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Recently, Cont introduced a quantitative framework for measuring model uncertainty in the context of derivative pricing [*Model uncertainty and its impact on the pricing of derivative instruments*, Math. Finance, 16(3) (2006), pp. 519–547]. Two measures of model uncertainty were proposed: one measure based on a coherent risk measure compatible with market prices of derivatives and another measure based on convex risk measures. We show in a discrete time, finite state probability setting, that the two measures introduced by Cont are closely related to calibrated option bounds studied recently by King et al. [*Calibrated option bounds*, Inf. J. Ther. Appl. Financ., 8(2) (2005), pp. 141–159]. The precise relationship is established through convex programming duality. As a result, the model uncertainty measures can be computed efficiently by solving convex programming or linear programming problems after a suitable discretization. Numerical results using *S*&*P* 500 options are given.

Keywords: model uncertainty; option pricing; incomplete markets; coherent risk measures; convex risk measures; calibrated option bounds; duality

AMS Subject Classifications: 91B28; 90C90

1. Introduction

Cont [6] reports that financial market participants usually distinguish between two types of risk, commonly referred to as 'market risk' and 'model risk' according to Routledge and Zin [23]. While market risk is quantified by the specification of a probabilistic model for the uncertain quantities, model risk is usually dealt with by a worst-case approach involving, e.g. stress testing of a portfolio. This distinction, noted by Knight [18], has led to the differentiation of *risk* from *ambiguity* where the former represents the probabilistic nature of future evolution of financial market instruments, while the latter refers to the possibility of several specifications to model these probabilistic phenomena. Decision making under ambiguity has been explored in [10,14] where its axiomatic foundations were established. The application of these developments to the behaviour of security prices was studied in [11,23]. More recently, coherent risk measures introduced by Artzner et al. [1] and further developed by Föllmer and Schied [12] were also important contributions to the literature on decision making under ambiguity. A thorough discussion of these approaches

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to decision making under ambiguity along with their shortcomings when applied in the pricing of derivative instruments can be found in [6].

The difficulties of using the aforementioned existing approaches to decision making under ambiguity in the context of derivative pricing stem from the following observations. In an arbitrage-free and complete financial market where the asset prices evolve according to some probability measure, the assumption of linearity of prices implies the existence of a unique equivalent measure such that the value of an option is computed as the expected value of its (discounted) pay-off under this equivalent measure that also makes discounted asset prices into a martingale; see e.g. Theorem 6.8 of Björk [4] or Section 1.4 of Pliska [21]. However, this equivalent martingale measure is not uniquely specified when the market is incomplete even when there is no ambiguity in the underlying instruments' price processes. Therefore, one faces the issue of choosing an appropriate martingale measure among infinitely many possibilities in valuing a future stochastic pay-off. To treat this problem, calibration techniques try to specify a single pricing measure that is optimal with respect to some selection criterion among those measures consistent with option prices observed in the market [15]. However, the result depends on the selection criterion used for calibration. Avellanedas et al. [2] and Avellanedas and Parás [3] do not advocate a single measure but an interval 'calibrated' to observed market prices for the contingent claim to be priced hence the term 'calibrated option bounds'. The bounds constituting the interval are based on an uncertain volatility model where the volatility process is assumed to stay within an uncertainty band. A requirement of the models of [2,3] is that the writer's and the buyer's prices of a contingent claim are differentiable functions of the cash-flows. However, this assumption may fail to hold in incomplete markets [17].

The problem of non-unique specification of a pricing rule is termed 'model uncertainty in option pricing' in [6], which proposed two measures of model uncertainty satisfying certain requirements for quantifying ambiguity in the context of pricing a contingent claim. Against this background, the purpose of the present article is to present the relationship between measures of model uncertainty introduced by Cont [6] and a recent method of computing calibrated option bounds studied recently by King et al. [17]. We show using convex duality theory that the two measures defined by Cont [6] are obtained directly from the calibrated option bounds approach of King et al. [17]. More precisely, the first measure of Cont, the coherent measure of model uncertainty, is obtained as the difference of the calibrated option bounds of King et al. Moreover, the calibrated option bounds as advocated by King et al. do not require differentiability of writer and buyer prices with respect to contingent claim cash flows and do not assume any specific form of the price process for the underlying. They are easily computable as the optimal values of convex optimization problems corresponding to the hedging policies of a writer and a buyer of the contingent claim under study where the writer (and/or buyer) also includes a static (buy-and-hold) strategy using the benchmark options traded in the market in addition to trading in the underlying. A similar result is obtained for the second measure of Cont based on convex risk measures with the additional restriction that the long- and short-static hedge positions in traded (benchmark) options are bounded in some suitable norm. For simplicity, the results are derived and remain valid, in a discrete time finite probability space framework, while Cont's development in [6] is given in continuous time. A direct consequence of our results is that the second measure of model uncertainty of Cont [6], based on convex risk measures, yields a number at least as large as the first measure based on coherent risk measures. To illustrate the numerical calculation of measures of model uncertainty using continuous optimization in keeping with the theme of this article, we adopt the experimental setting of [17]. More precisely, we use data on 48 European call-and-put options on the *S&P* 500 index to compute the uncertainty measures for each of the 48 options using the remaining 47 as benchmark. We use a Gauss–Hermite quadrature-based [19,20] scenario tree approximation to set up linear and non-linear convex optimization problems that we solve numerically using off-the-shelf optimization solvers. The scenario tree approximation can be used even if the price process has jumps, or is non-Markovian, or incorporates stochastic volatility, and therefore can accommodate the cases mentioned in [6] as potential sources of ambiguity in option pricing. Since the tree approximations and the models are built in the high-level modeling language GAMS [5], they are accessible to the numerical optimization and mathematical finance communities. Hence, the present article serves to further the bridge between numerical optimization and mathematical finance communities, by offering the former an entry point into mathematical finance where convex optimization can be useful, and the latter a simple tool for computing measures of model uncertainty, thereby complementing earlier related work in [9,16,17].

Another recent approach to pricing contingent claims in incomplete markets is through robust utility functions that represent investor preferences under ambiguity, especially when the investor is averse to risk and ambiguity. Schied [24] gives a detailed review of risk measures and associated robust optimization problems. Schied and Wu [25] study the duality theory of maximizing the robust utility functions for pricing contingent claims in incomplete markets.

The rest of this article is organized as follows. In Section 2, we review the model uncertainty and risk measures introduced by Cont [6]. We specify our market model in Section 3, and we describe the calibrated option bounds as well as the precise connections between the previous section. Section 4 gives the results of our numerical experiments using *S&P* 500 options. Some conclusions are given in Section 5.

2. Model uncertainty and risk measures

Cont [6] introduced a methodology for measuring model uncertainty using the following ingredients:

(1) Benchmark instruments: these are derivative instruments traded in the market with prices that can be observed. Let us denote the index set of available benchmark instruments by I (of cardinality K), their pay-offs with $(H^i)_{i\in I}$, and their observed market prices by $(C_i^*)_{i\in I}$. Typically, instead of a unique price, we have the bid-and-ask prices for buying and selling. Therefore we have $C_i^* \in [C_i^b, C_i^a]$.

(2) A set of arbitrage-free pricing models $\mathcal{Q}$, i.e. a set of risk-neutral probability measures $\mathbb{Q}$ on some suitable set of market scenarios $(\Omega, \mathcal{F})$ consistent with the market prices of benchmark instruments with the property that the discounted underlying asset(s) prices $(S_t)_{t\in[0,T]}$ is a martingale under each $\mathbb{Q} \in \mathcal{Q}$ with respect to $\mathcal{F}_t$ and

$$\forall \mathbb{Q} \in \mathcal{Q}, \quad \forall i \in I, \quad \mathbb{E}^{\mathbb{Q}}[|H^i|] < \infty, \quad \mathbb{E}^{\mathbb{Q}}[H^i] \in [C_i^b, C_i^a]. \tag{1}$$

Let us now define the set $\mathcal{C}$ of contingent claims with a well-defined price in all models:

$$\mathcal{C} = \left\{ H \in \mathcal{F}_T, \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[|H|] < \infty \right\}. \tag{2}$$

Let $(\phi_t)_{t \in [0,T]}$ represent a self-financing trading strategy with the stochastic integral $\int_0^t \phi_u \cdot \mathrm{d}S_u$ corresponding to the (discounted) gain from trading between 0 and t. Now Consider a mapping $\mu : \mathcal{C} \mapsto [0, \infty)$ representing the model uncertainty on a contingent claim which has pay-off X. Cont [6] imposes the following conditions on the model uncertainty measure μ:

(1) For liquid benchmark instruments, model uncertainty is at most equal to the absolute difference between bid and ask price, i.e. model uncertainty for benchmark instruments is already contained in bid–ask prices:

$$\forall i \in I, \quad \mu(H^i) \leq |C_i^a - C_i^b|. \tag{3}$$

(2) Hedging using the underlying asset(s) does not affect the model uncertainty measure:

$$\forall \phi \in \mathcal{S}, \quad \mu\left(X + \int_0^T \phi_t \cdot \mathrm{d}S_t \right) = \mu(X), \tag{4}$$

where $\mathcal{S}$ is the set of self-financing trading strategies. In particular, the value of a contingent claim which can be replicated by trading in the underlying has no model uncertainty, i.e.

$$\left[\exists x_0 \in \mathbb{R}, \exists \phi \in \mathcal{S}, \quad \forall \mathbb{Q} \in \mathcal{Q}, \ \mathbb{Q}\left(X = x_0 + \int_0^T \phi_t . \mathrm{d}S_t \right) = 1 \right] \Rightarrow \mu(X) = 0. \tag{5}$$

(3) Convexity: Any convex combination of the pay-offs of two contingent claims results in a model uncertainty measure value smaller or equal to the convex combination of model uncertainty measure values of individual contingent claims, i.e. diversification reduces model uncertainty measure value.

$$\forall X_1, X_2 \in \mathcal{C}, \quad \forall \lambda \in [0,1] \ \mu(\lambda X_1 + (1-\lambda)X_2) \leq \lambda \mu(X_1) + (1-\lambda)\mu(X_2). \tag{6}$$

(4) Static hedging using benchmark instruments:

$$\forall X \in \mathcal{C}, \quad \forall u \in \mathbb{R}^K, \quad \mu\left(X + \sum_{i=1}^K u_i H^i \right) \leq \mu(X) + \sum_{i=1}^K |u_i(C_i^a - C_i^b)|. \tag{7}$$

In particular, if the pay-off can be statically replicated by benchmark derivatives then the model uncertainty measure has a value which is at most the cost of the static replication:

$$\left[\exists u \in \mathbb{R}^K, X = \sum_{i=1}^K u_i H^i \right] \Longrightarrow \mu(X) \leq \sum_{i=1}^K |u_i| \ |C_i^a - C_i^b|. \tag{8}$$

As we deal in the present article with a discrete time representation of financial markets while the above requirements are formulated in more general, continuous time framework, we give the discrete time equivalents of requirements (4) and (5) using the terminology in Section 3.1.4 of Pliska [21]. Let φ_t represent the portfolio of underlying instrument(s) held between time points $t-1$ and t. Then the above requirements (4) and (5) translate into discrete time as

$$\forall \varphi \in \mathcal{S}, \quad \mu\left(X + \sum_{t=1}^{T} \varphi_t \cdot (S_t - S_{t-1})\right) = \mu(X), \tag{9}$$

$$\left[\exists x_0 \in \mathbb{R}, \quad \exists \varphi \in \mathcal{S}, \quad \forall \mathbb{Q} \in \mathcal{Q}, \ \mathbb{Q}\left(X = x_0 + \sum_{t=1}^{T} \varphi_t \cdot (S_t - S_{t-1})\right) = 1\right] \Longrightarrow \mu(X) = 0, \tag{10}$$

respectively.

Under the above requirements, the coherent measure of model uncertainty defined in [6] is the following number

$$\mu_{\mathcal{Q}}(X) = \overline{\pi}(X) - \underline{\pi}(X) \tag{11}$$

for $X \in \mathcal{C}$ and where

$$\overline{\pi}(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X], \quad \underline{\pi}(X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X]. \tag{12}$$

The mapping $X \mapsto \overline{\pi}(-X)$ defines a coherent risk measure in the sense of Föllmer and Schied [13].

The following was proved in Proposition 1 of [6].

PROPOSITION 1

(1) $\overline{\pi}$, $\underline{\pi}$ *assign values to the benchmark derivatives compatible with their market bid–ask prices*:

$$\forall i \in I, \quad C_i^b \leq \underline{\pi}(H^i) \leq \overline{\pi}(H^i) \leq C_i^a, \tag{13}$$

(2) $\mu_{\mathcal{Q}} : C \mapsto \mathbb{R}^+$ *defined by* (11) *is a measure of model uncertainty verifying* (3–8).

In [6], a measure of uncertainty based on convex risk measures in the sense of Föllmer and Schied [13] was also introduced. However, an important difference is that the set $\mathcal{Q}$ which represents a set of pricing rules consistent with the prices of benchmark instruments is replaced by $\mathcal{Q}'$ which is assumed to contain all measures that make the underlying asset prices a martingale.

This second measure of model uncertainty is defined using

$$\pi^*(X) = \sup_{\mathbb{Q}\in\mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] - \|C^* - \mathbb{E}^{\mathbb{Q}}[H]\|_p\} \tag{14}$$

$$\pi_*(X) = \inf_{\mathbb{Q}\in\mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] + \|C^* - \mathbb{E}^{\mathbb{Q}}[H]\|_p\} \tag{15}$$

assuming a unique price vector C^* for the benchmark instruments, and where $\|z\|_p = p\sqrt{\sum_{i=1}^n |z_i|^p}$ for some $z \in \mathbb{R}^K$ for $1 < p < \infty$. For $p = 1, \infty$, we deal with the penalty terms $\|C^* - \mathbb{E}^{\mathbb{Q}}[H]\|_1 = \sum_{i=1}^K |C_i^* - \mathbb{E}^{\mathbb{Q}}[H^i]|$ and $\|C^* - \mathbb{E}^{\mathbb{Q}}[H]\|_\infty = \max_{i=1,\ldots,K} |C_i^* - \mathbb{E}^{\mathbb{Q}}[H^i]|$, respectively. Then, the model uncertainty measure is defined as

$$\forall X \in \mathcal{C}, \quad \mu_*^p(X) = \pi^*(X) - \pi_*(X). \tag{16}$$

Allowing for bid-and-ask prices the associated bounds are defined as:

$$\pi^*(X) = \sup_{\mathbb{Q}\in\mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] - \|(\mathbb{E}^{\mathbb{Q}}[H] - C^a)_+\|_p - \|(C^b - \mathbb{E}^{\mathbb{Q}}[H])_+\|_p\} \tag{17}$$

and

$$\pi_*(X) = \inf_{\mathbb{Q}\in\mathcal{Q}'} \{\mathbb{E}^{\mathbb{Q}}[X] + \|(\mathbb{E}^{\mathbb{Q}}[H] - C^a)_+\|_p + \|(C^b - \mathbb{E}^{\mathbb{Q}}[H])_+\|_p\}, \tag{18}$$

where the operator $(\cdot)_+ = \max\{0, \cdot\}$ is applied to each component of the vectors $\mathbb{E}^{\mathbb{Q}}[H] - C^a$ and $C^b - \mathbb{E}^{\mathbb{Q}}[H]$. Instead of calibrating the martingale measure according to bid–ask prices of the benchmark instruments, the last two terms involving norms in the definition of the bounds above penalize deviations from bid–ask prices of the benchmark options. In the language of Föllmer and Schied, $\rho(X) = \pi^*(-X)$ is a convex risk measure [6,13]. Under some suitable assumptions including one which imposes that the set $\mathcal{Q}'$ contains a least one measure $\mathbb{Q}$ that gives

$$\mathbb{E}^{\mathbb{Q}}[H^i] \in [C_i^b, C_i^a] \quad \forall i \in I,$$

Cont [6] proves that the model uncertainty measure μ_* satisfies (3–6), and the appropriate modifications of (7) and (8); see Proposition 2 of [6] and the discussion therein.

3. Calibrated option bounds

Now, we adopt the setting of [17] by modeling security prices and other payments as discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms are sequences of real-valued vectors (asset values) over the discrete time periods $t = 0, 1, \ldots, T$. A detailed introduction to mathematical finance for discrete time, discrete state financial market structures can be found in the book by Pliska [21]. We assume that the market evolves as a discrete non-recombinant scenario tree, in which the partition of probability atoms $\omega \in \Omega$ generated by matching path records up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at level t in the tree. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, \ldots, T$ has a unique parent denoted $a(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$ has a non-empty set of child nodes $\mathcal{D}(n) \subset \mathcal{N}_{t+1}$. The uniqueness of the parent node makes the scenario tree non-recombinant, an essential feature in specifying incomplete market models [9]. We denote the set of all nodes in the tree by $\mathcal{N}$. The probability distribution P is obtained by attaching positive weights p_n to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n\in\mathcal{N}_T} p_n = 1$. For each non-terminal (intermediate level) node in the tree we have, recursively,

$$p_n = \sum_{m\in\mathcal{D}(n)} p_m, \quad \forall\, n \in \mathcal{N}_t,\ t = T-1, \ldots, 0.$$

Hence, each intermediate node has a probability mass equal to the combined mass of the paths passing through it. The ratios p_m/p_n, $m \in \mathcal{D}(n)$ are the conditional probabilities that the child node m is visited, given that the parent node $n = a(m)$ has been visited. We note that no particular form is assumed for P, i.e. the price process could have jumps, it could be non-Markovian, or it may incorporate stochastic volatility.

A random variable X is a real-valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathcal{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbb{R}\}$ is either the empty set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathcal{N}_t$ if it can be assigned a value on each node of $\mathcal{N}_t$ that is consistent with its definition on Ω [16]. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^P[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathcal{N}_t$ is given by the expression

$$\mathbb{E}^P[X_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{D}(n)} \frac{p_m}{p_n} X_m.$$

Under the light of the above definitions, the market consists of $J+1$ market-traded securities indexed by $j = 0, 1, \ldots, J$ with prices at node n given by the vector $S_n = (S_n^0, S_n^1, \ldots, S_n^J)$. We assume that the security indexed by 0 has strictly positive prices at each node of the scenario tree. This asset corresponds to the risk-free asset in the classical valuation framework. Choosing this security as the numéraire, we can scale the prices at each node where we obtain $S_n^0 = 1$ for all nodes $n \in \mathcal{N}$. For the sake of simplicity, we will assume that the prices have already been scaled with respect to the numéraire.

The amount of security j held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted θ_n^j. Therefore, to each state $n \in \mathcal{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^{J+1}$. The value of the portfolio at state n (discounted with respect to the numéraire) is

$$S_n \cdot \theta_n = \sum_{j=0}^{J} S_n^j \theta_n^j.$$

We will say that the vector process $\{S_t\}$ is called a vector-valued martingale under Q, and Q is called a martingale probability measure for the process if there exists a probability measure $Q = \{q_n\}_{n \in \mathcal{N}_T}$ such that

$$S_t = \mathbb{E}^Q[S_{t+1}|\mathcal{N}_t] \quad (t \leq T-1). \tag{19}$$

By a contingent claim we mean a stochastic cash-flow $F \in \mathcal{C}$ which in our present setting is characterized by (discounted) pay-outs $\{F_n\}_{n \in \mathcal{N}}$ that depend on the price process S of the underlying securities. King et al. [17] formulate the problem of the *writer* of the contingent claim F as computing the smallest amount of initial cash outlay required to hedge the pay-outs generated by the contingent claim by self-financing transactions so as to end up with a non-negative wealth position almost surely at the expiry date of the contingent claim. This initial cash outlay is the optimal value of the optimization problem

$$\begin{aligned} &\min V \\ \text{s.t.} \quad &S_0 \cdot \theta_0 = V \\ &S_n \cdot (\theta_n - \theta_{a(n)}) = -F_n, \quad \forall n \in \mathcal{N}_t, t \geq 1 \\ &S_n \cdot \theta_n \geq 0, \quad \forall n \in \mathcal{N}_T. \end{aligned}$$

When there are other options (benchmark derivatives) available for trading and they are used for static hedging purposes in the above model, one obtains the writer's problem (WC):

$$\begin{aligned} &\min V \\ \text{s.t.}\quad & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- = V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = H_n \cdot (\xi_+ - \xi_-) - F_n, \quad \forall n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n \geq 0, \quad \forall n \in \mathcal{N}_T, \\ & \xi_+, \xi_- \geq 0, \end{aligned}$$

where H^k, $k = 1, \dots, K$ represent the benchmark derivatives with bid–ask prices C_k^b and C_k^a, and (already discounted) pay-offs H_n^k, for all $n \in \mathcal{N}$ (i.e. H_n is a K-vector for all n), and the vectors ξ_+, $\xi_- \in \mathbb{R}^K$ are the amounts bought and shorted of each benchmark derivative instrument. Denote the optimal value in this problem by $V_w(F)$.

The hedging strategy of the buyer, which is the opposite of the writer, is obtained from the optimal solution of the following problem (BC):

$$\begin{aligned} &\max V \\ \text{s.t.}\quad & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- = -V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = H_n \cdot (\xi_+ - \xi_-) + F_n, \quad \forall n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n \geq 0, \quad \forall n \in \mathcal{N}_T, \\ & \xi_+, \xi_- \geq 0. \end{aligned}$$

Denote the optimal value of the above problem by $V_b(F)$.

The numbers $V_w(F)$ and $V_b(F)$ correspond to the *calibrated option bounds* that originated in [2,3], and further developed in [17]. In this approach to computing bounds for option prices market-traded options are used in the trading strategies of the seller and the buyer resulting in price measures (pricing rules) that are consistent with the observed market prices exactly as advocated in the previous section for the measure of model uncertainty. Therefore, our first observation is the following proposition.

PROPOSITION 2 *For each $F \in \mathcal{C}$, we have $\mu_{\mathcal{Q}}(F) = V_w(F) - V_b(F)$.*

Proof From [17], the dual of (WC) is the following linear programming problem in variables $\{y_n\}_{n \in \mathcal{N}}$:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} y_n F_n \\ \text{s.t.}\quad & y_0 = 1 \\ & y_m S_m = \sum_{n \in \mathcal{D}(m)} y_n S_n, \quad \forall m \in \mathcal{N}_t,\ 0 \leq t \leq T-1 \\ & \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} y_n H_n \leq C^a \\ & \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} y_n H_n \geq C^b \\ & y_n \geq 0 \quad \forall n \in \mathcal{N}_t. \end{aligned}$$

By Theorem 4.1 of [17], the dual problem is equivalently expressed as

$$\sup_{Q \in \mathcal{M}_C} \mathbb{E}^Q\left[\sum_{t=1}^{T} F_t\right]$$

where $\mathcal{M}_C = \{Q \in \mathcal{M} | C^b \leq \mathbb{E}^Q[\sum_{t=1}^{T} H_t] \leq C^a\}$ with $\mathcal{M}$ denoting the set of all martingale probability measures (not necessarily equivalent to P), i.e. the set of all q_n, $n \in \mathcal{N}$ satisfying

$$q_n \geq 0, \quad n \in \mathcal{N}_T,$$

$$q_n S_n = \sum_{m \in D(n)} q_m S_m, \quad n \in \mathcal{N}_t, \ t \leq T-1,$$

$$q_0 = 1$$

(cf Proposition 1 of [16]). Therefore, in our finite probability space, discrete time setting $\mathcal{M}_C$ and $\mathcal{Q}$ coincide. ■

Note that both problems WC and BC involved in computing $\mu_{\mathcal{Q}}$ are linear programming problems that can be routinely solved using available software as we shall see below in Section 4. We also remark that the property expressed in inequalities (13) of Proposition 1 is immediately obtained from problems WC and BC as follows. If we are computing $V_w(H^i)$ and $V_b(H^i)$ for the benchmark contingent claim $i \in [1, \ldots, K]$ it suffices to hold one unit long (or short, respectively) of the benchmark contingent claim and nothing else in the portfolio. More precisely, we make the corresponding entry ξ_+^i of the vector ξ_+ equal to one, and we set all other variables equal to zero, which results in a feasible solution to problem (WC), and hence an upper bound to $V_W(H^i)$ equal to C_i^a. Similarly, a short position of one unit, i.e. $\xi_-^i = 1$, with all other variables at zero constitutes a feasible solution to problem (BC) with a lower bound equal to C_i^b. A final observation which will be useful in the proof of Proposition 3 below is that $\mathcal{Q}'$ defined in Section 2 coincides with the set $\mathcal{M}$ in the proof of Proposition 2 in our finite state probability and discrete time context.

We now turn our attention to the second measure of model uncertainty based on convex risk measures. Let us fix some q such that $1 \leq q \leq \infty$, and consider in the discrete time, finite state framework of calibrated option bounds the following writer's optimal hedging problem CWC:

$$\begin{aligned}
&\inf V \\
&\text{s.t.} \quad S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- = V \\
&S_n \cdot (\theta_n - \theta_{a(n)}) = H_n \cdot (\xi_+ - \xi_-) - F_n, \quad \forall n \in \mathcal{N}_t, t \geq 1 \\
&S_n \cdot \theta_n \geq 0, \quad \forall n \in \mathcal{N}_T, \\
&\xi_+, \xi_- \geq 0, \\
&\|\xi_+\|_q \leq 1, \\
&\|\xi_-\|_q \leq 1,
\end{aligned}$$

with optimal value $VC_w^q(F)$, and the buyer's hedging problem CBC

$$\begin{aligned}
&\sup V \\
\text{s.t.}\quad & S_0\cdot\theta_0 + C^a\cdot\xi_+ - C^b\cdot\xi_- = -V \\
& S_n\cdot(\theta_n - \theta_{a(n)}) = H_n\cdot(\xi_+ - \xi_-) + F_n, \quad \forall n\in\mathcal{N}_t, t\geq 1 \\
& S_n\cdot\theta_n \geq 0, \quad \forall n\in\mathcal{N}_T, \\
& \xi_+,\xi_- \geq 0, \\
& \|\xi_+\|_q \leq 1, \\
& \|\xi_-\|_q \leq 1,
\end{aligned}$$

with optimal value $VC_b^q(F)$. We notice that the above optimization problems are almost identical to those of the previous section with the additional restriction that the long and short static hedge positions in traded (benchmark) options are bounded in some suitable norm. This is reminiscent of the study of Stockbridge [26] which considers the superhedging problem for option pricing while limiting the short positions in the underlying and the bond. This reference gives a stochastic process interpretation of the resulting dual as well.

Now, we can state the following observation.

PROPOSITION 3 *For* $F\in\mathcal{C}$, *and* $1\leq q\leq\infty$ *we have*

(1) $\mu_*^p(F) = VC_w^q(F) - VC_b^q(F)$,
(2) $\mu_{\mathcal{Q}}(F) \leq \mu_*^p(F)$,

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof Using Lagrange duality it is immediate to verify that the convex programming dual of CWC is given by

$$\sup_{Q\in\mathcal{M}}\left\{\mathbb{E}^Q\left[\sum_{t=1}^T F_t\right] - \|(\mathbb{E}^Q[H]-C^a)_+\|_p - \|(C^b - \mathbb{E}^Q[H])_+\|_p\right\} \tag{20}$$

and that of BC is given as

$$\inf_{Q\in\mathcal{M}}\left\{\mathbb{E}^Q\left[\sum_{t=1}^T F_t\right] + \|(\mathbb{E}^Q[H]-C^a)_+\|_p + \|(C^b - \mathbb{E}^Q[H])_+\|_p\right\}, \tag{21}$$

where $\mathbb{E}^Q[H]$ is a K-vector with the i-th component equal to $\mathbb{E}^Q[\sum_{t=1}^T H_t^i]$. The easiest way to see this duality relation is to re-write, e.g. (20), first as

$$\sup_{Q\in\mathcal{M}}\left\{\mathbb{E}^Q\left[\sum_{t=1}^T F_t\right] + \inf_{\|\xi_+\|_q\leq 1,\xi_+\geq 0}\xi_+^T(C^a - \mathbb{E}^Q[H]) + \inf_{\|\xi_-\|_q\leq 1,\xi_-\geq 0}\xi_-^T(\mathbb{E}^Q[H]-C^b)\right\}$$

using a dual representation of norms where $1/p+1/q=1$ (the non-negativity of ξ_+, ξ_- arises due to the $(\cdot)_+$ operator). This is equivalent to

$$\sup_{Q\in\mathcal{M}}\inf_{\|\xi_+\|_q\leq 1,\xi_+\geq 0,\|\xi_-\|_q\leq 1,\xi_-\geq 0}\left\{\mathbb{E}^Q\left[\sum_{t=1}^T F_t\right] + \xi_+^T(C^a - \mathbb{E}^Q[H]) + \xi_-^T(\mathbb{E}^Q[H]-C^b)\right\}.$$

Using Corollary 37.3.2 of [22] we can now exchange inf and sup since the set on which inf is taken is bounded, i.e. the previous expression is equal to

$$\inf_{\|\xi_+\|_q\leq 1,\xi_+\geq 0,\|\xi_-\|_q\leq 1,\xi_-\geq 0}\sup_{Q\in\mathcal{M}}\left\{\mathbb{E}^Q\left[\sum_{t=1}^T F_t\right]+\xi_+^T(C^a-\mathbb{E}^Q[H])+\xi_-^T(\mathbb{E}^Q[H]-C^b)\right\}.$$

Now, recalling the polyhedral description of $\mathcal{M}$ from the proof of Proposition 2, and proceeding to evaluate the inner sup using linear programming duality, one obtains the dual (or, primal) problem CWC. For the other bound, one writes (21) equivalently as

$$\inf_{Q\in\mathcal{M}}\left\{\mathbb{E}^Q\left[\sum_{t=1}^T F_t\right]+\sup_{\|\xi_+\|_q\leq 1,\xi_+\geq 0}\xi_+^T(\mathbb{E}^Q[H]-C^a)+\sup_{\|\xi_-\|_q\leq 1,\xi_-\geq 0}\xi_-^T(C^b-\mathbb{E}^Q[H])\right\}.$$

The rest of the argument is similar to the one above and leads to CBC as the dual problem. This proves part 1. Part 2 now follows from the observation that the problems CWC and CBC are more tightly constrained compared to their counterparts of Section 3. ∎

Notice that for the typical choices of the norm, e.g. for $p=1$ and $p=\infty$ the writer's hedging problem becomes polyhedral convex programs:

$$\begin{aligned}&\inf V\\ \text{s.t.}\quad & S_0\cdot\theta_0+C^a\cdot\xi_+-C^b\cdot\xi_-=V\\ & S_n\cdot(\theta_n-\theta_{a(n)})=H_n\cdot(\xi_+-\xi_-)-F_n,\quad \forall n\in\mathcal{N}_t, t\geq 1\\ & S_n\cdot\theta_n\geq 0,\quad \forall n\in\mathcal{N}_T,\\ & \xi_+,\xi_-\geq 0,\\ & \|\xi_+\|_\infty\leq 1,\\ & \|\xi_-\|_\infty\leq 1,\end{aligned}$$

which is reminiscent of the Stockbridge [26] superhedging problem with finite limits on borrowing and shorting, and

$$\begin{aligned}&\inf V\\ \text{s.t}\quad & S_0\cdot\theta_0+C^a\cdot\xi_+-C^b\cdot\xi_-=V\\ & S_n\cdot(\theta_n-\theta_{a(n)})=H_n\cdot(\xi_+-\xi_-)-F_n,\quad \forall n\in\mathcal{N}_t, t\geq 1\\ & S_n\cdot\theta_n\geq 0,\quad \forall n\in\mathcal{N}_T,\\ & \xi_+,\xi_-\geq 0,\\ & \|\xi_+\|_1\leq 1,\\ & \|\xi_-\|_1\leq 1.\end{aligned}$$

Both the above problems can be transformed to linear programming problems. For the case $p=2$, we are facing the convex programming problem with Euclidean unit-ball restrictions:

$$\begin{aligned}&\inf V\\ \text{s.t.}\quad & S_0\cdot\theta_0+C^a\cdot\xi_+-C^b\cdot\xi_-=V\\ & S_n\cdot(\theta_n-\theta_{a(n)})=H_n\cdot(\xi_+-\xi_-)-F_n,\quad \forall n\in\mathcal{N}_t, t\geq 1\end{aligned}$$

$$S_n \cdot \theta_n \geq 0, \quad \forall n \in \mathcal{N}_T,$$
$$\xi_+, \xi_- \geq 0,$$
$$\|\xi_+\|_2 \leq 1,$$
$$\|\xi_-\|_2 \leq 1.$$

All three problems above are efficiently processed using available optimization methods and software.

A variation on this theme is to consider weighted versions of the penalty terms in the definition of bounds

$$\pi^*(X) = \sup_{\mathbb{Q}\in\mathcal{Q}'}\{\mathbb{E}^{\mathbb{Q}}[X] - \|W(\mathbb{E}^{\mathbb{Q}}[H] - C^a)_+\|_p - \|W(C^b - \mathbb{E}^{\mathbb{Q}}[H])_+\|_p\} \tag{22}$$

and

$$\pi_*(X) = \inf_{\mathbb{Q}\in\mathcal{Q}'}\{\mathbb{E}^{\mathbb{Q}}[X] + \|W(\mathbb{E}^{\mathbb{Q}}[H] - C^a)_+\|_p + \|W(C^b - \mathbb{E}^{\mathbb{Q}}[H])_+\|_p\}, \tag{23}$$

where W is $K \times K$ diagonal matrix with positive diagonal entries; see Section 5 of [6] for a discussion. In this case, the dual problems are simply modified as

$$\begin{aligned} &\inf V \\ \text{s.t.}\quad & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- = V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = H_n \cdot (\xi_+ - \xi_-) - F_n, \quad \forall n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n \geq 0, \quad \forall\, n \in \mathcal{N}_T, \xi_+, \xi_- \\ & \|W^{-1}\xi_+\|_q \leq 1, \\ & \|W^{-1}\xi_-\|_q \leq 1, \end{aligned}$$

for π^* and

$$\begin{aligned} &\sup V \\ \text{s.t.}\quad & S_0 \cdot \theta_0 + C^a \cdot \xi_+ - C^b \cdot \xi_- = -V \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = H_n \cdot (\xi_+ - \xi_-) + F_n, \quad \forall n \in \mathcal{N}_t, t \geq 1 \\ & S_n \cdot \theta_n \geq 0, \quad \forall n \in \mathcal{N}_T, \\ & \xi_+, \xi_- \geq 0, \\ & \|W^{-1}\xi_+\|_q \leq 1, \\ & \|W^{-1}\xi_-\|_q \leq 1 \end{aligned}$$

for π_*.

4. Numerical results

In this section, we report the results of computational work to calculate the model uncertainty measures μ_Q and μ_*^P for *S&P* 500 index options on 10 September 2002 using data and the discretization procedure from [17]. While the options used in our computational tests are all liquid options with known bid and ask prices our purposes are to demonstrate the computational viability of the approach in a practical setting, and to check that the measures of model uncertainty are in agreement with the uncertainty contained in the bid and ask prices for liquid options. We consider 48 European call-and-put options with maturities equal to 17, 37 and 100 days, respectively. The data for these 48 European call and put options are listed below in Table 1 where 'Strike' denotes the strike price and 'Maturity' the maturity date in days, C^b and C^a the bid and ask prices using the notation of Section 2.

To compute the model uncertainty measures μ_Q and μ_*^p for *S&P* 500 options, we use the *S&P* 500 index values as S^1 in the notation of Section 3. Hence, we work with the vector of traded securities $S=(1, S^1)$. We assume that the value S^1 of the *S&P* 500 index evolves as a geometric Brownian motion with daily drift d and volatility σ. Let l be the length of period t in days. Then, the logarithm $\zeta_t = \ln S_t^1$ evolves according to

$$\zeta_t = \zeta_{t-1} + d_t + \epsilon_t$$

where $d_t = l_t d$, and ϵ_t is normally distributed with zero mean and standard deviation $\sigma_t = \sqrt{l_t}\sigma$. Using given parameters ζ_0, the initial value of ζ, l_t, $t=1,\ldots,T$, d and σ, we construct a scenario tree approximation to the stochastic process ζ_t using Gauss–Hermite quadrature as advocated in [17,19,20]. The scenario tree generation procedure consists in using Gauss–Hermite quadrature to obtain a sample $(\epsilon_1^{i_1})_{i_1=1}^{\nu_1}$ of dimension ν_1 with associated positive probabilities $(\pi_1^{i_1})_{i=1}^{\nu_1}$. Hence, we obtain an approximation of possible values of the index at time $t=1$ using the equation

$$\zeta_1^{i_1} = \zeta_0 + d_1 + \epsilon_1^{i_1}, \quad i_1 = 1,\ldots,\nu_1.$$

For time period $t=2$ we generate a sample $(\epsilon_2^{i_2})_{i_2=1}^{\nu_2}$ of dimension ν_2 with associated poitive probabilities $(\pi_2^{i_2})_{i_2=1}^{\nu_2}$ to get the possible values of the logarithmic index as

$$\zeta_2^{i_1,i_2} = \zeta_1^{i_1} + d_2 + \epsilon_2^{i_2}, \quad i_1 = 1,\ldots,\nu_1,\ i_2 = 1,\ldots,\nu_2.$$

Repeating this procedure for all time points up to time T, we obtain a scenario tree where the nodes $\mathcal{N}_t$ at time t are labelled by the t-tuple $(i_1,\ldots,i_t)$. In the notation of Section 3, we have that the set $\mathcal{N}$ of all nodes in the tree are given as the union of all nodes for each time point t, i.e. $\mathcal{N}=\mathcal{N}_1\cup\cdots\cup\mathcal{N}_T$. The parent node $a(i_1,\ldots,i_t)$ of $(i_1,\ldots,i_t)$ is the node labelled $(i_1,\ldots,i_{t-1})$; the child nodes $\mathcal{D}(i_1,\ldots,i_t)$ of the node $(i_1,\ldots,i_t)$ is the set $\{(i_1,\ldots,i_{t+1})\in\mathcal{N}_{t+1} | i_{t+1}\in\{1,\ldots,\nu_{t+1}\}\}$. Finally the probability distribution P for the leaf nodes is specified as $p(i_1,\ldots,i_T) = \pi_1^{i_1}\cdots\pi_T^{i_T}$, and $S_n = e^{\zeta_n}$ for all $n\in\mathcal{N}$. This completes the specification of the scenario tree. As the number of branches ν increases, the tree converges weakly to a discrete time geometric Brownian motion as shown in [20].

We compute the model uncertainty measures for each of the 48 options using the remaining 47 options in the static buy-hold positions for calibration. More precisely, we take F to be each of the 48 options, while the remaining 47 options represent the list of benchmark derivatives H^k, $k=1,\ldots,47$ in the notation of optimization problems WC (CWC) and BC (CBC) solved 48 times each. Our trading dates are assumed to be 0, 17, 37 and 100 days, i.e. the maturity dates of the list of options. Therefore, we have a three-stage optimization model, where we set $\nu_1=50$, $\nu_2=10$ and $\nu_3=10$, resulting in 5000 scenarios. We use $d=0.0001$, $\sigma=0.013175735$ and $S_0=909.58$, which was the closing price on 10 September 2002 (Pennanen, private communication).

We programmed the scenario tree and model generation in the high level modeling language GAMS [5], and used the linear programming and convex non-linear

Table 1. Computational results with *S&P* 500 options.

Option No.	Type	Strike	Maturity	C_b	C_a	μ_Q	μ_*^2	μ_*^1
1	Call	890	17	31.5	33.5	1.02	1.02	1.02
2	Call	900	17	24.4	26.4	1.38	2.18	1.97
3	Call	905	17	21.2	23.2	1.29	1.38	1.37
4	Call	910	17	18.5	20.1	1.37	1.40	1.38
5	Call	915	17	15.8	17.4	1.43	1.43	1.43
6	Call	925	17	11.2	12.6	2.34	2.43	2.38
7	Call	935	17	7.6	8.6	1.41	1.41	1.41
8	Call	950	17	3.8	4.6	1.40	1.61	1.57
9	Call	955	17	3	3.7	0.90	0.90	0.90
10	Call	975	17	0.95	1.45	1.01	1.01	1.01
11	Call	980	17	0.65	1.15	0.78	0.78	0.78
12	Call	900	37	42.3	44.3	2.00	2.00	2.00
13	Call	925	37	28.2	29.6	2.00	2.00	2.00
14	Call	950	37	17.5	19	5.16	6.24	5.96
15	Call	875	100	77.1	79.1	2.00	2.00	2.00
16	Call	900	100	61.6	63.6	2.00	2.00	2.00
17	Call	950	100	35.8	37.8	7.02	7.58	7.02
18	Call	975	100	26	28	5.03	5.17	5.12
19	Call	995	100	19.9	21.5	4.75	4.75	4.75
20	Call	1025	100	12.6	14.2	8.42	8.76	8.6
21	Call	1100	100	3.4	3.8	12.80	12.80	12.80
22	Put	750	17	0.4	0.6	1.15	1.15	1.15
23	Put	790	17	1	1.3	0.57	0.57	0.57
24	Put	800	17	1.3	1.65	0.58	0.58	0.58
25	Put	825	17	2.5	2.85	0.68	0.68	0.68
26	Put	830	17	2.6	3.1	0.41	0.41	0.41
27	Put	840	17	3.4	3.8	0.70	0.71	0.71
28	Put	850	17	3.9	4.7	0.40	0.40	0.40
29	Put	860	17	5.5	5.8	1.33	1.33	1.33
30	Put	875	17	7.2	7.8	1.18	1.22	1.35
31	Put	885	17	9.4	10.4	0.91	0.92	0.92
32	Put	750	37	5.5	5.9	2.84	4.09	4.07
33	Put	775	37	6.9	7.7	1.62	1.69	1.67
34	Put	800	37	9.3	10	3.33	3.52	3.50
35	Put	850	37	16.7	18.3	6.04	6.39	6.08
36	Put	875	37	23	24.3	3.88	4.58	3.98
37	Put	900	37	31	33	1.33	1.33	1.33
38	Put	925	37	41.8	43.8	1.40	1.40	1.40
39	Put	975	37	73	75	4.67	5.78	5.26
40	Put	995	37	88.9	90.9	6.99	6.99	6.99
41	Put	650	100	5.7	6.7	5.98	6.98	6.97
42	Put	700	100	9.2	10.2	4.60	4.70	4.63
43	Put	750	100	14.7	15.8	4.40	4.57	4.43
44	Put	775	100	17.6	19.2	2.80	3.42	3.33
45	Put	800	100	21.7	23.7	4.50	5.00	4.72
46	Put	850	100	33.3	35.3	3.76	5.92	5.38
47	Put	875	100	40.9	42.9	1.38	1.38	1.38
48	Put	900	100	50.3	52.3	2.00	2.00	2.00

programming solvers available through GAMS. We report the results in Table 1 in the three rightmost columns with headings μ_Q, μ_*^2 and μ_*^1, respectively. To compute μ_*^2 for each option we solve two non-linear convex programming problems using the non-linear programming solver CONOPT [8]. We solve four linear programs to compute the measures μ_Q, and μ_*^1 for each option using CPLEX [7]. Each optimization problem has approximately 10,500 constraints and 11,200 variables with slight variations among different models. The optimization problems are typically solved on the average in few minutes, and at most within 10 min of computing time. It is reassuring to note that the measures of model uncertainty, while sometimes exceeding the difference between the ask-and-bid price given for the option, in most cases (roughly two-thirds of the 48 cases) remain close to this value, except for those options that are deep out-of-the-money, e.g. call options numbered 17–21, and put options numbered 41 and 42. A possible explanation for this phenomenon could be that the number of options available to hedge a given option may not constitute a good enough hedge for its cash flows. A similar observation is made in [17]. This phenomenon occurs for instance in option number 21 to which a good hedge among other options cannot be found. Tight measures of model uncertainty seem to be possible for a given option when there are several options available for hedging with strikes and maturities close to the strike and maturity of the option in question. For instance, options numbered 1–5 yield measures of model uncertainty very close to the difference between ask and bid prices by finding good hedges using similar options, e.g. options 2, 28 and 30 for option 1. For hedging option 2, options 1, 3, 4, 7, 29 and 30 are used, and so forth.

As predicted by Proposition 3, the uncertainty measures based on convex risk measures μ_*^2, μ_*^1 are at least as large as the measure μ_Q while μ_*^2 may be slightly larger than μ_*^1 in some cases.

5. Conclusions

In this article, we proved the relationship between two measures of model uncertainty introduced by Cont [6] in order to quantify the ambiguity inherent in contingent claim prices in incomplete prices and calibrated option bounds of King et al. [17]. We developed our results in a discrete time, finite state probability framework in order to take advantage of convex and linear programming duality theory. We demonstrated the computational feasibility of computing the measures of model uncertainty using the calibrated option bounds on *S&P* 500 options used as a benchmark. This computational approach is general enough to accommodate different forms of security price processes, and is numerically reliable and efficient. It is our hope that the computational framework of this study may lead to further, similar studies in pricing contingent claims in incomplete markets using numerical optimization tools.

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