1}}{G}\right) = -\frac{G_t}{G}, \qquad D^{-1}\left(H_{t_1} - \frac{G_{t_1}H}{G}\right) = -\frac{H_t}{G}\,. \tag{32}$$

It is seen from the first equation of (32) that $\frac{G_t}{G}$ is a function that depends only on variable t, even though functions G and G_t depend on variables t and t_1. Denote $\frac{G_t}{G} =: a(t)$. Then $\frac{G_{t_1}}{G} = -a(t_1)$. The last two equations imply that $G = A_1(t_1)e^{\tilde{a}(t)} = A_2(t)e^{-\tilde{a}(t_1)}$ for some functions $A_1(t_1)$ and $A_2(t)$ and $\tilde{a}(t) = \int_0^t a(\tau)d\tau$. Noticing that $A_1(t_1)e^{\tilde{a}(t_1)} = A_2(t)e^{-\tilde{a}(t)}$, we conclude that $A_1(t_1)e^{\tilde{a}(t_1)}$ is a constant. Denoting $\gamma := A_1(t_1)e^{\tilde{a}(t_1)}$ and $G_1(t) := e^{-\tilde{a}(t)}$, we have

$$G(t,t_1) = \gamma\frac{G_1(t_1)}{G_1(t)} \qquad \text{and, therefore,} \qquad f(t,t_1,t_x) = \gamma\frac{G_1(t_1)}{G_1(t)}t_x + H. \tag{33}$$

The second equation of (32) implies that

$$\frac{H_t}{G} = -\mu(t) \quad \text{and} \quad H_{t_1} - \frac{G_{t_1}H}{G} = \mu(t_1) \tag{34}$$

for some function $\mu(t)$. Using (33), the second equation in (34) can be rewritten as $H_{t_1} - \frac{G_1'(t_1)H}{G_1(t_1)} = \mu(t_1)$, or the same, as $\left\{\frac{H(t,t_1)}{G_1(t_1)}\right\}_{t_1} = \frac{\mu(t_1)}{G_1(t_1)}$. It means that

$$H(t,t_1) = G_1(t_1)H_1(t_1) + G_1(t_1)H_2(t) \tag{35}$$

for some functions $H_1(t_1)$ and $H_2(t)$. By substituting $H(t,t_1)$ from (35), $G(t,t_1)$ from (33) into the second equation of (34) and making all cancellations we have,

$$G_1(t_1)H_1'(t_1) = \mu(t_1), \quad \text{or the same,} \quad G_1(t)H_1'(t) = \mu(t)\,. \tag{36}$$

By substituting $G(t,t_1)$ from (33) and $H(t,t_1)$ from (35) into the first equation of (34), we have

$$H_2'(t)G_1(t) = -\gamma\mu(t)\,. \tag{37}$$

Combining together (36) and (37) we obtain that $H_2'(t)G_1(t) = -\gamma G_1(t)H_1'(t)$, or the same, $H_2'(t) = -\gamma H_1'(t)$, or $(H_2(t) + \gamma H_1(t))' = 0$ that implies that $H_2(t) = -\gamma H_1(t) + \eta$ for some constant η. Therefore,

$$f(t, t_1, t_x) = \gamma \frac{G_1(t_1)}{G_1(t)} t_x + G_1(t_1)H_1(t_1) - \gamma G_1(t_1)H_1(t) + \eta G_1(t_1)\,. \tag{38}$$

Note that only properties (29) and (14) were used to obtain representation (38) for $f(t, t_1, t_x)$. Using (30) and (14) we have $0 = \frac{\gamma G_1(t_1)}{G_1(t)} \{-H_1'(t)G_1(t) + G_1(t_1)H_1'(t_1)\}$, i.e. $-H_1'(t)G_1(t) + G_1(t_1)H_1'(t_1) = 0$. This implies that $H_1'(t)G_1(t) = c$, where c is some constant. Substituting $G_1(t) = \frac{c}{H_1'(t)}$ into (38) we have,

$$f(t, t_1, t_x) = \gamma \frac{H_1'(t)}{H_1'(t_1)} t_x + c\frac{H_1(t_1)}{H_1'(t_1)} - \gamma c \frac{H_1(t)}{H'(t_1)} + \eta \frac{c}{H_1'(t_1)}\,. \tag{39}$$

By using substitution $s = H_1(t)$ equation (39) is reduced to $s_{1x} = \gamma s_x + cs_1 - c\gamma s + \eta c$. Introducing $\tilde{x} = cx$ allows to rewrite the last equation as $s_{1\tilde{x}} = \gamma s_{\tilde{x}} + s_1 - \gamma s + \eta$. If $\gamma = 1$ substitution $s = \tau - n\eta$ reduces the equation to $\tau_{1\tilde{x}} = \tau_{\tilde{x}} + \tau_1 - \tau$. If $\gamma \neq 1$, substitution $s = \gamma^n \tau + \eta \frac{\gamma^n - 1}{1-\gamma}$ reduces the equation to $\tau_{1\tilde{x}} = \tau_{\tilde{x}} + \tau_1 - \tau$. $\square$

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