

SOLUTIONS OF THE EXTENDED KADOMTSEV– PETVIASHVILI–BOUSSINESQ EQUATION BY THE HIROTA DIRECT METHOD

ASLI PEKCAN

*Department of Mathematics, Faculty of Science
Bilkent University, 06800 Ankara, Turkey
asli@fen.bilkent.edu.tr*

Received 29 January 2007

Accepted 29 August 2008

We show that we can apply the Hirota direct method to some non-integrable equations. For this purpose, we consider the extended Kadomtsev–Petviashvili–Boussinesq (eKPBo) equation with M variable which is

$$(u_{xxx} - 6uu_x)_x + a_{11}u_{xx} + 2 \sum_{k=2}^M a_{1k}u_{xxk} + \sum_{i,j=2}^M a_{ij}u_{x_ix_j} = 0,$$

where $a_{ij} = a_{ji}$ are constants and $x_i = (x, t, y, z, \dots, x_M)$. We will give the results for $M = 3$ and a detailed work on this equation for $M = 4$. Then we will generalize the results for any integer $M > 4$.

Keywords: The Hirota direct method; non-integrable equations; exact solutions; solitons; Kadomtsev–Petviashvili equation; Boussinesq equation.

1. Introduction

The Hirota direct method is one of the famous method to construct multi-soliton solutions of integrable nonlinear partial differential equations. Hirota gave the exact solution of Korteweg-de Vries (KdV) equation for multiple collisions of solitons by using the Hirota direct method in 1971 [1]. In his successive articles, he dealt with many other nonlinear evolution equations such as modified Korteweg-de Vries (mKdV) [2], sine-Gordon (sG) [3], nonlinear Schrödinger (nLS) [4] and Toda lattice (TL) [5] equations. Hirota method was also applied to Kadomtsev–Petviashvili (KP) and Boussinesq (Bo) equations. KP equation is

$$(u_{xxx} - 6uu_x)_x + u_{xt} + 3u_{yy} = 0 \quad (1.1)$$

and it has been solved in [6]. Bo equation which is

$$(u_{xxx} - 6uu_x)_x + u_{xx} - u_{tt} = 0 \quad (1.2)$$

has been solved by again Hirota [7]. Both of these equations are in the class of KdV-type equations.

The first step of the Hirota direct method is to transform the nonlinear partial differential or difference equation into a quadratic form in dependent variables. The new form of the equation is called “bilinear form”. In the second step, we write the bilinear form of the equation as a polynomial of a special differential operator called Hirota D-operator. This polynomial of D-operator is called “Hirota bilinear form”. In fact, some equations may not be written in Hirota bilinear form but perhaps in trilinear or multilinear forms [8]. The last step of the method is using the finite perturbation

expansion in Hirota bilinear form. The coefficients of the perturbation parameter and its powers are analyzed separately. Depending upon the finite perturbation expansion one finds one-, two-, ... and N -soliton solutions.

The KdV-type equations which have Hirota bilinear form possess one- and two-soliton solutions [9] automatically. The first difficulty arises at three-soliton solutions. In order that an equation to have three-soliton solution, it should satisfy certain condition, called three-soliton solution condition. This condition was used as a powerful tool to search the integrability of the equations by Hietarinta [10]. Hietarinta also used this condition to produce new integrable equations in his articles [9, 11–13]. KP and Bo satisfy such condition immediately. For some of the equations although they have Hirota bilinear form, this condition is not satisfied directly. Three-soliton solution condition can be used to find exact solutions of such differential equations.

In this work we will consider a KdV-type equation unifying KP and Bo. A simple form of such an equation was first considered by Johnson [14]. Johnson analyzed the equation

$$(u_{xxx} - 6uu_x)_x + u_{xx} - u_{tt} + u_{yy} = 0, \quad (1.3)$$

which he called the two dimensional Boussinesq equation. It is introduced to describe the wave propagation of gravity waves on the surface of the water of constant depth. This equation has one- and two-soliton and resonant solutions. Also, even the two dimensional Boussinesq equation does not have distributed solution, under some transformations and assumptions on its parameters it can be transformable to KP which has distributed solution.

Here we further generalize Johnson's equation as

$$(u_{xxx} - 6uu_x)_x + a_{11}u_{xx} + 2 \sum_{k=2}^M a_{1k}u_{xx_k} + \sum_{i,j=2}^M a_{ij}u_{x_i x_j} = 0, \quad (1.4)$$

where $a_{ij} = a_{ji}$ are constants and $x_i = (x, t, y, z, \dots, x_M)$. We call this equation as M -dimensional extended Kadomtsev–Petviashvili–Boussinesq (eKPBo) equation. Here we will analyze (1.4) for $M = 3$, $M = 4$ and for any integer $M > 4$.

2. $M = 3$, Three Dimensional eKPBo

Three dimensional eKPBo is

$$(u_{xxx} - 6uu_x)_x + a_{11}u_{xx} + 2a_{12}u_{xt} + 2a_{13}u_{xy} + a_{22}u_{tt} + 2a_{23}u_{ty} + a_{33}u_{yy} = 0. \quad (2.1)$$

The second line of the equation can be simplified by letting

$$\begin{aligned} t' &= a_1 t + b_1 y, \\ y' &= a_2 t + b_2 y, \end{aligned} \quad (2.2)$$

where a_1, b_1, a_2 and b_2 are some constants. Then the equation becomes

$$(u_{xxx} - 6uu_x)_x + a_{11}u_{xx} + 2\bar{a}_{12}u_{xt'} + 2\bar{a}_{13}u_{xy'} + \bar{a}_{22}u_{t't'} + 2\bar{a}_{23}u_{t'y'} + \bar{a}_{33}u_{y'y'} = 0, \quad (2.3)$$

where

$$\begin{aligned} \bar{a}_{12} &= a_1 a_{12} + b_1 a_{13}, \\ \bar{a}_{13} &= a_2 a_{12} + b_2 a_{13}, \\ \bar{a}_{22} &= a_1^2 a_{22} + 2a_1 b_1 a_{23} + b_1^2 a_{33}, \\ \bar{a}_{23} &= a_1 a_2 a_{22} + a_2 b_1 a_{23} + a_1 b_2 a_{23} + b_1 b_2 a_{33}, \\ \bar{a}_{33} &= a_2^2 a_{22} + 2a_2 b_2 a_{23} + b_2^2 a_{33}. \end{aligned} \quad (2.4)$$

Under the conditions $a_{23}^2 = a_{22}a_{33}$ and $\frac{a_1}{b_1} = -\frac{a_{23}}{a_{22}}$ we have $\bar{a}_{22} = \bar{a}_{23} = 0$ so (2.3) turns out to be

$$(u_{xxx} - 6uu_x)_x + a_{11}u_{xx} + 2\bar{a}_{12}u_{xt'} + 2\bar{a}_{13}u_{xy'} + \bar{a}_{33}u_{y'y'} = 0. \quad (2.5)$$

This equation can be transformable to KP. If we consider $a_{23}^2 = a_{22}a_{33}$ and $\frac{a_2}{b_2} = -\frac{a_{33}}{a_{23}}$ we have $\bar{a}_{23} = \bar{a}_{33} = 0$ so (2.3) becomes

$$(u_{xxx} - 6uu_x)_x + a_{11}u_{xx} + 2\bar{a}_{12}u_{xt'} + 2\bar{a}_{13}u_{xy'} + \bar{a}_{22}u_{t't'} = 0. \quad (2.6)$$

This is equivalent to KP if $\bar{a}_{12} \neq 0$ and $\bar{a}_{13} \neq 0$. If they are zero then the equation becomes Bo.

Lemma 1. For $M = 3$, if we have the condition $a_{23}^2 = a_{22}a_{33}$, then Eq. (1.4) can be transformable to either KP or Bo.

Now we will give the application of the Hirota method on four dimensional eKPBo.

3. $M = 4$, Four Dimensional EKPBo

Here we apply the Hirota method by using the properties of Hirota D-operator and steps given in [15] to Eq. (1.4) with four variables.

Step 1. Bilinearization: We bilinearize the equation i.e. transform it to a quadratic form in dependent variable by the transformation

$$u(x, t, y, z) = -2\partial_x^2 \log f(x, t, y, z), \quad (3.1)$$

so the bilinear form of the equation is

$$f_{xxxx}f - 4f_x f_{xxx} + 3f_{xx}^2 + \sum_{i,j=1}^4 a_{ij}(ff_{x_i x_j} - f_{x_i} f_{x_j}) = 0. \quad (3.2)$$

Step 2. Transformation to the Hirota bilinear form: We use Hirota D-operator which is simply defined as

$$\begin{aligned} D_{x_1} D_{x_2} \{f \cdot f\} &= (\partial_{x_1} - \partial_{x'_1})(\partial_{x_2} - \partial_{x'_2})f(x_1, x_2) \cdot f(x'_1, x'_2) \quad (x_1 = x'_1, x_2 = x'_2) \\ &= 2(ff_{x_1 x_2} - f_{x_1} f_{x_2}). \end{aligned} \quad (3.3)$$

By using some sort of combination of D-operator we write Hirota bilinear form of the equation as

$$P(D)\{f \cdot f\} = \left(D_x^4 + \sum_{i,j=1}^M a_{ij} D_{x_i x_j} \right) \{f \cdot f\} = 0, \quad (3.4)$$

for $M = 4$.

Step 3. Application of the Hirota perturbation: We insert $f = 1 + \sum_{n=1}^N \varepsilon^n f_n$ into Eq. (3.4) so we have

$$P(D)\{f \cdot f\} = P(D)\{1.1\} + \varepsilon P(D)\{f_1.1 + 1.f_1\} + \cdots + \varepsilon^{2N} P(D)\{f_N.f_N\} = 0. \quad (3.5)$$

Here ε is a constant called the perturbation parameter.

Step 4. Examination of the coefficients of the perturbation parameter ε : We make the coefficients of ε^m , $m = 1, 2, \dots, N$ appeared in (3.5) to vanish. Here we shall consider only the case $N = 3$. Note that since the equation is not integrable except for some conditions, we call the solutions obtained by using the Hirota method as restricted N -soliton solution of the equation for $N \geq 3$. Before passing

to restricted three-soliton solution of eKPBo with four variables, let us give one- and two-soliton solutions of it. One-soliton solution of eKPBo is

$$u(x, t, y, z) = -\frac{k_1^2}{2\cosh^2(\frac{\theta_1}{2})}. \tag{3.6}$$

Here $\theta_1 = (l_1)_1x + (l_2)_1t + (l_3)_1y + (l_4)_1z + \alpha_1$ where we denote $l_1 = k$. The constants $k_1, (l_2)_1, (l_3)_1$ and $(l_4)_1$ satisfy $k_1^4 + \sum_{i,j=1}^4 a_{ij}(l_i)_1(l_j)_1 = 0$. Two-soliton solution of eKPBo is

$$u(x, t, y, z) = -2\frac{E(x, t, y, z)}{F(x, t, y, z)},$$

where

$$E(x, t, y, z) = k_1^2e^{\theta_1} + k_2^2e^{\theta_2} + [(k_1 - k_2)^2 + A(1, 2)((k_1 + k_2)^2 + k_1^2e^{\theta_2} + k_2^2e^{\theta_1})]e^{\theta_1+\theta_2}$$

and

$$F(x, t, y, z) = (1 + e^{\theta_1} + e^{\theta_2} + A(1, 2)e^{\theta_1+\theta_2})^2$$

for $\theta_n = k_nx + (l_2)_nt + (l_3)_ny + (l_4)_nz + \alpha_n, n = 1, 2$ and $A(1, 2) = R(1, 2)/S(1, 2)$ where,

$$R(1, 2) = (k_1 - k_2)^4 + \sum_{i,j=1}^4 a_{ij}[(l_i)_1 - (l_i)_2][(l_j)_1 - (l_j)_2],$$

$$S(1, 2) = (k_1 + k_2)^4 + \sum_{i,j=1}^4 a_{ij}[(l_i)_1 + (l_i)_2][(l_j)_1 + (l_j)_2].$$

Now we apply the Hirota direct method to four dimensional eKPBo with the anzats which is used to construct three-soliton solutions. We take

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3,$$

where $f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$ with $\theta_n = (l_1)_nx + (l_2)_nt + (l_3)_ny + (l_4)_nz + \alpha_n$ where $l_1 = k$ for $n = 1, 2, 3$ and insert it into (3.5). The coefficient of ε^0 is identically zero. By the coefficient of ε^1 , we have the relation

$$P(\vec{p}_n) = k_n^4 + \sum_{i,j=1}^4 a_{ij}(l_i)_n(l_j)_n = 0, \tag{3.7}$$

where $\vec{p}_n = (k_n, (l_2)_n, (l_3)_n, (l_4)_n)$ for $n = 1, 2, 3$. This relation is called the dispersion relation. From the coefficient of ε^2 we get

$$-P(\partial)f_2 = \sum_{n<m}^{(3)} e^{\theta_n+\theta_m} \left\{ (k_n - k_m)^4 + \sum_{i,j=1}^4 a_{ij}[(l_i)_n - (l_i)_m][(l_j)_n - (l_j)_m] \right\}, \tag{3.8}$$

where (3) indicates the summation of all possible combinations of the three elements with $n < m$ for $n, m = 1, 2, 3, 4$. Thus to satisfy the equation, f_2 should be of the form

$$f_2 = A(1, 2)e^{\theta_1+\theta_2} + A(1, 3)e^{\theta_1+\theta_3} + A(2, 3)e^{\theta_2+\theta_3}. \tag{3.9}$$

We insert f_2 into Eq. (3.8) so we get $A(n, m)$ as

$$A(n, m) = -\frac{P(\vec{p}_n - \vec{p}_m)}{P(\vec{p}_n + \vec{p}_m)}, \tag{3.10}$$

where $n, m = 1, 2, 3, 4$ with $n < m$. From the coefficient of ε^3 we get

$$-P(\partial)\{f_3\} = e^{\theta_1+\theta_2+\theta_3} [A(1, 2)P(\vec{p}_3 - \vec{p}_2 - \vec{p}_1) + A(1, 3)P(\vec{p}_2 - \vec{p}_1 - \vec{p}_3) + A(2, 3)P(\vec{p}_1 - \vec{p}_2 - \vec{p}_3)]. \tag{3.11}$$

Hence f_3 is of the form $f_3 = Be^{\theta_1+\theta_2+\theta_3}$ where B is found as

$$B = -[A(1, 2)P(\vec{p}_3 - \vec{p}_1 - \vec{p}_2) + A(1, 3)P(\vec{p}_2 - \vec{p}_1 - \vec{p}_3) + A(2, 3)P(\vec{p}_1 - \vec{p}_2 - \vec{p}_3)]/P(\vec{p}_1 + \vec{p}_2 + \vec{p}_3). \tag{3.12}$$

The coefficient of ε^4 gives us the coefficient B as

$$B = A(1, 2)A(1, 3)A(2, 3). \tag{3.13}$$

To be consistent, the two expressions for B should be equivalent. This is satisfied when the following condition holds:

$$\begin{aligned} &P(\vec{p}_1 - \vec{p}_2)P(\vec{p}_1 - \vec{p}_3)P(\vec{p}_2 - \vec{p}_3)P(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \\ &+ P(\vec{p}_1 - \vec{p}_2)P(\vec{p}_1 + \vec{p}_3)P(\vec{p}_2 + \vec{p}_3)P(\vec{p}_3 - \vec{p}_1 - \vec{p}_2) \\ &+ P(\vec{p}_1 - \vec{p}_3)P(\vec{p}_1 + \vec{p}_2)P(\vec{p}_2 + \vec{p}_3)P(\vec{p}_2 - \vec{p}_1 - \vec{p}_3) \\ &+ P(\vec{p}_2 - \vec{p}_3)P(\vec{p}_1 + \vec{p}_2)P(\vec{p}_1 + \vec{p}_3)P(\vec{p}_1 - \vec{p}_2 - \vec{p}_3) = 0. \end{aligned}$$

This condition which we call restricted three-soliton solution condition (*R3SC*) can also be written as

$$\sum_{\sigma_r=\pm 1} P(\sigma_1\vec{p}_1 + \sigma_2\vec{p}_2 + \sigma_3\vec{p}_3)P(\sigma_1\vec{p}_1 - \sigma_2\vec{p}_2)P(\sigma_2\vec{p}_2 - \sigma_3\vec{p}_3)P(\sigma_1\vec{p}_1 - \sigma_3\vec{p}_3) = 0, \tag{3.14}$$

for $r = 1, 2, 3$. After some simplifications (*R3SC*) for four dimensional eKPBo turns out to be

$$\begin{aligned} &k_1^2 k_2^2 k_3^2 [(a_{22}a_{33} - a_{23}^2) \det(K, L_2, L_3)^2 + (a_{22}a_{44} - a_{24}^2) \det(K, L_2, L_4)^2 \\ &+ (a_{33}a_{44} - a_{34}^2) \det(K, L_3, L_4)^2 + 2(a_{22}a_{34} - a_{23}a_{24}) \det(K, L_2, L_4) \det(K, L_2, L_3) \\ &+ 2(a_{33}a_{24} - a_{23}a_{34}) \det(K, L_2, L_3) \det(K, L_4, L_3) \\ &+ 2(a_{44}a_{23} - a_{24}a_{34}) \det(K, L_4, L_2) \det(K, L_4, L_3)] = 0, \end{aligned} \tag{3.15}$$

where $K = (k_1, k_2, k_3)^T$ and $L_r = ((l_r)_1, (l_r)_2, (l_r)_3)^T$ for $r = 2, 3, 4$. In compact form, we can write the above equation as

$$k_1^2 k_2^2 k_3^2 \sum_{i,j,m,n=2}^4 a_{ij} a_{mn} \det(K, L_i, L_m) \det(K, L_j, L_n) = 0, \tag{3.16}$$

for $i \neq m, j \neq n$. Note that $a_{i_1 i_2} = a_{i_2 i_1}$ for $i_1, i_2 = 2, 3, 4$.

Finally, the coefficients of ε^5 and ε^6 vanish trivially. We have completed the application of the method and found the function f as

$$f = 1 + \varepsilon(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) + \varepsilon^2 \left(\sum_{i,j=1}^3 A_{i,j} e^{\theta_i+\theta_j} \right) + \varepsilon^3 (Be^{\theta_1+\theta_2+\theta_3}), \tag{3.17}$$

where $i < j$. Without loss of generality, we set $\varepsilon = 1$. Then by using (3.1) with this f we get a restricted (by (3.16)) three-soliton solution.

4. Restricted Three-Soliton Solution Conditions

Even though we have given the application of the Hirota method only for four dimensional eKPBo in detail, it is not hard to see the facts for eKPBo with $M = 3$ and $M > 4$ variables. Here we will give restricted three-soliton solution conditions for eKPBo and we analyze the cases that these conditions are satisfied.

(a) $M = 3$ variables:

Restricted three-soliton solution condition of three dimensional eKPBo equation is

$$k_1^2 k_2^2 k_3^2 (a_{22} a_{33} - a_{23}^2) \det(K, L_2, L_3) = 0, \tag{4.1}$$

where $K = (k_1, k_2, k_3)^T$ and $L_r = ((l_r)_1, (l_r)_2, (l_r)_3)^T$ for $r = 2, 3$. As we see this condition satisfied when $a_{23}^2 = a_{22} a_{33}$. This relation makes three dimensional eKPBo transformable to either integrable KP or Bo equations. Except this case, we do not have integrable equations. Other cases satisfying (4.1) are

Case 1. Any one of $k_i = 0$, $i = 1, 2, 3$, the rest are different.

Case 2. The parameter vectors (K, L_2, L_3) are linearly dependent.

(b) $M = 4$ variables:

Restricted three-soliton solution condition (3.16) is equivalent to

$$k_1^2 k_2^2 k_3^2 \sum_{i,j,m,n=2}^4 C_{ij} \det(K, L_i, L_m) \det(K, L_j, L_n) = 0, \tag{4.2}$$

where C_{ij} 's are the cofactors of the coefficient matrix

$$\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Let us denote $\det(K, L_i, L_j) = \epsilon_{ijk} \rho_k$ for $i, j, k = 2, 3, 4$ where ϵ_{ijk} is Levi-Civita symbol. It is possible to write (4.2) as

$$\begin{aligned} k_1^2 k_2^2 k_3^2 \sum_{i,j=2}^4 C_{ij} (\epsilon_{imk} \rho_k) (\epsilon_{jml} \rho_l) &= k_1^2 k_2^2 k_3^2 \sum_{i,j=2}^4 C_{ij} \rho_k \rho_l \epsilon_{imk} \epsilon_{jml} \\ &= k_1^2 k_2^2 k_3^2 \sum_{i,j=2}^4 \overline{C}_{ij} \rho_i \rho_j = 0, \end{aligned} \tag{4.3}$$

where $m, k, l = 2, 3, 4$, C is the matrix of C_{ij} , $\overline{C}_{ij} = C_{ij} - \text{tr}(C) \delta_{ij}$ and δ_{ij} is the Kronecker delta, $i, j = 2, 3, 4$.

Example. Let us take $a_{ij} = \delta_{ij}$. In this case, $C_{ij} = \delta_{ij}$. So $\overline{C}_{ij} = -2\delta_{ij}$. As we see to satisfy (4.3) we should have $\rho_i^2 = 0$ so $\rho_i = 0$ for any $i = 2, 3, 4$.

The cases that (4.3) is satisfied are the followings:

Case 1. Any one of $k_i = 0$, $i = 1, 2, 3$, the rest are different.

Case 2. Let \overline{C} be the matrix of \overline{C}_{ij} . Suppose that \overline{C} is a nonnegative (nonpositive) matrix i.e. eigenvalues of the matrix are all positive (negative) or zero. Then (4.3) is satisfied when $\rho_i = 0$ for any $i = 2, 3, 4$. This implies that $\det(K, L_i, L_j) = 0$, $i, j = 2, 3, 4$ or the parameter vectors (K, L_2, L_3, L_4) are parallel in one of such ways:

- (i) All vectors are parallel,
- (ii) L_i is parallel to L_j and L_m is parallel to L_n $i, j, m, n = 1, 2, 3, 4$, where all the indices are different (Note that we denote $L_1 = k$),
- (iii) Only three of the parameter vectors are parallel to each other.

(c) M variables, $M > 4$:

For eKPBo equation with M variables we have (R3SC) similar to (3.16),

$$k_1^2 k_2^2 k_3^2 \sum_{i,j,m,n=2}^M a_{ij} a_{mn} \det(K, L_i, L_m) \det(K, L_j, L_n) = 0, \quad (4.4)$$

where $i \neq m, j \neq n, a_{i_1 i_2} = a_{i_2 i_1}$ for $i_1, i_2 = 2, 3, \dots, M$, $K = (k_1, k_2, k_3)^T$ and $L_r = ((l_r)_1, (l_r)_2, (l_r)_3)^T$ for $r = 2, 3, \dots, M$.

Case 1. Any one of $k_i = 0, i = 1, 2, 3$, the rest are different.

Case 2. Let us consider the set of parameter vectors $(K, L_2, L_3, \dots, L_M)$ as a union of two disjoint subsets. If all vectors belonging to the same subset are parallel to each other, then the condition (4.4) is satisfied.

Case 3. All the parameter vectors $(K, L_2, L_3, \dots, L_M)$ are parallel to each other.

Remark. When **Case 2** is satisfied for any $M \geq 3$, then the solution of (1.4) becomes two-dimensional. If we have **Case 3** then solution turns to be one-dimensional.

5. Restricted Three-Soliton Solution of EKPBo

Application of the Hirota direct method to eKPBo gives us the functions $f_i, i = 1, 2, 3$ in $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3$ and some additional conditions. Finally we take $\varepsilon = 1$ and insert f into the bilinearizing transformation. Hence restricted three-soliton solution of M dimensional eKPBo with the condition (4.4) satisfied takes the form

$$u(x, t, y, z, \dots, x_M) = -2 \frac{T(x, t, y, z, \dots, x_M)}{V(x, t, y, z, \dots, x_M)},$$

where

$$\begin{aligned} T(x, t, y, z, \dots, x_M) = & k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + k_3^2 e^{\theta_3} \\ & + e^{2\theta_1 + \theta_2 + \theta_3} [A(1, 2)A(1, 3)(k_2 - k_3)^2 + B(k_2 + k_3)^2] \\ & + e^{\theta_1 + \theta_2 + 2\theta_3} [A(1, 3)A(2, 3)(k_1 - k_2)^2 + B(k_1 + k_2)^2] \\ & + e^{\theta_1 + 2\theta_2 + \theta_3} [A(1, 2)A(2, 3)(k_1 - k_3)^2 + B(k_1 + k_3)^2] \\ & + e^{\theta_1 + \theta_2} [(k_1 - k_2)^2 + A(1, 2)(k_1^2 e^{\theta_2} + k_2^2 e^{\theta_1} + (k_1 + k_2)^2)] \\ & + e^{\theta_1 + \theta_3} [(k_1 - k_3)^2 + A(1, 3)(k_1^2 e^{\theta_3} + k_3^2 e^{\theta_1} + (k_1 + k_3)^2)] \\ & + e^{\theta_2 + \theta_3} [(k_2 - k_3)^2 + A(2, 3)(k_2^2 e^{\theta_3} + k_3^2 e^{\theta_2} + (k_2 + k_3)^2)] \\ & + e^{\theta_1 + \theta_2 + \theta_3} [A(1, 2)(k_1^2 + k_2^2 + k_3^2 + 2k_1 k_2 - 2k_1 k_3 - 2k_2 k_3) \\ & + A(1, 3)(k_1^2 + k_2^2 + k_3^2 + 2k_1 k_3 - 2k_1 k_2 - 2k_2 k_3) \\ & + A(2, 3)(k_1^2 + k_2^2 + k_3^2 + 2k_2 k_3 - 2k_1 k_2 - 2k_1 k_3) \\ & + B(k_1^2 + k_2^2 + k_3^2 + 2k_1 k_2 + 2k_1 k_3 + 2k_2 k_3) \\ & + B(A(1, 2)k_3^2 e^{\theta_1 + \theta_2} + A(1, 3)k_2^2 e^{\theta_1 + \theta_3} + A(2, 3)k_1^2 e^{\theta_2 + \theta_3})] \end{aligned}$$

and

$$\begin{aligned} V(x, t, y, z, \dots, x_M) = & [1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1, 2)e^{\theta_1 + \theta_2} \\ & + A(1, 3)e^{\theta_1 + \theta_3} + A(2, 3)e^{\theta_2 + \theta_3} + B e^{\theta_1 + \theta_2 + \theta_3}]^2. \end{aligned}$$

Here $\theta_n = k_n x + \sum_{i=2}^M (l_i)_n x_i, n = 1, 2, 3$ and

$$A(i, j) = -\frac{(k_i - k_j)^4 + \sum_{i,j=1}^3 [(l_n)_i - (l_n)_j][(l_m)_i - (l_m)_j]}{(k_i + k_j)^4 + \sum_{i,j=1}^3 [(l_n)_i + (l_n)_j][(l_m)_i + (l_m)_j]} \quad (5.1)$$

with $l_1 = k$ and $i, j = 1, 2, 3$ for $i < j, n, m = 1, 2, \dots, M$ and B is as in (3.13).

6. Explicit Solutions of EKPBo

Here, for illustration, we show the graphs of the solutions of the equation

$$(u_{xxx} - 6uu_x)_x + u_{xx} + u_{ty} - u_{yy} = 0. \quad (6.1)$$

The Eq. (6.1) is one of the three-dimensional eKPBo equations. We give the graphs of restricted three-soliton solutions of this equation. In order to determine the constants k_i , w_i and l_i we use the

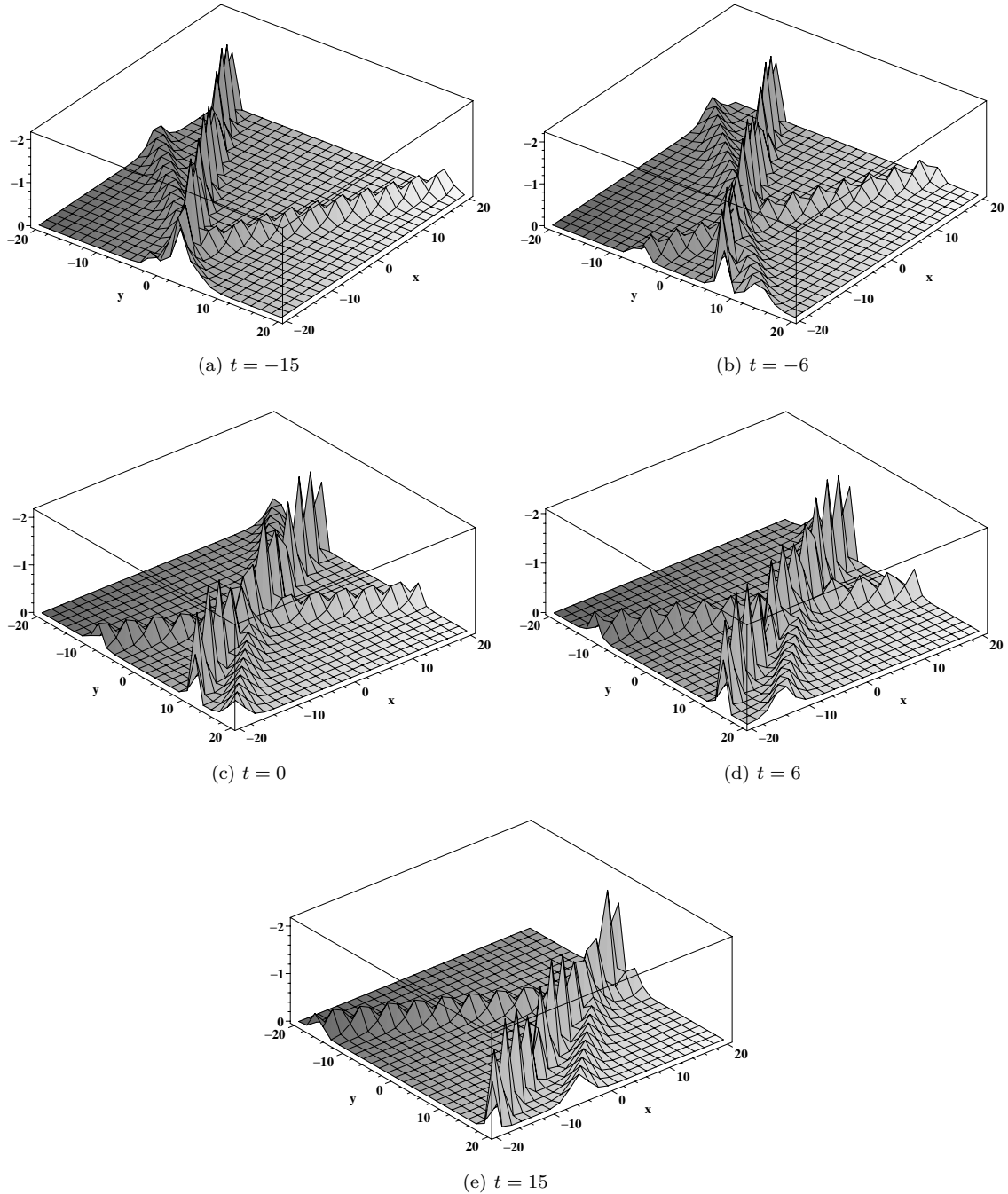


Fig. 1. The behavior of restricted three-soliton solution at different times.

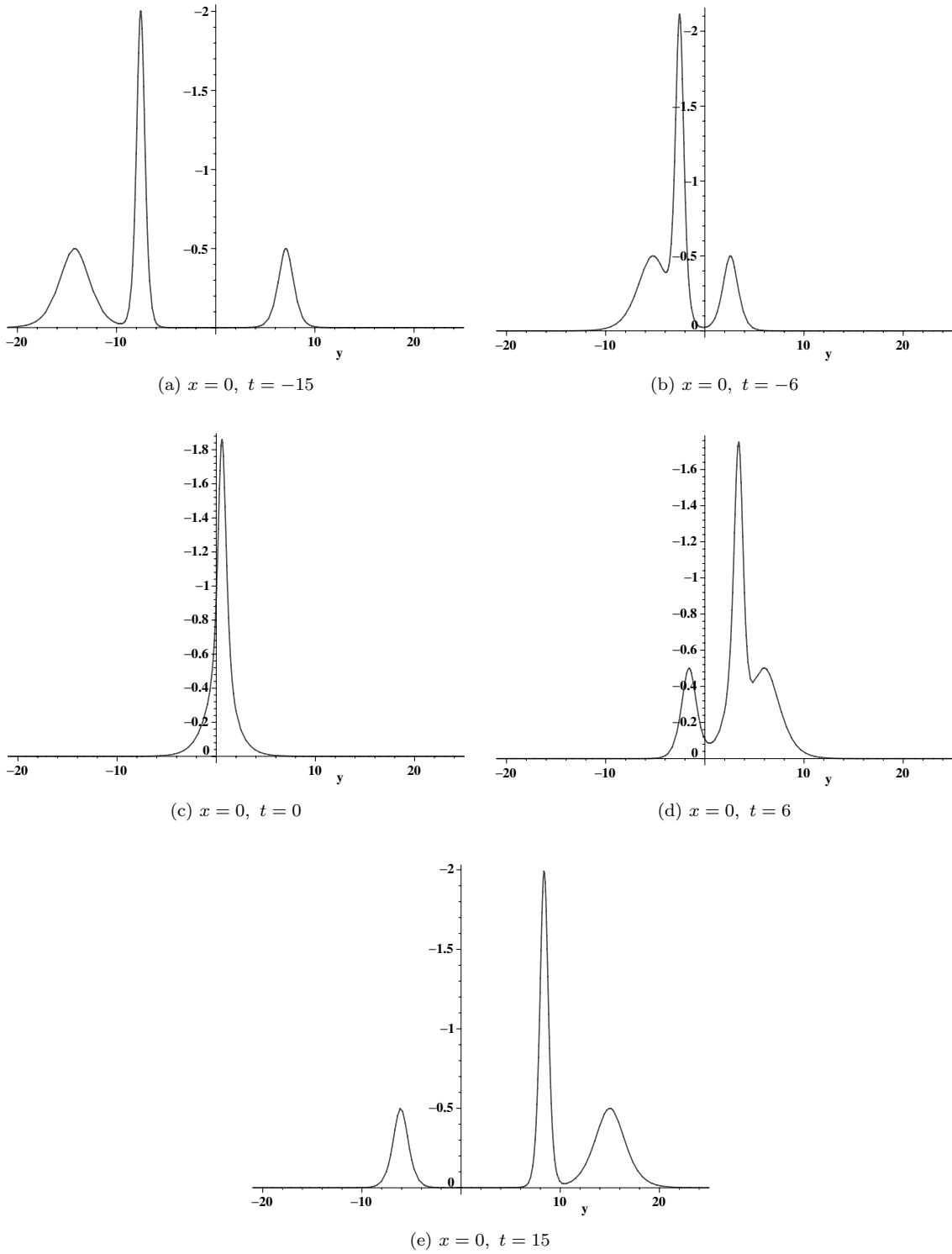


Fig. 2. The projection of the graphs in Fig. 1 with $x = 0$.

dispersion relation of (6.1) that is

$$k_i^4 + k_i^2 + w_i l_i - l_i^2 = 0, \quad i = 1, 2, 3, \quad (6.2)$$

and (R3SC) given in (4.1). We present the graphs in two groups. The first group that consists Figs. 1 and 2 is plotted due to the **Case 2**. Figures 3 and 4, which constitute the second group are plotted due to the **Case 1** of restricted three-soliton solution conditions for three-dimensional

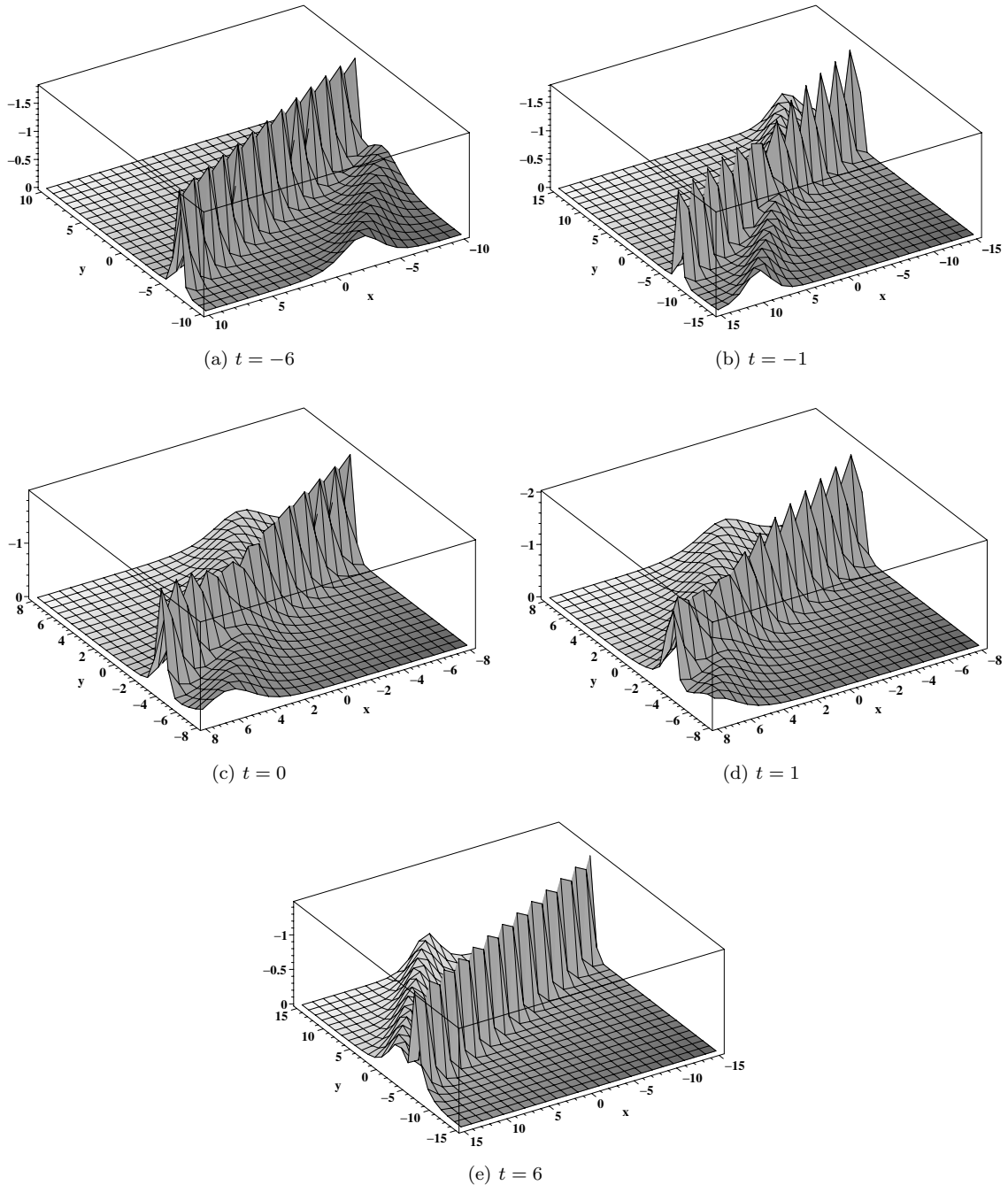


Fig. 3. Three-dimensional graphs of restricted three-soliton solution of eKPBo with $k_1 = 0$.

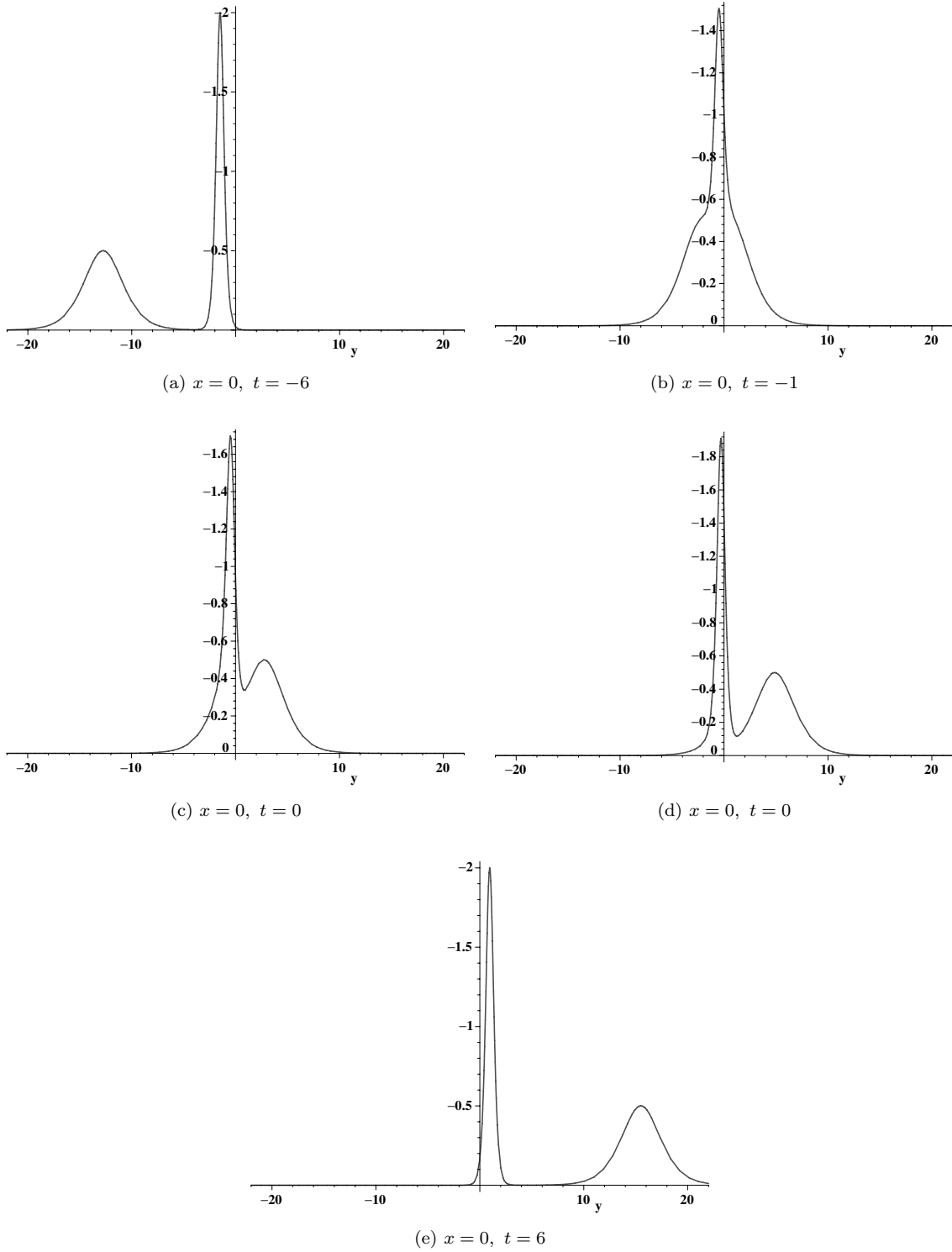


Fig. 4. The projection of the graphs in Fig. 3 with $x = 0$.

eKPBo. According to these, we determine the constants for the first group as

$$\begin{aligned}k_1 &= 1, k_2 = 1, k_3 = -2, \\w_1 &= -1, w_2 = -1, w_3 = 2, \\l_1 &= -2, l_2 = 1, l_3 = 1 - \sqrt{21}.\end{aligned}$$

In Fig. 1, we note that our solution does not seem to have solitonic behavior. But in Fig. 2, when the graphs are projected, we see the perfect movements of three waves. Indeed, they have solitonic property.

Now we pass to the second group of graphs. The constants k_i, w_i and $l_i, i = 1, 2, 3$ are

$$\begin{aligned}k_1 &= 0, k_2 = 2, k_3 = -1, \\w_1 &= -2, w_2 = -1, w_3 = 17/10, \\l_1 &= -2, l_2 = 4, l_3 = -4/5.\end{aligned}$$

We plotted Figs. 3 and 4 by taking $k_1 = 0$. This makes the solution to lose one wave from the graphs. Note that in Fig. 3, unlike Fig. 1 we have two waves and they seem to have solitonic property.

7. Conclusion

In this work, we have generalized the two dimensional Boussinesq equation given in (1.3). We have studied on the most general nonlinear partial differential equation depending on four variables and written in the form

$$(D_x^4 + \text{quadratic part})\{f \cdot f\} = 0. \quad (7.1)$$

We called this equation as extended Kadomtsev–Petviashvili–Boussinesq (eKPBo) equation. We noted that it reduces to the KP and the Boussinesq (Bo) equations under some conditions on the constants of the equation.

We applied the Hirota direct method to eKPBo equation. EKPBo equation is a KdV type-equation. Since every KdV type-equation having Hirota bilinear form has one- and two-soliton solutions immediately, we dealt with three-soliton solutions of eKPBo. We have shown that to have three-soliton solution, it should satisfy a condition, which we called restricted three-soliton solution condition (*R3SC*). Our equation is not integrable except for some cases. Hence it does not satisfy (*R3SC*) automatically. So we have analyzed the cases which make this condition to hold. We have seen that there is also a simple form of (*R3SC*) for M dimensional eKPBo. We have also given the general form of restricted three-soliton solution of M dimensional eKPBo under the condition (4.4). Finally, for some specific values of the parameters k_i, w_i and $l_i, i = 1, 2, 3$, we have plotted restricted three-soliton solution of three dimensional eKPBo for different values of time parameter t .

Acknowledgments

This work is partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK).

References

- [1] R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, *Phys. Rev. Lett.* **27** (1971) 1192–1194.
- [2] R. Hirota, Exact solution of the modified Korteweg-de Vries equation for multiple collisions of solitons, *J. Phys. Soc. Japan* **33** (1972) 1456–1458.
- [3] R. Hirota, Exact solution of the Sine-Gordon equation for multiple collisions of solitons, *J. Phys. Soc. Japan* **33** (1972) 1459–1463.

- [4] R. Hirota, Exact envelope-soliton solutions of a nonlinear wave equation, *J. Math. Phys.* **14** (1973) 805–809.
- [5] R. Hirota, Exact N -soliton solution of a nonlinear lumped network equation, *J. Phys. Soc. Japan* **35** (1973) 286–288.
- [6] R. Hirota, Y. Ohta and J. Satsuma, Solutions of the Kadomtsev–Petviashvili equation and the two-dimensional Toda equations, *J. Phys. Soc. Japan* **57** (1988) 1901–1904.
- [7] R. Hirota, Exact N -soliton solutions of the wave equation of long waves on shallow-water and in nonlinear lattices, *J. Math. Phys.* **14** (1973) 810–814.
- [8] B. Grammaticos, A. Ramani and J. Hietarinta, Multilinear operators: The natural extension of Hirota’s bilinear formalism, *Phys. Lett. A* **190** (1994) 65–70.
- [9] J. Hietarinta, A search of bilinear equations passing Hirota’s three-soliton condition: I. KdV-type bilinear equations, *J. Math. Phys.* **28** (1987) 1732–1742.
- [10] J. Hietarinta, Searching for integrable PDE’s by testing Hirota’s three-soliton condition, *Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation, ISSAC’91*, Stephen M. Watt (Association for Computing Machinery, 1991) 295–300.
- [11] J. Hietarinta, A search of bilinear equations passing Hirota’s three-soliton condition: II. mKdV-type bilinear equations, *J. Math. Phys.* **28** (1987) 2094–2101.
- [12] J. Hietarinta, A search of bilinear equations passing Hirota’s three-soliton condition: III. Sine-Gordon-type bilinear equations, *J. Math. Phys.* **28** (1987) 2586–2592.
- [13] J. Hietarinta, A search of bilinear equations passing Hirota’s three-soliton condition: IV. Complex bilinear equations, *J. Math. Phys.* **29** (1988) 628–635.
- [14] R. S. Johnson, A two-dimensional Boussinesq equation for water waves and some of its solutions, *J. Fluid Mech.* **323** (1996) 65–78.
- [15] J. Hietarinta, Introduction to the bilinear method, *Integrability of Nonlinear Systems*, eds. Y. Kosman-Schwarzbach, B. Grammaticos and K. M. Tamizhmani, Springer Lecture Notes in Physics **495** (1997) 95–103, arxiv.org/abs/solv-int/9708006.