Fundamental groups of symmetric sextics. II

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FUNDAMENTAL GROUPS OF SYMMETRIC SEXTICS. II

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Abstract. We study the moduli spaces and compute the fundamental groups of plane sextics of torus type with the set of inner singularities $2\mathbf{A}_8$ or $\mathbf{A}_{17}$. We also compute the fundamental groups of a number of other sextics, both of and not of torus type. The groups found are simplest possible, *i.e.*, $\mathbb{Z}_2 * \mathbb{Z}_3$ and $\mathbb{Z}_6$, respectively.

1. Introduction

1.1. Principal results. This work concludes the series of papers [8], [9], [10], where we attempt to classify and to compute the fundamental groups of irreducible plane sextics of torus type. Recall that a sextic B is said to be of *torus type* if its equation can be represented in the form $p^3 + q^2 = 0$, where p and q are certain homogeneous polynomials of degree 2 and 3, respectively. Alternatively, $B \subset \mathbb{P}^2$ is of torus type if and only if it is the ramification locus of a projection to $\mathbb{P}^2$ of a cubic surface in $\mathbb{P}^3$. A representation of the equation in the form $p^3+q^2 = 0$ (up to the obvious equivalence) is called a *torus structure* of B. A singular point P of B is called *inner* (*outer*) with respect to a torus structure (p, q) if P does (respectively, does not) belong to the intersection of the conic $\{p = 0\}$ and the cubic $\{q = 0\}$. Each sextic B considered in this paper has a unique torus structure, see [5]; hence, we can speak about inner and outer singular points of B. To indicate the difference, we will use the notation $(\Sigma_{\text{inner}}) \oplus \Sigma_{\text{outer}}$ in the listings. (Note that simple singular points of a sextic are conveniently identified with their resolution lattices in the homology of the covering $K3$-surface; for this reason, we use the direct summation symbol $\oplus$ in the notation.)

Another special class is formed by the so called $\mathbb{D}_{2n}$*-sextics*, *i.e.*, irreducible plane sextics whose fundamental group factors to the dihedral group $\mathbb{D}_{2n}$. Due to [5], the $\mathbb{D}_6$-sextics are precisely those of torus type (see also [24]), and the other possible values are $n = 5$ or 7. All $\mathbb{D}_{10}$- and $\mathbb{D}_{14}$-sextics are classified and most fundamental groups are computed in [7] (see also [14]) and [11].

First, sextics of torus type appeared in O. Zariski [25]. For the modern state of the subject and further references, see M. Oka, D. T. Pho [20], [21], H. Tokunaga [23], and A. Degtyarev [5]. According to [25], the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ of any sextic of torus type factors to the reduced braid group $\bar{\mathbb{B}}_3 := \mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong PSL(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ (which is the group of the 'simplest' curve, the six cuspidal sextic, constructed in [25]). We show that, in fact, for most irreducible sextics of torus type, the group equals $\bar{\mathbb{B}}_3$. (A summary of known cases is found in Section 7.1. At present, there are 15 sets of singularities for which the group is still unknown.)

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Our principal results in this paper are Theorems 1.1.1 and 1.1.3 below, classifying and computing the fundamental group of sextics with the set of singularities $(2\mathbf{A}_8)$ and their degenerations.

1.1.1. Theorem. *An irreducible plane sextic of torus type with inner singular points* $2\mathbf{A}_8$ *or* $\mathbf{A}_{17}$ *has one of the following eight sets of singularities*:

$$\begin{gathered}(\mathbf{A}_{17})\oplus\mathbf{A}_2,\quad(\mathbf{A}_{17})\oplus\mathbf{A}_1,\quad(\mathbf{A}_{17}),\\(2\mathbf{A}_8)\oplus\mathbf{A}_3,\quad(2\mathbf{A}_8)\oplus\mathbf{A}_2,\quad(2\mathbf{A}_8)\oplus2\mathbf{A}_1,\quad(2\mathbf{A}_8)\oplus\mathbf{A}_1,\quad(2\mathbf{A}_8).\end{gathered}$$

The moduli space of irreducible sextics of torus type realizing each of the sets of singularities above is unirational; *in particular, it is connected.*

To complete Theorem 1.1.1, we also consider reducible sextics of torus type with a type $\mathbf{A}_{17}$ singular point; they split into two cubics.

1.1.2. Theorem. *A reducible plane sextic of torus type with a type* $\mathbf{A}_{17}$ *singular point has one of the following four sets of singularities*:

$$(\mathbf{A}_{17})\oplus\mathbf{A}_2,\quad(\mathbf{A}_{17})\oplus2\mathbf{A}_1,\quad(\mathbf{A}_{17})\oplus\mathbf{A}_1,\quad(\mathbf{A}_{17}).$$

Each of these sets of singularities is realized by a single connected deformation family of reducible plane sextics of torus type.

Theorems 1.1.1 and 1.1.2 are proved in Sections 2.4 and 5.3, respectively. Another family of reducible sextics of torus type, those splitting into a quartic and a conic, is considered in §5, see Theorems 5.1.1 and 5.2.1.

1.1.3. Theorem. *The fundamental group* $\pi_1(\mathbb{P}^2\setminus B)$ *of each plane sextic* B *as in Theorem* 1.1.1 *equals* $\bar{\mathbb{B}}_3$.

This theorem is proved in Section 3.6.

1.2. Other results. In the proof of Theorems 1.1.1 and 1.1.3, we use the approach of [7], [8], [10] (see also Oka [18]), representing the sextics in question as double coverings of a certain rigid (maximal in the sense of [4]) trigonal curve $\bar{B}$ in the Hirzebruch surface Σ_2 (see Section 2.1). According to [9], there are four maximal trigonal curves admitting a torus structure. Two of them are studied in [8] and [10], one is considered here (see $\bar{B}_1$ in Section 2.2), and the fourth one is reducible (see $\bar{B}_2$ in Section 2.5). We extend to the remaining reducible curve $\bar{B}_2$ the results of [9] and show that it corresponds to sextics of torus type splitting into a quartic and a conic, see Theorem 5.1.1 for the precise statement. As a consequence, we obtain a deformation classification of such reducible sextics, see Theorem 5.2.1, and compute their fundamental groups, see §4. However, we do not make any attempt to simplify the presentations obtained; we merely summarize the results in Theorem 5.2.3 and Remark 5.2.4.

The double covering construction involving the reducible curve $\bar{B}_2$ makes use of two sections: the linear component of $\bar{B}_2$ and the ramification locus. Interchanging the sections, we obtain another family of reducible sextics whose groups are found with almost no extra work, see §6. The geometry of these curves is briefly discussed in Section 6.6; it involves yet another pair of reducible maximal trigonal curves found in [9].

Instead of simplifying the groups of reducible sextics, we perturb the curves and obtain the groups of irreducible ones. The perturbations are constructed using Proposition 5.1.1 in [8], stating that any induced subgraph of the combined Dynkin graph os a sextic B can be realized by a perturbation of B. This procedure gives rise to a few new sextics of torus type (the items marked as 'see 4.?' in Table 1 in §7) and a number of sextics not of torus type (Table 3 in §7). Incorporating the results of [8] and [10], we obtain the fundamental groups of all but 15 irreducible sextics of torus type (most of the groups are $\bar{\mathbb{B}}_3$, see Section 7.1 for details) and 768 other sextics not covered by M. V. Nori's theorem [17] (all groups are abelian). *Extremal* (in the sense of this paper, *i.e.*, not degenerating to a larger set of singularities *with known fundamental group*) sets of singularities with known groups are listed in §7.

Strictly speaking, for most configurations of singularities, the connectedness of the equisingular moduli space is still unknown. For this reason, we state most results in the form of existence only. However, there is a strong arithmetical evidence for the following conjecture, which would imply the connectedness.

1.2.1. Conjecture. *The equisingular moduli space of irreducible plane sextics with any* non-maximal *configuration of simple singularities is connected.*

Here, the *configuration of singularities* is the set of singularities enriched with certain information on the mutual position of the singular points, see [3] for the precise definition. According to [5], in the case of irreducible sextics, all extra information needed is whether the curve is or is not a $\mathbb{D}_{2n}$-sextic for some n. A configuration of singularities is *non-maximal* if it extends to a larger configuration of singularities still realized by plane sextics.

1.3. Contents of the paper.

In §2, we explain the double covering construction used in the proofs, introduce the maximal trigonal curves $\bar{B}_1$ and $\bar{B}_2$, and study the sections of Σ_2 that are in a special position with respect to one of these curves. Theorem 1.1.1 is proved here.

§3 deals with the proof of Theorem 1.1.3. We sketch out Zariski–van Kampen's method [15] in the special case of the ruling of Σ_2 (section 3.1), explain how the braid monodromy is computed (Section 3.2), and compute the groups of the two maximal sextics (Sections 3.3 and 3.4). Then, we study the local perturbations of a few simple singularities (Section 3.5) and global perturbations of sextics of torus type (Section 3.6), computing the groups of the other sextics listed in Theorem 1.1.3.

In §4, we compute the groups of sextics of torus type splitting into a quartic and a conic (using the same approach as in §3). In §6, the representations obtained are modified to produce the groups of a few other reducible sextics. In all cases, we are only interested in the curves whose perturbations contain new irreducible sextics.

§5 is a digression: we establish a geometric correspondence between the curve $\bar{B}_2$ and sextics of torus type splitting into a quartic and a conic (Theorem 5.1.1). As a consequence, we give a complete classification of such sextics, see Theorem 5.2.1. Theorem 1.1.2 is also proved here.

In §7, we give a brief summary of the results of [8], [10], and this paper. In particular, we give a list of the 15 sets of singularities of sextics of torus type, for which the fundamental group is still unknown, and discuss the so called classical Zariski pairs (Section 7.3).

1.4. Acknowledgements.

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2. THE TRIGONAL MODELS

In Section 2.1, we explain the double covering construction used to produce symmetric plane sextics from trigonal curves in the Hirzebruch surface Σ_2. Then, we study two particular rigid trigonal curves $\bar{B}_1$ and $\bar{B}_2$, with the sets of singularities $\mathbf{A}_8$ and $\mathbf{A}_5 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, respectively. Most calculations below are straightforward; they were done using `Maple`.

Note that, according to [9], there are four maximal trigonal curves in Σ_2 that admit a torus structure. Two of them are considered in [8] and [10]; the curves $\bar{B}_1$ and $\bar{B}_2$ studied here are the remaining two.

2.1. The double covering construction. Denote by $\Sigma_2 \to \mathbb{P}^1$ the Hirzebruch surface (*i.e.*, geometrically ruled rational surface) with an exceptional section E of self-intersection (-2). (One can think of Σ_2 as the minimal resolution of singularities of the quadratic cone in $\mathbb{P}^3$.) When speaking about affine coordinates (x, y) in Σ_2, we always assume that E is given by $y = \infty$.

Any section of Σ_2 disjoint from E has the form

$$y = s(x) := ax^2 + bx + c, \qquad a, b, c \in \mathbb{C}. \tag{2.1.1}$$

Given such a section $\bar{L}$, the double covering of the cone Σ_2/E ramified at E/E and $\bar{L}$ is the projective plane $\mathbb{P}^2$, and the deck translation of the covering is an involutive automorphism $c\colon \mathbb{P}^2 \to \mathbb{P}^2$. Conversely, any involution $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ has a fixed line L_c and an isolated fixed point O_c, and the quotient $\mathbb{P}^2(O_c)/c$ is the Hirzebruch surface Σ_2. (Here, $\mathbb{P}^2(O_c)$ stands for the plane $\mathbb{P}^2$ blown up at O_c.)

For the purpose of this paper, a *trigonal curve* is a curve $\bar{B} \subset \Sigma_2$ disjoint from E and intersecting each fiber at three points. (Alternatively, $\bar{B}$ is a curve in $|3E + 6F|$ not containing E, where F is a fiber.) Any trigonal curve is given by a polynomial of the form

$$f(x, y) := y^3 + r_2(x)y^2 + r_4(x)y + r_6(x), \qquad \deg r_i = i. \tag{2.1.2}$$

A *torus structure* on the trigonal curve given by (2.1.2) is a decomposition of the form

$$f(x, y) = \bigl(y + q_2(x)\bigr)^3 + \bigl(q_1(x)y + q_3(x)\bigr)^2, \qquad \deg q_i = i. \tag{2.1.3}$$

A trigonal curve admitting a torus structure is said to be of *torus type*.

Given a trigonal curve $\bar{B}$ and a section $\bar{L}$ not contained in $\bar{B}$, the pull-back of $\bar{B}$ under the double covering $\mathbb{P}^2 \to \Sigma_2/E$ ramified at E/E and $\bar{L}$, see above, is a plane sextic $B \subset \mathbb{P}^2$; we denote it by $\operatorname{Dbl}_{\bar{L}} \bar{B}$ and call it the *double* of $\bar{B}$ ramified at $\bar{L}$. In appropriate affine coordinates (x, y) in $\mathbb{P}^2$, the double is given by the equation

$$f(x, y^2 + s(x)) = 0, \tag{2.1.4}$$

where f and s are as in (2.1.2) and (2.1.1), respectively. If $\bar{B}$ is of torus type, so is $\operatorname{Dbl}_{\bar{L}} \bar{B}$ for any section $\bar{L}$. The relation between the singularities of $\bar{B} + \bar{L}$ and those of $\operatorname{Dbl}_{\bar{L}} \bar{B}$ is studied in [7].

The sextic $B = \operatorname{Dbl}_{\bar{L}} \bar{B}$ has an involutive *symmetry*, *i.e.*, an automorphism $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ preserving B. Conversely, any sextic B with an involutive symmetry c such that $O_c \notin B$ is the double of a trigonal curve.

2.2. The curve $\bar{B}_1$ (the set of singularities $\mathbf{A}_8$). The trigonal curve $\bar{B}_1 \subset \Sigma_2$ with the set of singularities $\mathbf{A}_8$ is a maximal trigonal curve in the sense of [4]; its skeleton is shown in Figure 1, left. Alternatively, $\bar{B}_1$ can be obtained by a birational transformation from a plane quartic with a type $\mathbf{A}_6$ singular point. The curve is plotted (in black) in Figures 3 and 4 below.

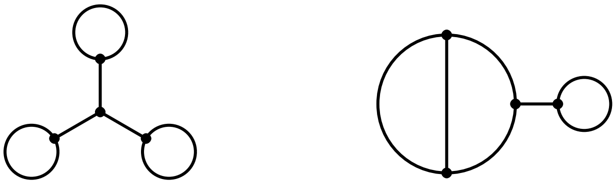

FIGURE 1. The skeletons of $\bar{B}_1$ and $\bar{B}_2$.

In appropriate affine coordinates (x, y) in Σ_2, the curve is given by the polynomial

$$f_1(x, y) := -y^3 + y^2 - x^3(2y - x^3). \tag{2.2.1}$$

It has a unique torus structure, given by

$$f_1(x, y) = (-y)^3 + (y - x^3)^2,$$

and a parametrization

$$x = x_t := \frac{t}{t^3 + 1}, \qquad y = y_t := \frac{1}{(t^3 + 1)^2}.$$

The discriminant of f_1 with respect to y is $-x^9(27x^3 - 4)$. Thus, $\bar{B}_1$ has a type $\mathbf{A}_8$ singular point $P_0 = (0, 0)$ (corresponding to $t = \infty$) and three vertical tangency points

$$P_1 = \Big(\frac{\sqrt[3]{4}}{3}, \frac{4}{9}\Big), \quad P_\pm = \Big(\epsilon_\pm \frac{\sqrt[3]{4}}{3}, \frac{4}{9}\Big)$$

(corresponding to the roots of the equation $2t^3 = 1$), where $\epsilon_\pm = (-1 \pm i\sqrt{3})/2$ are the primitive cubic roots of unity.

The surface Σ_2 has three automorphisms $(x, y) \mapsto (\epsilon x, y)$, $\epsilon^3 = 1$, preserving $\bar{B}_1$ (the *symmetries* of $\bar{B}_1$) and three real structures $\operatorname{conj}_\epsilon \colon (x, y) \mapsto (\epsilon^2 \bar{x}, \bar{y})$, $\epsilon^3 = 1$, with respect to which $\bar{B}_1$ is real. The real part (*i.e.*, the fixed point set) of $\operatorname{conj}_\epsilon$ is $(\epsilon\mathbb{R}, \mathbb{R})$. In the sequel, we use the abbreviation $\operatorname{conj} := \operatorname{conj}_1$ and $\operatorname{conj}_\pm := \operatorname{conj}_{\epsilon_\pm}$. It is easy to see that a conj-real section (2.1.1), $a, b, c \in \mathbb{R}$, $a \neq 0$, intersects the real part $(\epsilon_\pm\mathbb{R}, \mathbb{R})$ of $\operatorname{conj}_\pm$ at two points: $(0, c)$ and $\Big(\epsilon_\pm \dfrac{b}{a}, -\dfrac{b^2}{a} + c\Big)$. (If $a = 0$ and $b \neq 0$, the only intersection point is $(0, c)$. If $a = b = 0$, the section is real with respect to all three real structures.)

2.3. Special sections. Pick a section $\bar{L}$ and consider the double $B = \operatorname{Dbl}_{\bar{L}} \bar{B}_1$. It is a sextic of torus type, see 2.1. If $\bar{L}$ is generic, the set of singularities of B is $(2\mathbf{A}_8)$; otherwise, the singularities of B are recovered from those of $\bar{B}_1 + \bar{L}$ using the results of [7]. Below, for each configuration of $\bar{B}_1 + \bar{L}$, we parametrize the space of sections $\bar{L}$ that are in the desired position with respect to $\bar{B}_1$; for the reader's convenience, we also indicate the corresponding set of singularities of B.

A section (2.1.1) passes through the cusp P_0 of $\bar{B}_1$ (the set of singularities $(\mathbf{A}_{17})$) if and only if $c = 0$.

A section (2.1.1) is tangent to $\bar{B}_1$ at a point $(x_t(t), y_t(t))$, $2t^3 \neq 1$, (the set of singularities $(2\mathbf{A}_8) \oplus \mathbf{A}_1$) if and only if $t^3 \neq -1$ and

$$b = -\frac{2t(2t^3-1)a - 6t^2}{(2t^3-1)(t^3+1)}, \quad c = \frac{t^2(2t^3-1)a - (4t^3+1)}{(2t^3-1)(t^3+1)^2} \tag{2.3.1}$$

or $t^3 = -1$ and $a = -t$, $b = -2t^2/3$. Such a section passes through the cusp P_0 (the set of singularities $(\mathbf{A}_{17}) \oplus \mathbf{A}_1$) if and only if

$$a = \frac{4t^3+1}{t^2(2t^3-1)}, \quad b = -\frac{2}{t(2t^3-1)}, \quad c = 0.$$

A section (2.1.1) is inflection tangent to $\bar{B}_1$ at a point $(x_t(t), y_t(t))$, $2t^3 \neq 1$, (the set of singularities $(2\mathbf{A}_8) \oplus \mathbf{A}_2$) if and only if

$$a = \frac{3t(8t^6+t^3+2)}{(2t^3-1)^3}, \quad b = -\frac{6t^2(4t^3+1)}{(2t^3-1)^3}, \quad c = \frac{8t^3-1}{(2t^3-1)^3}.$$

There are three inflection tangents passing through the cusp P_0 of $\bar{B}_1$ (the set of singularities $(\mathbf{A}_{17}) \oplus \mathbf{A}_2$):

$$t = \frac{\epsilon}{2}, \quad (a,b,c) = \Bigl(-8\epsilon, \frac{16\epsilon^2}{3}, 0\Bigr), \quad \epsilon^3 = 1. \tag{2.3.2}$$

Clearly, the three tangents (2.3.2) are interchanged by the symmetries of $\bar{B}_1$. The tangent $\bar{L}$ corresponding to the real value $\epsilon = 1$ is conj-real; it is shown in grey in Figure 3 below. The tangent intersects $\bar{B}_1$ at P_0 and the following two points:

- inflection tangency at $t = \frac{1}{2}$, $(x,y) = \Bigl(\frac{4}{9}, \frac{64}{81}\Bigr) \approx (0.44, 0.79)$;
- transversal intersection at $t = -\frac{3}{2}$, $(x,y) = \Bigl(\frac{12}{19}, \frac{64}{361}\Bigr) \approx (0.63, 0.78)$.

The intersection of $\bar{L}$ with the real part $\operatorname{Fix}\operatorname{conj}_+$ is at $(x,y) = \Bigl(-\frac{2\epsilon_+}{3}, \frac{32}{9}\Bigr)$.

Next lemma deals with the case when a section $\bar{L}$ as in (2.1.1) is double tangent to $\bar{B}_1$ (the set of singularities $(2\mathbf{A}_8) \oplus 2\mathbf{A}_1$).

2.3.3. Lemma. *There exists a section $\bar{L}$ of Σ_2 tangent to $\bar{B}_1$ at two distinct points $(x_t(t_i), y_t(t_i))$, $i = 1, 2$, $t_1 \neq t_2$, $2t_i^3 \neq 1$, if and only if $2(t_1+t_2)^3 = -1$. A pair t_1, t_2 as above determines the double tangent $\bar{L}$ uniquely.*

Proof. It suffices to substitute $t = t_1$ and $t = t_2$ to (2.3.1), equate the resulting expressions for b and c, and solve for a the linear system obtained. The relation $2(t_1+t_2)^3 = -1$ is the condition for the compatibility of the two equations. □

Letting $t_1 = t_2$ in Lemma 2.3.3, we obtain three sections having a point of quadruple intersection with $\bar{B}_1$ (the set of singularities $(2\mathbf{A}_8) \oplus \mathbf{A}_3$):

$$t = \frac{\delta}{2}, \quad (a,b,c) = \Bigl(-\frac{56\delta}{27}, \frac{64\delta^2}{81}, \frac{256}{243}\Bigr), \quad \delta^3 = -\frac{1}{2}. \tag{2.3.4}$$

The three sections (2.3.4) are interchanged by the symmetries of $\bar{B}_1$. The section $\bar{L}$ corresponding to the real value $\delta = -\sqrt[3]{4}/2$ is conj-real; it is shown in grey in Figure 4 below. This section intersects $\bar{B}_1$ at the following points:

- quadruple intersection at $t = -\dfrac{\sqrt[3]{4}}{4}$ over $x = -\dfrac{4\sqrt[3]{4}}{15} \approx -0.42$;
- transversal intersection at $t = \left(\dfrac{1}{2} \pm \dfrac{3}{4}i\right)\sqrt[3]{4}$ over $x = -\left(\dfrac{44}{327} \pm \dfrac{48}{109}i\right)\sqrt[3]{4}$.

The intersection of $\bar{L}$ with $\operatorname{Fix}\operatorname{conj}_+$ is at $(x, y) = \left(\dfrac{4\sqrt[3]{4}\epsilon_+}{21}, \dfrac{512}{567}\right) \approx (0.30\epsilon_+, 0.90)$. Note that the three points of $\bar{B}_1$ in the fiber over $x = 4\sqrt[3]{4}\epsilon_+/21$ are $y \approx 0.024$, 0.034, and 0.94.

2.4. Proof of Theorem 1.1.1. Let Σ be a set of singularities as in Theorem 1.1.1, and let $\mathcal{M}(\Sigma)$ be the moduli space of irreducible sextics of torus type realizing Σ. Due to [9], each sextic $\bar{B}$ in question has a unique involutive stable symmetry c, and the quotient B/c is a trigonal curve $\bar{B} \subset \Sigma_2$ with a single singular point of type $\mathbf{A}_8$. Conversely, one has $B = \operatorname{Dbl}_{\bar{L}} \bar{B}$ for an appropriate section $\bar{L}$ (the image of L_c). Hence, $\mathcal{M}(\Sigma)$ can be identified with the moduli space of pairs $(\bar{B}, \bar{L})$, where $\bar{B} \subset \Sigma_2$ is a trigonal curve with the set of singularities $\mathbf{A}_8$ and $\bar{L}$ is a section of Σ_2 in a certain prescribed position with respect to $\bar{B}$. Furthermore, since any curve $\bar{B}$ as above is isomorphic to the curve $\bar{B}_1$ considered in Section 2.2 (see [9]) and the group of symmetries of $\bar{B}_1$ is $\mathbb{Z}_3$, there is a cyclic triple ramified covering $\tilde{\mathcal{M}}(\Sigma) \to \mathcal{M}(\Sigma)$, where $\tilde{\mathcal{M}}(\Sigma)$ is the space of sections $\bar{L}$ forming a prescribed configuration with $\bar{B}_1$.

The spaces $\tilde{\mathcal{M}}(\Sigma)$ are described in Section 2.3. In each case, $\tilde{\mathcal{M}}(\Sigma)$ either is rational or consists of three rational components (for $\Sigma = (\mathbf{A}_{17} \oplus \mathbf{A}_2$, $(2\mathbf{A}_8) \oplus \mathbf{A}_3$, or $(2\mathbf{A}_8) \oplus 2\mathbf{A}_1)$. In the former case, $\mathcal{M}(\Sigma)$ is unirational; in the latter case, the three components are interchanged by the symmetries of $\bar{B}_1$ (the deck translation of the covering) and hence $\mathcal{M}(\Sigma)$ is rational. $\square$

2.5. The curve $\bar{B}_2$ (the set of singularities $(\mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_1$). The trigonal curve $\bar{B}_2 \subset \Sigma_2$ with the set of singularities $(\mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_1$ is a maximal trigonal curve; its skeleton is shown in Figure 1, right, and the curve is plotted (in black) in Figures 8–11 below. Alternatively, the curve $\bar{B}_2$ can be obtained by a birational transformation from a pair of conics inflection tangent to each other or from a plane quartic splitting into a cuspidal cubic and a tangent to it. In appropriate affine coordinates (x, y) in Σ_2, the curve is given by the polynomial

$$f_2(x, y) = (y^2 - x)(y - l(x)), \quad \text{where} \quad l(x) = -x^2 + \frac{3}{2}x + \frac{3}{16}.$$

It splits into a hyperelliptic curve $\bar{B}_2' = \{x = y^2\}$ with a cusp R_∞ over $x = \infty$ and a section $\bar{L}' = \{y = l(x)\}$. The two components have a point of inflection tangency at $R_5 = (1/4, 1/2)$ and a point of transversal intersection at $R_1 = (9/4, -3/2)$, forming the singular points of $\bar{B}_2$ of types $\mathbf{A}_5$ and $\mathbf{A}_1$, respectively. The only torus structure of $\bar{B}_2$ is given by

$$64 f_2(x, y) = (4y - 4x - 1)^3 + (8xy + 6y - 12x - 1)^2.$$

Pick a section $\bar{L}$ as in (2.1.1) and consider the sextic $B = \operatorname{Dbl}_{\bar{L}} \bar{B}_2$. It is of torus type, see 2.1. Furthermore, B splits into a quartic B_4 and conic B_2 (the pull-backs of $\bar{B}_2'$ and $\bar{L}'$, respectively), which may further be reducible. If $\bar{L}$ is generic, B_4 has two cusps and two points of inflection tangency with B_2.

A section $\bar{L}$ as in (2.1.1) is inflection tangent to $\bar{B}_2'$ at a point (t^2,t), $t\ne 0$ (the quartic B_4 has three cusps) if and only if

$$(a,b,c)=\Bigl(-\frac{1}{8t^3},\frac{3}{4t},\frac{3t}{8}\Bigr).$$

Such a section passes through R_5 if and only if

$$(a,b,c)=\Bigl(27,-\frac{9}{2},-\frac{1}{16}\Bigr). \tag{2.5.1}$$

It is shown in grey in Figure 8 below (where R_1 is missing). The section is inflection tangent to $\bar{B}_2'$ at $Q_5=(1/36,-1/6)$ and intersects $\bar{L}'$ at $Q_1=(-1/28,13/98)$. The double B has the set of singularities $(\mathbf{A}_{11}\oplus 2\mathbf{A}_2)\oplus\mathbf{A}_2\oplus 2\mathbf{A}_1$; it splits into a three cuspidal quartic and a conic.

Making $\bar{L}'$ and $\bar{L}$ trade rôles, *i.e.*, considering the sextic given by

$$\bigl((y^2+l(x))^2-x\bigr)\bigl(y^2+l(x)-s(x)\bigr)=0, \tag{2.5.2}$$

we obtain a curve $B=\operatorname{Dbl}_{\bar{L}'}(\bar{B}_2'+\bar{L})$ with the set of singularities $(2\mathbf{A}_5\oplus 2\mathbf{A}_2)\oplus\mathbf{D}_5$, also splitting into a three cuspidal quartic and a conic. (Note that B is still of torus type, as the trigonal curve $\bar{B}_2'+\bar{L}$ is isomorphic to $\bar{B}_2$ *via* $(x,y)\mapsto(9x,-3y)$.)

A section $\bar{L}$ passes through R_5 and is tangent to $\bar{B}_2'$ at R_1 if and only if

$$(a,b,c)=\Bigl(\frac{1}{3},-\frac{11}{6},\frac{15}{16}\Bigr). \tag{2.5.3}$$

It is shown in grey in Figure 9 below. (There is an extra point $Q_1=(25/4,5/2)$ of transversal intersection of $\bar{L}$ and the upper branch of $\bar{B}_2'$; it is missing in the figure.) The double B has the set of singularities $(\mathbf{A}_{11}\oplus 2\mathbf{A}_2)\oplus\mathbf{D}_4$; it splits into a quartic with the set of singularities $2\mathbf{A}_2\oplus\mathbf{A}_1$ and a conic.

The section $\bar{L}$ passing through R_5 and tangent to $\bar{B}_2'$ at R_∞ is given by

$$y=1/2, \tag{2.5.4}$$

see the solid grey line in Figure 10. The section intersects $\bar{L}'$ transversally at a point $Q_1=(5/4,1/2)$. The double B has the set of singularities $(\mathbf{E}_6\oplus\mathbf{A}_{11})\oplus 2\mathbf{A}_1$; it splits into a quartic with a type $\mathbf{E}_6$ singular point and a conic.

The section $\bar{L}$ passing through R_1 and tangent to $\bar{B}$ at R_∞ is given by

$$y=-3/2, \tag{2.5.5}$$

see the dotted grey line in Figure 10. The section intersects $\bar{L}'$ transversally at a point $Q_1=(-3/4,-3/2)$. The double B has the set of singularities $(\mathbf{E}_6\oplus 2\mathbf{A}_5)\oplus\mathbf{A}_3$; it splits into a quartic with a type $\mathbf{E}_6$ singular point and a conic.

A section $\bar{L}$ passes through R_∞ and is tangent to $\bar{B}_2'$ at R_1 if and only if

$$(a,b,c)=\Bigl(0,-\frac{1}{3},-\frac{3}{4}\Bigr), \tag{2.5.6}$$

see the solid grey line in Figure 11. The section intersects $\bar{L}'$ transversally at the point $Q_1 = (-5/12, -11/18)$. The double B has the set of singularities $(3\mathbf{A}_5) \oplus \mathbf{D}_4$, splitting into three conics inflection tangent to each other and having a common point.

A section $\bar{L}$ passes through P_1, P_5, and R_∞ if and only if

$$(a, b, c) = \Bigl(0, -1, \frac{3}{4}\Bigr), \tag{2.5.7}$$

see the dotted grey line in Figure 11. The double B has the set of singularities $(\mathbf{A}_{11} \oplus \mathbf{A}_5) \oplus \mathbf{A}_3$; it splits into a quartic with a type $\mathbf{A}_5$ singular point and a conic.

2.6. Other degenerations. Here, we consider other possible degeneration of a section $\bar{L}$ with respect to $\bar{B}_2$, each time showing that the space of sections admits a rational parametrization. We omit obviously linear conditions, like passing through one or several of the points R_1, R_5, R_∞.

As above, we fix the notation $B = \operatorname{Dbl}_{\bar{L}} \bar{B}_2$ and the splitting $B = B_4 + B_2$ into the pull-backs of $\bar{B}'_2$ and $\bar{L}'$, respectively.

A section $\bar{L}$ is tangent to $\bar{B}'_2$ at a point (t^2, t), $t \neq 0$, (the quartic B_4 has an extra node) if and only if

$$b = -\frac{4at^3 - 1}{2t}, \quad c = at^4 + \frac{1}{2}t.$$

Such a section passes through R_5, R_1, or R_∞ if and only if, respectively,

$$a = -\frac{2}{t(2t+1)^2}, \quad a = -\frac{2}{t(2t-3)^2}, \quad \text{or} \quad a = 0.$$

(The corresponding degenerations of B are, respectively, the confluence of two points of inflection tangency of B_4 and B_2 into a single point of 6-fold intersection, the confluence of two points of transversal intersection of B_4 and B_2 into a single tacnode $\mathbf{A}_3$, and the confluence of two cusps of B_4 into a single type $\mathbf{A}_5$ singular point.) The section $\bar{L}$ cannot pass through two of the points R_5, R_1, R_∞ unless one of them is the tangency point.

A section $\bar{L}$ is tangent to the $\bar{L}'$ component of $\bar{B}_2$ (the conic B_2 splits into two lines) if and only if $-16ac + 3a + 4b^2 - 12b - 16c + 12 = 0$. (Clearly, this equation defines a rational subvariety in the space of triples (a, b, c).) Such a section cannot pass through R_5 or R_1 (unless it is the tangency point); it passes through R_∞ if and only if $a = 0$ and $4c = b^2 - 3b + 3$.

A section tangent to B'_2 at (t^2, t) is also tangent to $\bar{L}'$ if and only if

$$a = -\frac{1}{t^2(2t+3)}, \quad b = \frac{3(2t+1)}{2t(2t+3)}, \quad c = \frac{3t}{2(2t+3)}.$$

When $t \to \infty$, it tends to the section $y = 3/4$ tangent to $\bar{B}'_2$ at R_∞ and tangent to $\bar{L}'$ (the set of singularities $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus 3\mathbf{A}_1$).

3. PROOF OF THEOREM 1.1.3

In the rest of the paper, we compute the fundamental groups $\pi_1(\mathbb{P}^2 \smallsetminus B)$ of various sextics B of the form $\operatorname{Dbl}_{\bar{L}} \bar{B}$, see 2.1. We start with $\bar{B} = \bar{B}_1$, see 2.2.

Sections 3.1 and 3.2 contain an outline of the approach used to compute the groups. In 3.3 and 3.4, we study the two maximal doubles of $\bar{B}_1$. In 3.5, we discuss the perturbations of a few simple singular points. These results are applied in 3.6 to prove Theorem 1.1.3.

3.1. Preliminaries. Let $\bar{B} = \bar{B}_1 \subset \Sigma_2$ be the trigonal curve as in 2.2, and let $\bar{L}$ be a section of Σ_2. We start with the group $\bar{\pi}_1 := \pi_1(\Sigma_2 \smallsetminus (\bar{B}_1 \cup \bar{L} \cup E))$ and compute it, applying the classical Zariski–van Kampen method [15] to the ruling of Σ_2 (the pencil $\{x = \text{const}\}$ in the notation of §2).

Pick and fix a real section $S = \{y = \text{const} \gg 0\}$ of Σ_2 and a real nonsingular fiber $F = \{x = \tau\}$, where $\tau > 0$ is sufficiently small. Let $P = F \cap S$, and pick a basis α, β, γ, δ for the group $\pi_F := \pi_1(F \smallsetminus (\bar{B}_1 \cup \bar{L} \cup E), P)$ as shown in black in Figure 2, left. (In all cases considered below, all intersection points are real; the black loops are oriented counterclockwise.) Alternatively, denote α, β, γ, δ by η_i, $i = 1, \ldots, 4$, numbering them consecutively according to the decreasing of the y-coordinate. (For example, in Figure 2 one has $\alpha = \eta_1$, $\delta = \eta_2$, $\beta = \eta_3$, $\gamma = \eta_4$.) We always assume that α, β, γ are small loops about the three points of $F \cap \bar{B}_1$, numbered consecutively, whereas δ is a loop about $F \cap \bar{L}$. Thus, the position of δ in the sequence $(\alpha, \beta, \gamma, \delta)$ may change; this position is important for some relations.

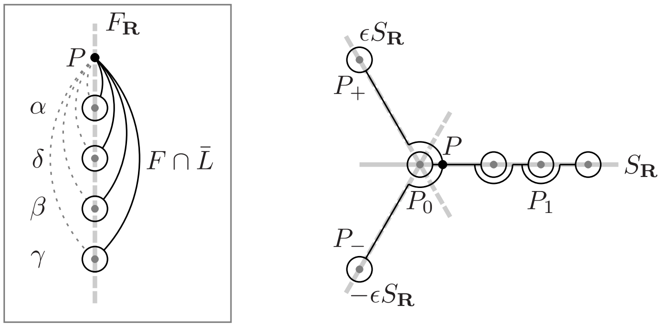

FIGURE 2. The basis α, β, γ, δ and the loops ξ_i

The braid group $\mathbb{B}_4$ acts on π_F: we denote by σ_1, σ_2, σ_3 the standard generators of $\mathbb{B}_4$ and consider the right action defined by $\sigma_i\colon \eta_i \mapsto \eta_i \eta_{i+1} \eta_i^{-1}$, $\eta_{i+1} \mapsto \eta_i$.

Let $F_1, \ldots, F_k$ be the singular fibers of $\bar{B}_1 + \bar{L}$. (Recall that singular are the fiber $\{x = 0\}$ through P_0, the vertical tangents through P_1 and $P_\pm$, and the fibers through the points of intersection of $\bar{B}_1$ and $\bar{L}$.) Let $\xi_1, \ldots, \xi_k$ be a basis for the group $\pi_1(S \smallsetminus (\bigcup F_i \cup \{x = \infty\}), P)$ similar to that shown in Figure 2, left: each ξ_i is a small loop about $S \cap F_i$ connected to P by a segment, circumventing the interfering fibers in the counterclockwise direction. (In the figures, we consider the section $\bar{L}$ given by (2.3.2); necessary modifications for the other cases are discussed below. The bold grey lines in the figures represent the real parts $\operatorname{Fix} \operatorname{conj}_\epsilon$, $\epsilon^3 = 1$.) For each i, let $m_i \in \mathbb{B}_4$ be the *braid monodromy* along ξ_i, *i.e.*, the automorphism of π_F obtained by dragging F along ξ_i while keeping the base point on ξ_i. Then, the Zariski–van Kampen theorem [15] states that

$$(3.1.1) \qquad \bar{\pi}_1 = \bigl\langle \alpha, \beta, \gamma, \delta \bigm| m_i = \mathrm{id},\ i = 1, \ldots, k,\ (\eta_1 \eta_2 \eta_3 \eta_4)^2 = 1 \bigr\rangle.$$

Here, each *braid relation* $m_i = \mathrm{id}$ should be understood as a quadruple of relations $m_i(\alpha) = \alpha$, $m_i(\beta) = \beta$, $m_i(\gamma) = \gamma$, $m_i(\delta) = \delta$; the precise form of the *relation at infinity* $(\eta_1 \eta_2 \eta_3 \eta_4)^2 = 1$ depends on the order of the generators.

Now, the passage to the group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus B)$ is straightforward (see [7] for details): one has $\pi_1 = \operatorname{Ker}[\kappa\colon \bar{\pi}_1/\delta^2 \to \mathbb{Z}_2]$, where $\kappa\colon \alpha, \beta, \gamma \mapsto 0$ and $\kappa\colon \delta \mapsto 1$. In terms of the presentations, we have the following statement.

3.1.2. Lemma. *If $\bar\pi_1$ is given by $\langle\alpha,\beta,\gamma,\delta \mid R_j=1,\ j=1,\ldots,s\rangle$, then*

$$\pi_1=\langle\alpha,\bar\alpha,\beta,\bar\beta,\gamma,\bar\gamma \mid R'_j=\bar R'_j=1,\ j=1,\ldots,s\rangle,$$

where bar stands for the conjugation by δ, $\bar w=\delta w\delta$, and R'_j is obtained from R_j, $j=1,\ldots,s$ by letting $\delta^2=1$ and expressing the result in terms of α, $\bar\alpha$, ... ☐

3.1.3. Remark. Note that $^-\colon w\mapsto\bar w=\delta w\delta$ is an involutive automorphism of π_1. Hence, whenever a relation $R=1$ holds in π_1, the relation $\bar R=1$ also holds.

3.2. Computing the braid monodromy. In this section, we make a few general remarks that facilitate the computation of the braid monodromy.

According to [7], in the presence of the relation at infinity, (any) one of the braid relations can be ignored. We will ignore the relation arising from the monodromy around the cusp P_0.

Since the curves $\bar B_1$ and $\bar L$, the initial fiber F, and the base point P are all chosen conj-real, the conjugation conj induces an automorphism $\operatorname{conj}_*\colon\bar\pi_1\to\bar\pi_1$. Hence, for each pair $F_\pm$ of conjugate singular fibers, it suffices to compute the monodromy m_+ about F_+; the relations for F_- are obtained from those for F_+ by applying conj_*. The images under conj_* of the generators α, β, γ, δ are shown in Figure 2, left: the loops are oriented in the *clockwise* direction and connected to P by the dotted grey paths. Thus, the action of conj_* is as follows:

$$\eta_1\mapsto\eta_1^{-1},\quad \eta_2\mapsto\eta_1\eta_2^{-1}\eta_1^{-1},\quad \eta_3\mapsto(\eta_1\eta_2)\eta_3^{-1}(\eta_1\eta_2)^{-1},$$
$$\eta_4\mapsto(\eta_1\eta_2\eta_3)\eta_4^{-1}(\eta_1\eta_2\eta_3)^{-1}.$$

Its precise form in terms of α, β, γ, δ depends on the order of the generators.

The relations arising from conj-real singular fibers are easily computed using the plots: the monodromy along a small circle about the fiber (or along a semicircle circumventing another singular fiber) is found using a local normal form of the singularity, and along a segment of the real line the four points of $\bar B_1+\bar L$ can be traced as all but at most two of them are real. In the computation below, we merely indicate the resulting relations. (Clearly, it does not really matter whether the interfering singular fibers are circumvented in the counterclockwise or clockwise direction; each time, we choose the more convenient one.)

The monodromy m_+ about P_+ has the form $\sigma_3^3m'_+\sigma_3^{-3}$, where m'_+ is the monodromy along the small loop about P_+ connected to the point over $x=\epsilon_+\tau$ by a conj_+-real segment I_+. (For m'_+, we choose the generators α', β', γ' in the fiber F' over $x=\epsilon_+\tau$ similar to Figure 2 and take for δ' the image of δ under the monodromy along the arc $x=\tau\exp(it)$, $t\in[0,2\pi/3]$. Note that the point $F'\cap\bar L$ has positive imaginary part, *i.e.*, in Figure 2 it would be located to the right from $F'_{\mathbb R}$.) Now, m'_+ is found similar to the monodromy about P_1, using a plot of the conj_+-real part of $\bar B$, which looks exactly the same as its conj-real part. However, when computing the braid along I_+, one should take into account the points of intersection of $\bar L$ and the conj_+-real part $(\epsilon_+\mathbb R,\mathbb R)$ of Σ_2.

The relations for the conjugate point P_- are obtained from those for P_+ by applying conj_*.

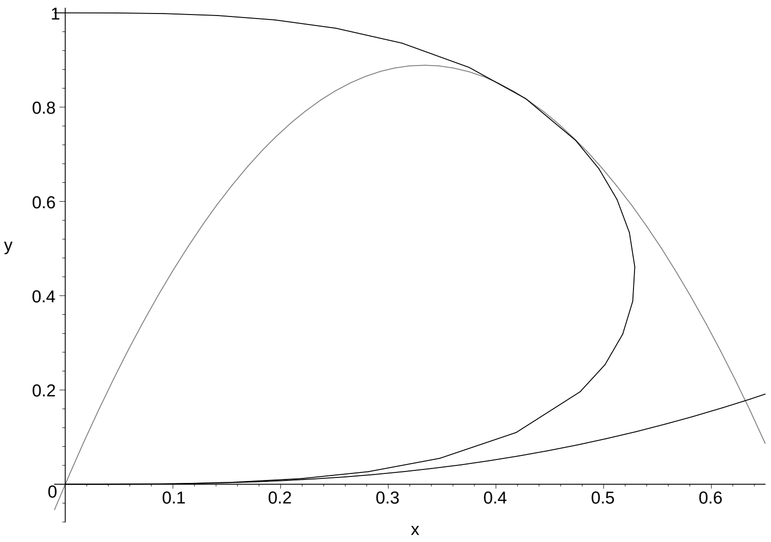

FIGURE 3. The set of singularities $(\mathbf{A}_{17})\oplus\mathbf{A}_2$

3.3. The set of singularities $(\mathbf{A}_{17})\oplus\mathbf{A}_2$. Take for $\bar L$ the section given by (2.3.2). The curve and the section are plotted in Figure 3.

The basis $(\eta_1,\eta_2,\eta_3,\eta_4)=(\alpha,\delta,\beta,\gamma)$ is as shown in Figure 2, and the relations are

$$
\begin{aligned}
&(\beta\gamma)^{-1}\gamma(\beta\gamma)=\alpha && \text{(the vertical tangent through } P_+\text{)},\\
&(\delta\beta\gamma\beta)\gamma(\delta\beta\gamma\beta)^{-1}=\alpha && \text{(the vertical tangent through } P_-\text{)},\\
&(\alpha\delta)^3=(\delta\alpha)^3 && \text{(the inflection tangency)},\\
&(\delta\alpha\delta)^{-1}\alpha(\delta\alpha\delta)=\beta && \text{(the vertical tangent through } P_1\text{)},\\
&[(\alpha\delta)^{-1}\delta(\alpha\delta),\beta\gamma\beta^{-1}]=1 && \text{(the transversal intersection)},\\
&(\alpha\delta\beta\gamma)^2=1 && \text{(the relation at infinity)}.
\end{aligned}
$$

(The monodromy m_+ about P_+ is computed as explained in Section 3.2; since $\bar L$ has no conj_+-real points over I_+, see Section 2.3, the result is $\sigma_3^3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-3}$.) Using Lemma 3.1.2, we obtain the following relations for π_1:

$$\gamma\beta\gamma=\beta\gamma\alpha,\quad \bar\gamma\bar\beta\bar\gamma=\bar\beta\bar\gamma\bar\alpha, \tag{3.3.1}$$

$$\beta\gamma\beta\gamma=\bar\alpha\beta\gamma\beta,\quad \bar\beta\bar\gamma\bar\beta\bar\gamma=\alpha\bar\beta\bar\gamma\bar\beta, \tag{3.3.2}$$

$$\alpha\bar\alpha\alpha=\bar\alpha\alpha\bar\alpha, \tag{3.3.3}$$

$$\bar\alpha^{-1}\alpha\bar\alpha=\beta,\quad \alpha^{-1}\bar\alpha\alpha=\bar\beta, \tag{3.3.4}$$

$$(\alpha\bar\beta)\bar\gamma(\alpha\bar\beta)^{-1}=(\bar\alpha\beta)\gamma(\bar\alpha\beta)^{-1}, \tag{3.3.5}$$

$$\alpha\bar\beta\bar\gamma\bar\alpha\beta\gamma=1. \tag{3.3.6}$$

From (3.3.4) it follows that $\bar\alpha\beta=\alpha\bar\alpha$ and $\alpha\bar\beta=\bar\alpha\alpha$. Substituting these expressions to (3.3.5), we obtain $(\bar\alpha\alpha)\bar\gamma(\bar\alpha\alpha)^{-1}=(\alpha\bar\alpha)\gamma(\alpha\bar\alpha)^{-1}$ or, replacing $\bar\alpha\alpha$ with $\alpha\bar\alpha\alpha\bar\alpha^{-1}$

from (3.3.3), $\alpha^{-1}\gamma\alpha = \bar\alpha^{-1}\bar\gamma\bar\alpha$. Thus, introducing $\gamma_1 = \alpha^{-1}\gamma\alpha$ instead of γ, we obtain $\bar\gamma_1 = \gamma_1$.

Use (3.3.4) and (3.3.3) again to get $\alpha^{-1}\beta\alpha = \bar\alpha$ and $\bar\alpha^{-1}\bar\beta\bar\alpha = \alpha$. Then, the conjugation of the first and second relations (3.3.1) by α and $\bar\alpha$, respectively, turns them into

$$\gamma_1\bar\alpha\gamma_1 = \bar\alpha\gamma_1\alpha \quad\text{and}\quad \gamma_1\alpha\gamma_1 = \alpha\gamma_1\bar\alpha. \tag{3.3.7}$$

Similarly, relations (3.3.2) turn into

$$\bar\alpha\gamma_1\bar\alpha\gamma_1 = \alpha^{-1}(\bar\alpha\alpha\bar\alpha)\gamma_1\bar\alpha \quad\text{and}\quad \alpha\gamma_1\alpha\gamma_1 = \bar\alpha^{-1}(\alpha\bar\alpha\alpha)\gamma_1\alpha;$$

using (3.3.3), they simplify to $\gamma_1\bar\alpha\gamma_1 = \alpha\gamma_1\bar\alpha$ and $\gamma_1\alpha\gamma_1 = \bar\alpha\gamma_1\alpha$. Comparing these expressions to (3.3.7), one concludes that $\alpha = \bar\alpha$. Hence, also $\beta = \bar\beta = \alpha$ and $\gamma = \bar\gamma$, and the map $\alpha, \bar\alpha, \beta, \bar\beta \mapsto \sigma_1$, $\gamma, \bar\gamma \mapsto \sigma_2$ establishes an isomorphism

$$\pi_1(\mathbb{P}^2 \setminus B) = \bar{\mathbb{B}}_3.$$

(Relation (3.3.6), which turns into $(\sigma_1^2\sigma_2)^2 = 1$, is equivalent to $(\sigma_1\sigma_2)^3 = 1$ in $\mathbb{B}_3$.)

3.4. The set of singularities $(2\mathbf{A}_8)\oplus\mathbf{A}_3$**.** Take for $\bar L$ the section given by (2.3.4). The curve and the section are shown in Figure 4.

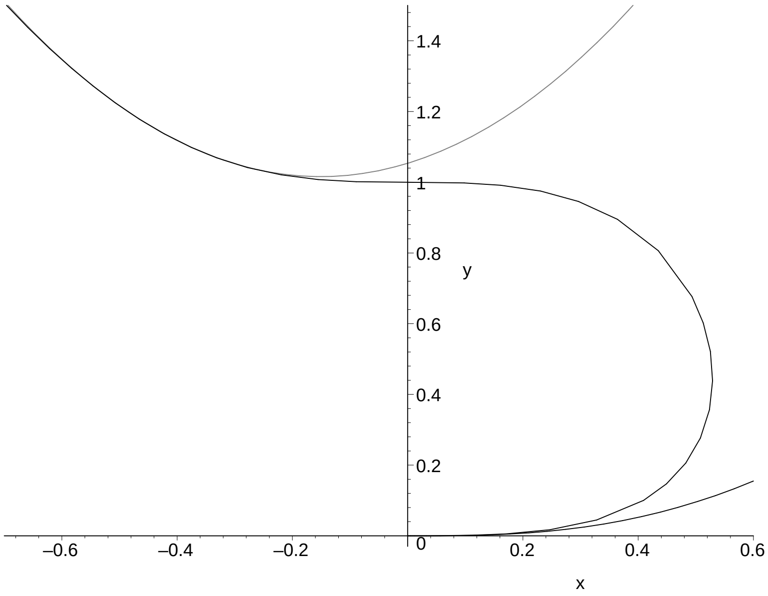

FIGURE 4. The set of singularities $(2\mathbf{A}_8) \oplus \mathbf{A}_3$

The singular fibers of $\bar B_1 + \bar L$ and the basis for $\pi_1(S \setminus (\bigcup F_i \cup \{x = \infty\}))$ are shown in Figure 5, where $Q_\pm$ are the two points of transversal intersection of $\bar B_1$ and $\bar L$, see Section 2.3. In the basis $(\eta_1, \eta_2, \eta_3, \eta_4) = (\delta, \alpha, \beta, \gamma)$ for the group π_F, the relations are

$$(\beta\gamma)^{-1}\gamma(\beta\gamma) = (\delta\alpha)^{-1}\alpha(\delta\alpha) \qquad \text{(the vertical tangent through } P_+\text{)},$$

$$\begin{aligned}
&(\beta\gamma\beta)\gamma(\beta\gamma\beta)^{-1}=\delta\alpha\delta^{-1} && \text{(the vertical tangent through } P_-\text{)},\\
&\alpha=\beta && \text{(the vertical tangent through } P_0\text{)},\\
&(\alpha\delta)^4=(\delta\alpha)^4 && \text{(the quadruple intersection point)},\\
&(\delta\alpha\beta\gamma)^2=1 && \text{(the relation at infinity)}.
\end{aligned}$$

The monodromy m_+ about P_+ is found as explained in Section 3.2. This time, $\bar{L}$ does have a conj_+-real point over I_+, see Section 2.3, which is located between the two upper branches of $\bar{B}_1$. Hence, $m_+=\sigma_3^3\sigma_1^2\sigma_2\sigma_1^{-2}\sigma_3^{-3}$.

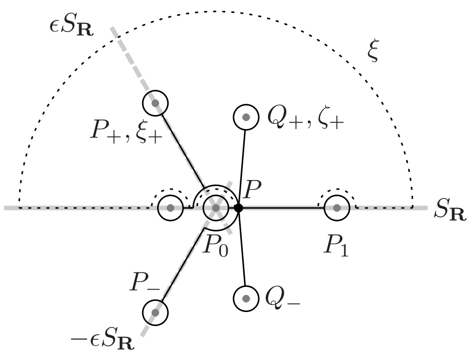

FIGURE 5. The basis in $\pi_1(S\smallsetminus(\bigcup F_i\cup\{x=\infty\}))$

To compute the monodromy n_+ along the loop ζ_+ about Q_+, observe that $\zeta_+\xi_+$ is homotopic to the loop ξ 'surrounding' the upper half-plane $\operatorname{Im}x>0$ (the dotted black loop in Figure 5). More precisely, the loop ξ is composed of a large semicircle $x=r\exp(it)$, $t\in[0,\pi]$, $r\gg 0$, and real segment $[-r,r]$ circumventing the real singular fibers in the *clockwise* direction. The braid over the real segment is found using local normal forms of the singularities and the plot of the real part of the curve; it is shown in Figure 6. The braid over the imaginary semicircle is the element $\Delta^2\in\mathbb{B}_4$ corresponding to one full turn. (Indeed, when x tends to infinity, $\bar{B}+\bar{L}$ has four quadratic branches with pairwise distinct leading coefficients.) Hence, the monodromy m along ξ is

$$m=(\sigma_2^{-1})(\sigma_3^{-1}\sigma_2)\Delta^2(\sigma_1^{-4})(\sigma_3^{-4}),$$

and $n_+=mm_+^{-1}$.

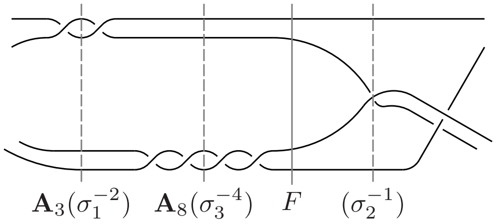

FIGURE 6. The braid m_+ (to be closed by Δ^2)

Equating $n_+(\delta) = \delta$, we obtain the relation

$$\delta\alpha\gamma^{-1}\beta\gamma\alpha^{-1}\delta\alpha\gamma^{-1}\beta\gamma\alpha^{-1}\delta^{-1} = \delta,$$

resulting from the monodromy about Q_+. The relation from Q_- is obtained by applying conj_*. Simplifying both relations, we get

$$[\alpha^{-1}\delta\alpha, \gamma^{-1}\beta\gamma] = 1, \tag{3.4.1}$$

$$[\beta^{-1}\delta\beta, \gamma\beta\gamma^{-1}] = 1. \tag{3.4.2}$$

Eliminate α using the relation $\alpha = \beta$, substitute $\beta^{-1}\delta\beta = \delta_1$, let $\delta_1^2 = 1$, and pass to the generators β, γ and $\bar\beta = \delta_1^{-1}\beta\delta_1$, $\bar\gamma = \delta_1^{-1}\gamma\delta_1$, *cf.* Lemma 3.1.2. We obtain the following set of relations for π_1:

$$\gamma\beta\gamma = \beta\gamma\bar\beta, \quad \bar\gamma\bar\beta\bar\gamma = \bar\beta\bar\gamma\beta, \tag{3.4.3}$$

$$\gamma\beta\gamma = \bar\beta\gamma\beta, \quad \bar\gamma\bar\beta\bar\gamma = \beta\bar\gamma\bar\beta, \tag{3.4.4}$$

$$(\beta\bar\beta)^2 = (\bar\beta\beta)^2, \tag{3.4.5}$$

$$\gamma\bar\beta\bar\gamma = \beta\gamma\beta, \quad \bar\gamma\beta\gamma = \bar\beta\bar\gamma\bar\beta, \tag{3.4.6}$$

$$\bar\gamma\bar\beta\gamma = \beta\gamma\beta, \quad \gamma\beta\bar\gamma = \bar\beta\bar\gamma\bar\beta, \tag{3.4.7}$$

$$\beta\gamma\beta\bar\beta\bar\gamma\bar\beta = 1. \tag{3.4.8}$$

Here, relation (3.4.1) turns into $\gamma^{-1}\beta\gamma = \bar\gamma^{-1}\bar\beta\bar\gamma$; in the presence of (3.4.3) it is equivalent to (3.4.6). Similarly, (3.4.2) turns into $\gamma\beta\gamma^{-1} = \bar\gamma\bar\beta\bar\gamma^{-1}$, which is equivalent to (3.4.7) in the presence of (3.4.4).

To simplify the group, consider the generators $u = \beta\gamma\beta$ and $v = \gamma\beta$, so that $\beta = uv^{-1}$ and $\gamma = v^2u^{-1}$. Then $\bar u = u^{-1}$ (from (3.4.8)) and $\bar v = u^2v^{-2}$ (from the first relation in (3.4.7)). Since the automorphism $w \mapsto \bar w$ is an involution, one must have $v = \bar u^2\bar v^{-2}$, *i.e.*,

$$u^2vu^2 = v^2u^{-2}v^2. \tag{3.4.9}$$

Equate the right hand sides of the first equations in (3.4.3) and (3.4.4) to obtain $uvu^{-2}v^2u^{-2} = u^{-1}v^2u^{-2}v$, or $[v, u^{-2}v^2u^{-2}] = 1$. Using (3.4.9), we get $[v, u^2] = 1$; hence $v^3 = u^6$ and $[u, v^3] = 1$. Then the first relation in (3.4.3) simplifies to $u^2 = 1$; hence also $v^3 = 1$, $\bar u = u$, and $\bar v = v$. Thus, $\bar\beta = \beta$, $\bar\gamma = \gamma$, and the map $\beta, \bar\beta \mapsto \sigma_1$, $\gamma, \bar\gamma \mapsto \sigma_2$ establishes an isomorphism

$$\pi_1(\mathbb{P}^2 \smallsetminus B) = \bar{\mathbb{B}}_3.$$

3.5. Perturbations of singular points. Consider an isolated singular point P of a plane curve B, pick a Milnor ball U about P, and consider a perturbation B_t, $t \in [0, 1]$, of $B = B_0$ transversal to ∂U. Let $B' = B_1$. We are interested in the perturbation epimorphism $\pi_1(U \smallsetminus B) \twoheadrightarrow \pi_1(U \smallsetminus B')$, *cf.* [25].

A perturbation B to B' is called *maximal* if the total Milnor number of $B' \cap U$ equals $\mu(P) - 1$. A perturbation is called *irreducible* if the normalization of $B' \cap U$ is connected (equivalently, if the abelianization of $\pi_1(U \smallsetminus B')$ is cyclic). If P is simple, any irreducible perturbation factors through a maximal irreducible one.

Let P be of type $\mathbf{A}_{3k-1}$ or $\mathbf{E}_6$. Then the ball U admits a regular $\mathbb{S}_3$-covering ramified at B. Let $\phi\colon \pi_1(U \smallsetminus B) \twoheadrightarrow \mathbb{S}_3$ be the corresponding epimorphism of the fundamental group. A perturbation B' of P is said to be of *torus type* if ϕ factors through the perturbation epimorphism above. From the results of [5] it follows that, if B is a sextic of torus type and P is an inner singularity of B, then B' is still of torus type if and only if the perturbation is of torus type.

According to E. Looijenga [16], the deformation classes (in the obvious sense) of perturbations of a simple singularity P are enumerated by the induced subgraphs of the Dynkin diagram of P (up to a certain equivalence, which is not important here). In the statements and proofs below, we merely indicate the result of the enumeration.

3.5.1. Lemma. *The maximal irreducible perturbation of a type* $\mathbf{A}_{11}$ *singular point are* $\mathbf{A}_8 \oplus \mathbf{A}_2$ *(of torus type) and* $\mathbf{A}_{10}$ *and* $\mathbf{A}_6 \oplus \mathbf{A}_4$ *(not of torus type). In the former case, the group* $\pi_1(U \smallsetminus B')$ *equals* $\mathbb{B}_3$*; in the latter case, it is cyclic.*

3.5.2. Lemma. *The maximal irreducible perturbation of a type* $\mathbf{A}_5$ *singular point are* $2\mathbf{A}_2$ *(of torus type) and* $\mathbf{A}_4$ *(not of torus type). In the former case, the group* $\pi_1(U \smallsetminus B')$ *equals* $\mathbb{B}_3$*; in the latter case, it is cyclic.*

3.5.3. Lemma. *The maximal irreducible perturbation of a type* $\mathbf{A}_7$ *singular point are* $\mathbf{A}_6$ *and* $\mathbf{A}_4 \oplus \mathbf{A}_2$*. For these perturbations, the group* $\pi_1(U \smallsetminus B')$ *is cyclic.*

Proof of Lemmas 3.5.1–3.5.3. All statements are a well known property of type $\mathbf{A}$ singular points: any perturbation of a type $\mathbf{A}_p$ singular point has the set of singularities $\bigoplus \mathbf{A}_{p_i}$ with $d = (p+1) - \sum(p_i+1) \geqslant 0$, and the group $\pi_1(U \smallsetminus B')$ is given by $\langle \alpha, \beta \,|\, \sigma^s\alpha = \alpha,\ \sigma^s\beta = \beta \rangle$, where σ is the standard generator of the braid group $\mathbb{B}_2$ acting on $\langle \alpha, \beta \rangle$ and $s = 1$ if $d > 0$ or $s = \mathrm{g.\,c.\,d.}(p_i+1)$ if $d = 0$.

A perturbation is reducible if and only if s above is even; a perturbation is of torus type is and only if $s = 0 \bmod 3$. ☐

3.5.4. Lemma. *The only maximal irreducible perturbation of a type* $\mathbf{D}_5$ *singular point is* $\mathbf{A}_4$*. For this perturbation, the group* $\pi_1(U \smallsetminus B')$ *is cyclic.*

Proof. The perturbation B_t can be realized by a family $C_t \subset \mathbb{C}^2$ of affine quartics with a point of quadruple intersection with the line at infinity, so that $(U, B_t) \cong (\mathbb{C}^2, C_t)$ for each $t \in [0,1]$. For a quartic C_1 with a type $\mathbf{A}_4$ singular point, one has $\pi_1(\mathbb{C}^2 \smallsetminus C_1) = \mathbb{Z}$, see [2]. ☐

3.5.5. Lemma. *For any nontrivial perturbation of a type* $\mathbf{D}_4$ *singular point, the group* $\pi_1(U \smallsetminus B')$ *is abelian.*

Proof. Any perturbation B_t of a type $\mathbf{D}_4$ singular point can be realized by a family $C_t \subset \mathbb{C}^2$ of affine cubics transversal to the line at infinity, so that $(U, B_t) \cong (\mathbb{C}^2, C_t)$ for each $t \in [0,1]$. Unless C_t is a triple of lines passing through a single point, the group $\pi_1(\mathbb{C}^2 \smallsetminus C_t)$ is abelian. ☐

3.5.6. Lemma. *The maximal irreducible perturbations of a type* $\mathbf{E}_7$ *singular point are* $\mathbf{E}_6$*,* $\mathbf{A}_6$*, and* $\mathbf{A}_4 \oplus \mathbf{A}_2$*. For the perturbation* $\mathbf{A}_4 \oplus \mathbf{A}_2$*, one has* $\pi_1(U \smallsetminus B') = \mathbb{Z} \times SL(2, \mathbb{F}_5)$*; for other irreducible perturbations, the group* $\pi_1(U \smallsetminus B')$ *is cyclic.*

Proof. Any perturbation B_t of a type $\mathbf{E}_7$ singular point can be realized by a family $C_t \subset \mathbb{C}^2$ of affine quartics inflection tangent to the line at infinity (see, *e.g.*, [1]), so that $(U, B_t) \cong (\mathbb{C}^2, C_t)$ for each $t \in [0,1]$. The groups $\pi_1(\mathbb{C}^2 \smallsetminus C_t)$ for such quartics are found in [2]. □

We need an explicit description of the epimorphism $\pi_1(U \smallsetminus B) \twoheadrightarrow \pi_1(U \smallsetminus B')$ for the perturbation $\mathbf{E}_7 \to \mathbf{A}_4 \oplus \mathbf{A}_2$. Consider the isotrivial trigonal curve $\bar{B} \subset \Sigma_2$ given by $y^3 + x^3 y = 0$. It has two singular fibers: a fiber of type $\tilde{\mathbf{E}}_7$ at $P = (0,0)$ and a fiber of type $\tilde{\mathbf{A}}_1^*$ over $x = \infty$. Let $\bar{U}$ be the affine part of Σ_2 (the complement of the exceptional section E and the fiber over $x = \infty$). Then the pair $(\bar{U}, \bar{B})$ is diffeomorphic to the pair (U, B) above, and the perturbation can be realized by deforming $\bar{B}$ to a maximal trigonal curve $\bar{B}'$ with singular fibers of types $\tilde{\mathbf{A}}_4$, $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{A}}_0^*$, and $\tilde{\mathbf{A}}_1^*$ (keeping the last one at infinity).

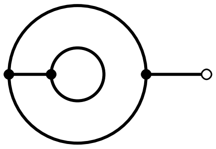

FIGURE 7. The skeleton of $\bar{B}'$

The skeleton of the perturbed curve $\bar{B}'$ is shown in Figure 7. It follows that $\bar{B}'$ and all its real fibers can be chosen real, and one can use the skeleton to sketch the real part of $\bar{B}'$ and compute the braid monodromy. (Alternatively, one can use the description of the braid monodromy in terms of the skeleton given in [4].) As a result, in the standard generators α, β, γ, *cf.* Figure 2, in the fiber over $x \ll 0$ (in which both $\bar{B}$ and $\bar{B}'$ have three real points), the relations for $\pi_1(\bar{U} \smallsetminus \bar{B}')$ are

$$\alpha\beta\alpha = \beta\alpha\beta, \quad \alpha\gamma\alpha = \gamma\alpha\beta, \quad \gamma\alpha\gamma = \beta\gamma\alpha. \tag{3.5.7}$$

3.6. Perturbations of sextics of torus type. Here, we state a few simple lemmas about irreducible plane sextics of torus type with the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B) = \bar{\mathbb{B}}_3$. (Note that, in fact, any sextic whose group is $\bar{\mathbb{B}}_3$ is irreducible and then, due to [5], it is of torus type.)

3.6.1. Lemma. *Let B be an irreducible plane sextic of torus type. Then any epimorphism $\phi\colon \mathbb{B}_3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus B)$ factors through an isomorphism $\bar{\mathbb{B}}_3 = \pi_1(\mathbb{P}^2 \smallsetminus B)$.*

Proof. Recall that $(\sigma_1\sigma_2)^3 \subset \mathbb{B}_3$ is a central element whose image in the abelianization $\mathbb{B}_3/[\mathbb{B}_3, \mathbb{B}_3] = \mathbb{Z}$ is 6. Since the abelianization of $\pi_1(\mathbb{P}^2 \smallsetminus B)$ is $\mathbb{Z}_6$, one has $(\sigma_1\sigma_2)^3 \in \operatorname{Ker}\phi$ and ϕ factors to an epimorphism $\bar{\mathbb{B}}_3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus B)$. On the other hand, there is an epimorphisms $\pi_1(\mathbb{P}^2 \smallsetminus B') \twoheadrightarrow \bar{\mathbb{B}}_3$ induced by the perturbation of B to Zariski's six cuspidal sextic, see [25]. Since the group $\bar{\mathbb{B}}_3 = \mathit{PSL}(2, \mathbb{Z})$ is obviously residually finite, hence Hopfian, both epimorphisms are isomorphisms. □

3.6.2. Corollary. *Let B be a sextic of torus type, $\pi_1(\mathbb{P}^2 \smallsetminus B) = \bar{\mathbb{B}}_3$, and let B' be a perturbation of B which is also of torus type. Then $\pi_1(\mathbb{P}^2 \smallsetminus B') = \bar{\mathbb{B}}_3$.* □

3.6.3. Lemma. *Let B be a sextic of torus type with $\pi_1(\mathbb{P}^2 \smallsetminus B) = \bar{\mathbb{B}}_3$, and let U be a Milnor ball about an inner singular point P of B. Then the inclusion homomorphism $\pi_1(U \smallsetminus B) \to \pi_1(\mathbb{P}^2 \smallsetminus B)$ is onto.*

Proof. Consider a perturbation B_t, $t \in [0,1]$, of $B = B_0$ to a six cuspidal sextic $B' = B_1$, transversal to ∂U. Pick a cusp $P' \in U$ of B', consider a Milnor ball U'

about P', and let $i\colon U'\smallsetminus B'\hookrightarrow U\smallsetminus B'$ and $j\colon U\smallsetminus B'\hookrightarrow \mathbb{P}^2\smallsetminus B'$ be the inclusions. The homomorphism $(j\circ i)_*\colon \pi_1(U'\smallsetminus B')\to\pi_1(\mathbb{P}^2\smallsetminus B')$ is onto, see [25]; hence, so is j_*. On the other hand, the perturbation $B\to B'$ induces an epimorphism $\pi_1(U\smallsetminus B)\twoheadrightarrow\pi_1(U\smallsetminus B')$ and an isomorphism $\pi_1(\mathbb{P}^2\smallsetminus B)=\pi_1(\mathbb{P}^2\smallsetminus B')$, see Corollary 3.6.2; hence, the inclusion homomorphism $\pi_1(U\smallsetminus B)\to\pi_1(\mathbb{P}^2\smallsetminus B)$ is onto. □

3.6.4. Lemma. *Let B be a sextic of torus type with simple singularities and with $\pi_1(\mathbb{P}^2\smallsetminus B)=\bar{\mathbb{B}}_3$, and let B' be a perturbation of B which is not of torus type. Then $\pi_1(\mathbb{P}^2\smallsetminus B')=\mathbb{Z}_6$.*

Proof. Since B' is not of torus type, there is an inner singular point P of B that undergoes a perturbation not of torus type. Let U be a Milnor ball about P. If P is of type $\mathbf{E}_6$, the group $\pi_1(U\smallsetminus B')$ is abelian, see [10], and the statement follows from Lemma 3.6.3. Assume that P is of type $\mathbf{A}_{3k-1}$. Then

$$\pi_1(U\smallsetminus B)=\langle\alpha,\beta\,|\,\sigma^{3k}=\mathrm{id}\rangle\quad\text{and}\quad\pi_1(U\smallsetminus B')=\langle\alpha,\beta\,|\,\sigma^{s}=\mathrm{id}\rangle$$

for some $s\neq 0 \bmod 3$, see the proof of Lemmas 3.5.1–3.5.3. On the other hand, in the group $\pi_1(\mathbb{P}^2\smallsetminus B)=\bar{\mathbb{B}}_3$, the generators α, β are subject to the braid relation $\sigma^3=\mathrm{id}$. (From the proof of Lemma 3.6.3, it follows that α, β are taken to the standard generators $\sigma_1,\sigma_2\in\bar{\mathbb{B}}_3$.) Hence, after the perturbation, one has a relation $\sigma=\mathrm{id}$, *i.e.*, $\alpha=\beta$ in $\pi_1(\mathbb{P}^2\smallsetminus B')$, and Lemma 3.6.3 applies. □

Proof of Theorem 1.1.3. Due to the results of Sections 3.3 and 3.4, Corollary 3.6.2 implies that, for any sextic B of torus type obtained by a perturbation from an irreducible sextic with the set of singularities $(\mathbf{A}_{17})\oplus\mathbf{A}_2$ or $(2\mathbf{A}_8)\oplus\mathbf{A}_3$, one has $\pi_1(\mathbb{P}^2\smallsetminus B)=\bar{\mathbb{B}}_3$. In particular, this statement covers all curves listed in Theorem 1.1.3, as their moduli spaces are connected, see Theorem 1.1.1. □

Now, consider a perturbation B' of B that is *not* of torus type. The extremal sets of singularities obtained in this way are

$$\begin{gathered}\mathbf{A}_{16}\oplus\mathbf{A}_2,\quad \mathbf{A}_{15}\oplus\mathbf{A}_2\oplus\mathbf{A}_1,\quad \mathbf{A}_{13}\oplus\mathbf{A}_3\oplus\mathbf{A}_2,\\ \mathbf{A}_{12}\oplus\mathbf{A}_4\oplus\mathbf{A}_2,\quad \mathbf{A}_{10}\oplus\mathbf{A}_6\oplus\mathbf{A}_2,\quad \mathbf{A}_9\oplus\mathbf{A}_7\oplus\mathbf{A}_2,\\ \mathbf{A}_8\oplus\mathbf{A}_7\oplus\mathbf{A}_3,\quad \mathbf{A}_8\oplus\mathbf{A}_6\oplus\mathbf{A}_3\oplus\mathbf{A}_1,\quad \mathbf{A}_8\oplus\mathbf{A}_4\oplus 2\mathbf{A}_3.\end{gathered}$$

Due to Lemma 3.6.4, each of these sets of singularities is realized by a sextic whose fundamental group is cyclic.

4. REDUCIBLE CURVES OF TORUS TYPE

Now, we consider the maximal reducible sextics of torus type of the form $B=\operatorname{Dbl}_{\bar L}\bar B_2$, where $\bar B_2$ is the trigonal curve as in 2.5. The computation of the group $\pi_1(\mathbb{P}^2\smallsetminus B)$ follows the outline in Sections 3.1 and 3.2, with an additional simplification due to the fact that the singular fibers of $\bar B_2+\bar L$ are all real; we systematically ignore the relation from the singular fiber at infinity.

Instead of simplifying the obtained presentations for $\pi_1(\mathbb{P}^2\smallsetminus B)$, we perturb B to an irreducible sextic B' and use Lemmas 3.5.1–3.5.6 to compute $\pi_1(\mathbb{P}^2\smallsetminus B')$. If B' is of torus type, we only prove that there is an epimorphism $\mathbb{B}_3\twoheadrightarrow\pi_1(\mathbb{P}^2\smallsetminus B)$; Lemma 3.6.1 implies that it is an isomorphism. (Similarly, if B' is not of torus type, we only prove that the group is abelian.) Furthermore, we only consider maximal irreducible perturbations; as the groups obtained are $\bar{\mathbb{B}}_3$ or $\mathbb{Z}_6$, the results extend to other perturbations using Corollary 3.6.2 and Lemma 3.6.4.

Here and in §6, without further references, the perturbations are constructed using Proposition 5.1.1 in [8], by perturbing the singular points independently.

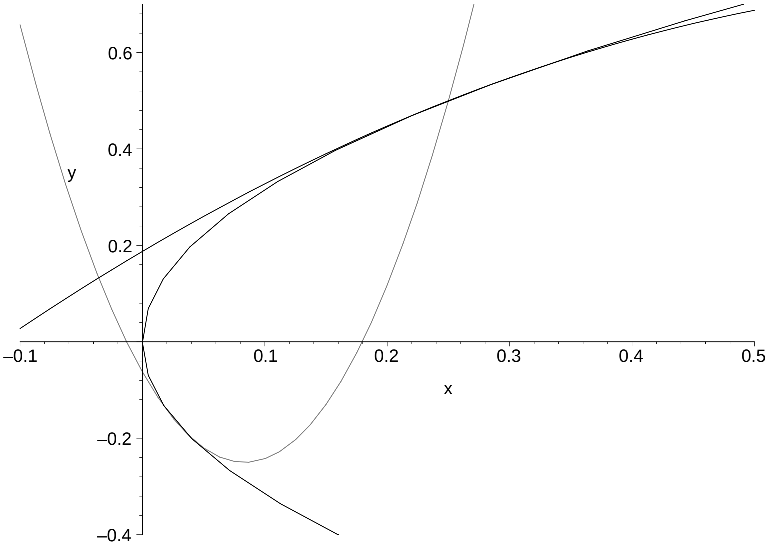

FIGURE 8. The set of singularities $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$

4.1. The set of singularities $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$. Take for $\bar{L}$ the section given by (2.5.1), see Figure 8 (where the point R_1 of transversal intersection of $\bar{L}'$ and the lower branch of $\bar{B}'_2$ is missing), and choose the generators $(\eta_1, \eta_2, \eta_3, \eta_4) = (\alpha, \beta, \delta, \gamma)$ in a real fiber F just to the left from R_5 (*e.g.*, over $x = 0.2$). The relations for $\bar{\pi}_1$ are:

$$\begin{aligned}
&\delta(\alpha\beta)^3 = (\beta\alpha\beta)\delta(\alpha\beta\alpha) && \text{(the fiber through } R_5\text{)},\\
&[\delta, \alpha\beta] = 1 && \text{(the fiber through } R_5\text{)},\\
&[(\beta\alpha\beta\delta)^{-1}\alpha(\beta\alpha\beta\delta), \gamma] = 1 && \text{(the fiber through } R_1\text{)},\\
&(\gamma\delta)^3 = (\delta\gamma)^3 && \text{(the fiber through } Q_5\text{)},\\
&\beta = (\delta\gamma\delta)\gamma(\delta\gamma\delta)^{-1} && \text{(the vertical tangent)},\\
&[\beta^{-1}\alpha\beta, (\delta\gamma)\delta(\delta\gamma)^{-1}] = 1 && \text{(the fiber through } Q_1\text{)},\\
&(\alpha\beta\delta\gamma)^2 = 1 && \text{(the relation at infinity)}.
\end{aligned}$$

Passing to the generators α, $\bar{\alpha}$, β, $\bar{\beta}$, γ, $\bar{\gamma}$, see Lemma 3.1.2, we obtain the following set of relations for $\pi_1(\mathbb{P}^2 \smallsetminus B)$, where B is a reducible sextic of torus type with the set of singularities $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$:

$$(\alpha\beta)^3 = \bar{\beta}\bar{\alpha}\bar{\beta}\alpha\beta\alpha, \quad (\bar{\alpha}\bar{\beta})^3 = \beta\alpha\beta\bar{\alpha}\bar{\beta}\bar{\alpha}, \tag{4.1.1}$$

$$\alpha\beta = \bar{\alpha}\bar{\beta}, \tag{4.1.2}$$

$$[(\beta\alpha\beta)^{-1}\alpha(\beta\alpha\beta),\bar\gamma]=1,\quad [(\bar\beta\bar\alpha\bar\beta)^{-1}\bar\alpha(\bar\beta\bar\alpha\bar\beta),\gamma]=1, \tag{4.1.3}$$

$$\gamma\bar\gamma\gamma=\bar\gamma\gamma\bar\gamma, \tag{4.1.4}$$

$$\beta=\bar\gamma\gamma\bar\gamma^{-1},\quad \bar\beta=\gamma\bar\gamma\gamma^{-1} \tag{4.1.5}$$

$$(\bar\beta\gamma)^{-1}\bar\alpha\bar\beta\gamma=(\beta\bar\gamma)^{-1}\alpha\beta\bar\gamma, \tag{4.1.6}$$

$$\alpha\beta\bar\gamma\bar\alpha\bar\beta\gamma=1. \tag{4.1.7}$$

In the presence of (4.1.1) and (4.1.2), relations (4.1.3) simplify to

$$[(\bar\alpha\bar\beta)\bar\alpha(\bar\alpha\bar\beta)^{-1},\bar\gamma]=1,\quad [(\alpha\beta)\alpha(\alpha\beta)^{-1},\gamma]=1. \tag{4.1.8}$$

Consider the perturbation $\mathbf{A}_{11}\mapsto\mathbf{A}_8\oplus\mathbf{A}_2$, producing an irreducible sextic B' of torus type with the set of singularities

$$(\mathbf{A}_8\oplus 3\mathbf{A}_2)\oplus\mathbf{A}_2\oplus 2\mathbf{A}_1.$$

This perturbation adds the braid relation $\alpha\beta\alpha=\beta\alpha\beta$. Then, the first relation in (4.1.1) simplifies to $\beta\alpha\beta=\bar\beta\bar\alpha\bar\beta$; in view of (4.1.2), this implies that $\beta=\bar\beta$. Similarly, using the second relation in (4.1.1), one obtains $\alpha=\bar\alpha$. Furthermore, in the presence of the braid relation, one has $(\alpha\beta)\alpha(\alpha\beta)^{-1}=\beta$; hence, (4.1.8) implies $[\beta,\bar\gamma]=[\beta,\gamma]=1$ and (4.1.5) yields $\gamma=\bar\gamma=\beta$. Thus, the map $\alpha,\bar\alpha\mapsto\sigma_1$, $\beta,\bar\beta,\gamma,\bar\gamma\mapsto\sigma_2$ establishes an isomorphism $\pi_1(\mathbb{P}^2\smallsetminus B')=\mathbb{B}_3$.

Now, perturb one of the nodes over R_1, producing an irreducible sextic B' of torus type with the set of singularities

$$(\mathbf{A}_{11}\oplus 2\mathbf{A}_2)\oplus\mathbf{A}_2\oplus\mathbf{A}_1.$$

This perturbation simplifies one of the two relations (4.1.8): for example, we can replace the first one with

$$\bar\gamma=(\bar\alpha\bar\beta)\bar\alpha(\bar\alpha\bar\beta)^{-1}. \tag{4.1.9}$$

To simplify the group, introduce the generators $u=\alpha\beta$ and $v=\alpha\beta\alpha$, so that $\alpha=u^{-1}v$ and $\beta=v^{-1}u^2$. Then $\bar u=u$ (from (4.1.2)) and $\bar v=u^{-3}vu^3$ (from the second relation in (4.1.1)). Hence, $\bar\alpha=u^{-4}vu^3$, $\bar\beta=u^{-3}v^{-1}u^5$, and the new relation (4.1.9) turns into $\bar\gamma=u^{-3}vu^2$; in view of (4.1.7), this implies $\gamma=u^{-3}v^{-1}u^2$. Substituting the expressions obtained to the second relation in (4.1.5), we arrive at $u^3=v^2$; hence, $\bar\alpha=\alpha=u^{-1}v$, $\bar\beta=\beta=\bar\gamma=v^{-1}u^2$, and $\gamma=v^{-3}u^2$. Substituting to the first relation in (4.1.5), we get $v^2=1$. Thus, also $u^3=1$, and we obtain an isomorphism $\pi_1(\mathbb{P}^2\smallsetminus B')=\bar{\mathbb{B}}_3$, given by $\alpha,\bar\alpha\mapsto\sigma_1$ and $\beta,\bar\beta,\gamma,\bar\gamma\mapsto\sigma_2$.

Finally, consider a maximal irreducible perturbation of the type $\mathbf{A}_{11}$ singular point that is not of torus type, producing irreducible plane sextics with the sets of singularities

$$\mathbf{A}_{10}\oplus 3\mathbf{A}_2\oplus 2\mathbf{A}_1,\quad \mathbf{A}_6\oplus\mathbf{A}_4\oplus 3\mathbf{A}_2\oplus 2\mathbf{A}_1$$

(see Lemma 3.5.1). According to Lemma 3.5.1, this perturbation introduces the additional relations $\alpha=\bar\alpha=\beta=\bar\beta$. Then, from (4.1.3) one has $[\alpha,\gamma]=[\alpha,\bar\gamma]=1$ and, due to (4.1.7), also $[\gamma,\bar\gamma]=1$. Hence, the group is abelian.

4.2. The set of singularities $(2\mathbf{A}_5\oplus 2\mathbf{A}_2)\oplus\mathbf{D}_5$. Consider the triple $\bar B_2'$, $\bar L'$, $\bar L$ as in Section 4.1, see Figure 8, but now let $\bar B=\bar B_2'+\bar L$ and $B=\operatorname{Dbl}_{\bar L'}(\bar B_2'+\bar L)$: it is a reducible sextic of torus type with the set of singularities $(2\mathbf{A}_5\oplus 2\mathbf{A}_2)\oplus\mathbf{D}_5$. The group $\bar\pi_1=\pi_1(\Sigma_2\smallsetminus(\bar B\cup\bar L'\cup E))$ is found in Section 4.1. Modifying Lemma 3.1.2, let $\alpha^2=1$ and pass to the generators β, $\bar\beta=\alpha\beta\alpha$, γ, $\bar\gamma=\alpha\gamma\alpha$, δ, $\bar\delta=\alpha\delta\alpha$. We obtain the following set of relations for the group $\pi_1(\mathbb{P}^2\smallsetminus B)$:

$$\bar\delta\beta\bar\beta\beta=\delta\bar\beta\beta\bar\beta, \tag{4.2.1}$$

$$\bar\delta\beta=\beta\delta,\quad \delta\bar\beta=\bar\beta\bar\delta, \tag{4.2.2}$$

$$(\beta\bar\beta\bar\delta)\bar\gamma(\beta\bar\beta\bar\delta)^{-1}=(\bar\beta\beta\delta)\gamma(\bar\beta\beta\delta)^{-1}, \tag{4.2.3}$$

$$(\gamma\delta)^3=(\delta\gamma)^3,\quad (\bar\gamma\bar\delta)^3=(\bar\delta\bar\gamma)^3, \tag{4.2.4}$$

$$\beta=(\delta\gamma\delta)\gamma(\delta\gamma\delta)^{-1},\quad \bar\beta=(\bar\delta\bar\gamma\bar\delta)\bar\gamma(\bar\delta\bar\gamma\bar\delta)^{-1}, \tag{4.2.5}$$

$$(\beta\delta\gamma)\delta(\beta\delta\gamma)^{-1}=(\bar\beta\bar\delta\bar\gamma)\bar\delta(\bar\beta\bar\delta\bar\gamma)^{-1}, \tag{4.2.6}$$

$$\bar\beta\bar\delta\bar\gamma\beta\delta\gamma=1. \tag{4.2.7}$$

(Here, (4.2.1) is simplified using (4.2.2).)

The perturbation $\mathbf{D}_5\mapsto\mathbf{A}_4$ produces an irreducible sextic B' of torus type with the set of singularities

$$(2\mathbf{A}_5\oplus 2\mathbf{A}_2)\oplus\mathbf{A}_4$$

and introduces the relation $\beta=\bar\beta=\delta=\bar\delta$, see Lemma 3.5.4. Then, due to (4.2.3), $\gamma=\bar\gamma$ and (4.2.5) simplifies to $\beta\gamma\beta=\gamma\beta\gamma$. Hence, the map $\beta,\bar\beta,\delta,\bar\delta\mapsto\sigma_1$, $\gamma,\bar\gamma\mapsto\sigma_2$ establishes an isomorphism $\pi_1(\mathbb{P}^2\smallsetminus B')=\mathbb{B}_3$.

The perturbation $\mathbf{A}_5\mapsto\mathbf{A}_4$ produces an irreducible sextic B' with the set of singularities

$$\mathbf{D}_5\oplus\mathbf{A}_5\oplus\mathbf{A}_4\oplus 2\mathbf{A}_2$$

and introduces the relation $\delta=\gamma$, see Lemma 3.5.2. Then, due to the first relation in (4.2.5), $\beta=\gamma$, relation (4.2.2) implies $\bar\delta=\gamma$ and $[\bar\beta,\gamma]=1$, and one has $\bar\beta=\gamma$ (from (4.2.1)) and $\bar\gamma=\gamma^{-5}$ (from (4.2.7)). Thus, the group is abelian.

4.3. The set of singularities $(\mathbf{A}_{11}\oplus 2\mathbf{A}_2)\oplus\mathbf{D}_4$. Take for $\bar L$ the section given by (2.5.3), see Figure 9 (where the point Q_1 of transversal intersection of $\bar L$ and the upper branch of $\bar B_2'$ is missing). Choose the generators $(\eta_1,\eta_2,\eta_3,\eta_4)=(\alpha,\beta,\delta,\gamma)$ in a real fiber F between R_5 and R_1 (*e.g.*, over $x=1$). The relations are:

$$\begin{aligned}
&\delta(\alpha\beta)^3=(\beta\alpha\beta)\delta(\alpha\beta\alpha) && \text{(the fiber through } R_5\text{)},\\
&[\delta,\alpha\beta]=1 && \text{(the fiber through } R_5\text{)},\\
&\beta(\delta\gamma)^2=\gamma\beta\delta\gamma\delta && \text{(the fiber through } R_1\text{)},\\
&[\beta,\delta\gamma]=1 && \text{(the fiber through } R_1\text{)},\\
&(\beta\alpha\beta\delta)^{-1}\alpha(\beta\alpha\beta\delta)=\gamma && \text{(the vertical tangent)},\\
&[\alpha,\gamma^{-1}\delta\gamma]=1 && \text{(the fiber through } Q_1\text{)},\\
&(\alpha\beta\delta\gamma)^2=1 && \text{(the relation at infinity)}.
\end{aligned}$$

Passing to the generators α, $\bar\alpha$, β, $\bar\beta$, γ, $\bar\gamma$, see Lemma 3.1.2, we obtain the following relations for the group $\pi_1(\mathbb{P}^2\smallsetminus B)$ of a reducible sextic B of torus type with the

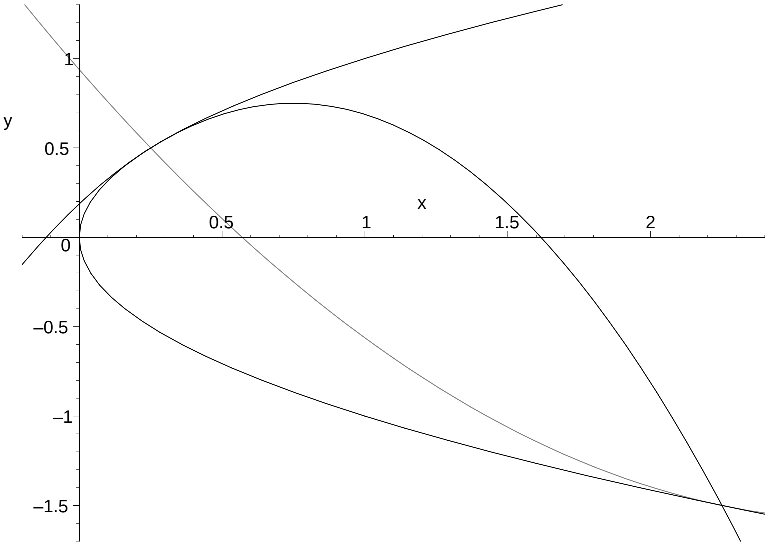

FIGURE 9. The set of singularities $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{D}_4$

set of singularities $(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{D}_4$:

$$(\alpha\beta)^3 = \bar\beta\bar\alpha\bar\beta\alpha\beta\alpha, \quad (\bar\alpha\bar\beta)^3 = \beta\alpha\beta\bar\alpha\bar\beta\bar\alpha, \tag{4.3.1}$$

$$\alpha\beta = \bar\alpha\bar\beta, \tag{4.3.2}$$

$$\beta\bar\gamma\gamma = \gamma\beta\bar\gamma, \quad \bar\beta\gamma\bar\gamma = \bar\gamma\bar\beta\gamma, \tag{4.3.3}$$

$$\beta\bar\gamma = \bar\gamma\bar\beta, \quad \bar\beta\gamma = \gamma\beta, \tag{4.3.4}$$

$$(\beta\alpha\beta)^{-1}\alpha(\beta\alpha\beta) = \bar\gamma, \quad (\bar\beta\bar\alpha\bar\beta)^{-1}\bar\alpha(\bar\beta\bar\alpha\bar\beta) = \gamma, \tag{4.3.5}$$

$$\alpha\gamma^{-1}\bar\gamma = \gamma^{-1}\bar\gamma\bar\alpha, \quad \bar\alpha\bar\gamma^{-1}\gamma = \bar\gamma^{-1}\gamma\alpha, \tag{4.3.6}$$

$$\alpha\beta\bar\gamma\bar\alpha\bar\beta\gamma = 1. \tag{4.3.7}$$

As above, in view of (4.3.1) and (4.3.2), relations (4.3.5) simplify to

$$(\bar\alpha\bar\beta)\bar\alpha(\bar\alpha\bar\beta)^{-1} = \bar\gamma, \quad (\alpha\beta)\alpha(\alpha\beta)^{-1} = \gamma. \tag{4.3.8}$$

The perturbation $\mathbf{A}_{11} \mapsto \mathbf{A}_8 \oplus \mathbf{A}_2$ produces an irreducible sextic B' of torus type with the set of singularities

$$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{D}_4.$$

The perturbation adds to the presentation a braid relation $\alpha\beta\alpha = \beta\alpha\beta$. Similar to Section 4.1 (the perturbation $\mathbf{A}_{11} \to \mathbf{A}_8 \oplus \mathbf{A}_2$), relations (4.3.1) and (4.3.2) imply that $\alpha = \bar\alpha$ and $\beta = \bar\beta$, and (4.3.8) turns into $\gamma = \bar\gamma = \beta$. Hence, there is an isomorphism $\pi_1(\mathbb{P}^2 \smallsetminus B') = \bar{\mathbb{B}}_3$ given by the map $\alpha, \bar\alpha \mapsto \sigma_1$, $\beta, \bar\beta, \gamma, \bar\gamma \mapsto \sigma_2$.

Now, perturb the type $\mathbf{D}_4$ point of B. After the perturbation, the generators β, $\bar\beta$, γ, and $\bar\gamma$ pairwise commute, see Lemma 3.5.5. It follows that $\beta = \bar\beta$ (from (4.3.4)), $\alpha = \bar\alpha$ (from (4.3.2)), and $\gamma = \bar\gamma$ (from (4.3.8)), and the presentation simplifies to

$$\bigl\langle \alpha, \beta \bigm| (\alpha\beta)^3 = (\beta\alpha)^3,\ (\alpha\beta\alpha)^2 = 1 \bigr\rangle. \tag{4.3.9}$$

This is, indeed, the fundamental group of a reducible curve of torus type whose set of singularities is

$$(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus 3\mathbf{A}_1, \quad (\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3, \quad \text{or} \quad (\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus 2\mathbf{A}_1.$$

To obtain an irreducible curve with the set of singularities

$$(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3,$$

we choose the perturbation $\mathbf{D}_4 \mapsto \mathbf{A}_3$ so that the generators β and γ become conjugate (hence equal) in $\pi_1(U_P \smallsetminus B')$. (Locally, we perturb a triple of lines to a conic and a line tangent to it, and the choice of a line to be kept as a separate component is governed by the choice of a subdiagram $\mathbf{A}_3 \subset \mathbf{D}_4$; any such choice can be realized by a perturbation of B.) Then, (4.3.8) implies the braid relation $\alpha\beta\alpha = \beta\alpha\beta$, and the map $\alpha, \bar{\alpha} \mapsto \sigma_1$, $\beta, \bar{\beta}, \gamma, \bar{\gamma} \mapsto \sigma_2$ establishes an isomorphism $\pi_1(\mathbb{P}^2 \smallsetminus B') = \bar{\mathbb{B}}_3$.

The perturbations $\mathbf{A}_{11} \mapsto \mathbf{A}_{10}$ or $\mathbf{A}_6 \oplus \mathbf{A}_4$, see Lemma 3.5.1, produce irreducible sextics with the sets of singularities

$$\mathbf{D}_4 \oplus \mathbf{A}_{10} \oplus 2\mathbf{A}_2, \quad \mathbf{D}_4 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$$

while adding the relation $\alpha = \bar{\alpha} = \beta = \bar{\beta}$. Then (4.3.5) implies $\gamma = \bar{\gamma} = \alpha$, and the group is abelian.

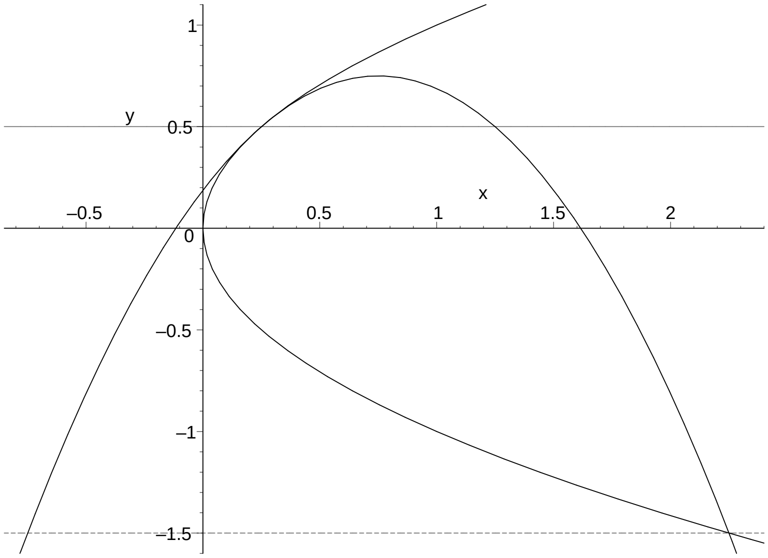

FIGURE 10. The sets $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus 2\mathbf{A}_1$ and $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$

4.4. The set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus 2\mathbf{A}_1$. Let $\bar{L}$ be the section given by (2.5.4), see the solid horizontal grey line in Figure 10. (The resulting set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus 2\mathbf{A}_1$ is erroneously missing in Oka [19].) Choosing the generators $(\eta_1, \eta_2, \eta_3, \eta_4) = (\alpha, \beta, \delta, \gamma)$ in a real fiber F between R_5 and Q_1 (*e.g.*, over $x = 1$), we obtain the relations

$$[\delta, \beta] = 1 \qquad \text{(the fiber through } Q_1\text{)},$$

$$[\delta, \alpha\beta] = 1 \qquad \text{(the fiber through } R_5\text{).}$$

(We only list the few relations needed in the sequel.) Hence, also $[\delta, \alpha] = 1$. The relation at the vertical tangent of $\bar{B}$ has the form $\gamma =$ (a word in α, β, δ); hence, we also have $[\delta, \gamma] = 1$. Thus, δ is a central element, and a presentation for the group $\pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup \bar{L} \cup E))$ can be obtained from the presentation in Section 4.3 by adding the relations $[\alpha, \delta] = [\beta, \delta] = [\gamma, \delta] = 1$. After eliminating $\gamma = (\beta\alpha)^{-1}\alpha(\beta\alpha)$, we get

$$\big\langle \alpha, \beta, \delta \bigm| (\alpha\beta)^3 = (\beta\alpha)^3,\ [\alpha, \delta] = [\beta, \delta] = 1,\ (\alpha\beta\alpha)^2\delta^2 = 1 \big\rangle. \tag{4.4.1}$$

The fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ of a reducible sextic B of torus type with the set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus 2\mathbf{A}_1$ is obtained from (4.4.1) by letting $\delta = 1$. The result is (4.3.9).

A perturbation of a node of B produces an irreducible sextic of torus type with the set of singularities

$$(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_1.$$

The additional relation introduced by this operation is $\beta = \gamma$. Hence, the resulting fundamental group is $\bar{\mathbb{B}}_3$, *cf.* the perturbation $\mathbf{D}_4 \mapsto \mathbf{A}_3$ in Section 4.3.

The perturbation $\mathbf{A}_{11} \mapsto \mathbf{A}_8 \oplus \mathbf{A}_2$ produces the set of singularities

$$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus 2\mathbf{A}_1.$$

The additional relation is $\alpha\beta\alpha = \beta\alpha\beta$, and the resulting group is $\bar{\mathbb{B}}_3$.

The perturbations $\mathbf{A}_{11} \mapsto \mathbf{A}_{10}$ or $\mathbf{A}_6 \oplus \mathbf{A}_4$ produce the sets of singularities

$$\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus 2\mathbf{A}_1, \quad \mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_1,$$

while adding to (4.3.9) the relation $\alpha = \beta$, see Lemma 3.5.1. Hence, the resulting group is abelian.

4.5. The set of singularities $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$. Let $\bar{L}$ be the section given by (2.5.5), see the dotted horizontal grey line in Figure 10. Choosing the generators $(\eta_1, \eta_2, \eta_3, \eta_4) = (\alpha, \beta, \gamma, \delta)$ in a real fiber F between R_5 and R_1 (*e.g.*, over $x = 1$), we obtain the relations

$$\begin{aligned}
&(\alpha\beta)^3 = (\beta\alpha)^3 && \text{(the fiber through } R_5\text{),}\\
&(\alpha\beta)\alpha(\alpha\beta)^{-1} = \gamma && \text{(the vertical tangent),}\\
&[\delta, \alpha\beta\alpha^{-1}] = 1 && \text{(the fiber through } Q_1\text{),}\\
&[\gamma\delta, \beta] = [\beta\gamma\delta, \gamma] = [\beta\gamma, \delta] = 1 && \text{(the fiber through } R_1\text{),}\\
&(\alpha\beta\gamma\delta)^2 = 1 && \text{(the relation at infinity).}
\end{aligned}$$

(The second and the third relations were simplified using the first one.) Passing to α, $\bar{\alpha}$, β, $\bar{\beta}$, γ, $\bar{\gamma}$, see Lemma 3.1.2, gives the following relations for the group of a reducible sextic B of torus type with the set of singularities $(\mathbf{E}_6 \oplus 2\mathbf{A}_5) \oplus \mathbf{A}_3$:

$$(\alpha\beta)^3 = (\beta\alpha)^3, \quad (\bar{\alpha}\bar{\beta})^3 = (\bar{\beta}\bar{\alpha})^3, \tag{4.5.1}$$

$$(\alpha\beta)\alpha(\alpha\beta)^{-1}=\gamma,\quad (\bar\alpha\bar\beta)\bar\alpha(\bar\alpha\bar\beta)^{-1}=\bar\gamma, \tag{4.5.2}$$

$$\alpha\beta\alpha^{-1}=\bar\alpha\bar\beta\bar\alpha^{-1}, \tag{4.5.3}$$

$$\gamma\bar\beta=\beta\gamma=\bar\beta\bar\gamma=\bar\gamma\beta, \tag{4.5.4}$$

$$\alpha\beta\gamma\bar\alpha\bar\beta\bar\gamma=1. \tag{4.5.5}$$

The perturbation $\mathbf{A}_5\mapsto\mathbf{A}_4$ of one of the type $\mathbf{A}_5$ singular points of B produces the set of singularities

$$\mathbf{E}_6\oplus\mathbf{A}_5\oplus\mathbf{A}_4\oplus\mathbf{A}_3$$

and adds the relation $\alpha=\beta$, see Lemma 3.5.2. Then $\gamma=\alpha$ (from (4.5.2)), $\bar\beta=\bar\gamma=\alpha$ (from (4.5.4)), and $\bar\alpha=\alpha^{-5}$ (from (4.5.5)). Hence, the group is abelian.

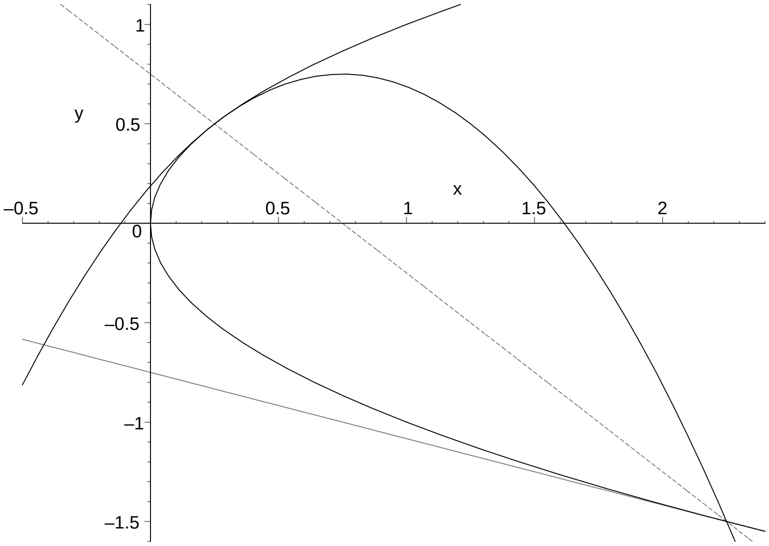

FIGURE 11. The sets $(3\mathbf{A}_5)\oplus\mathbf{D}_4$ and $(\mathbf{A}_{11}\oplus\mathbf{A}_5)\oplus\mathbf{A}_3$

4.6. The set of singularities $(3\mathbf{A}_5)\oplus\mathbf{D}_4$. Let $\bar L$ be the section given by (2.5.6), see the solid grey line in Figure 11. Choosing the generators α, β, γ, δ as in Section 4.5, we obtain the same set of relations, except that the relation at the fiber through R_1 should be replaced with

$$[\beta,\gamma\delta]=1,\quad \delta\beta\gamma\delta\gamma=\beta\gamma\delta\gamma\delta\quad\text{(the fiber through } R_1).$$

Hence, the relations for $\pi_1(\mathbb{P}^2\smallsetminus B)$ are (4.5.1)–(4.5.3), (4.5.5), and the relations

$$\beta\gamma=\gamma\bar\beta,\quad \bar\beta\bar\gamma=\bar\gamma\beta,\quad \bar\beta\bar\gamma\gamma=\beta\gamma\bar\gamma \tag{4.6.1}$$

replacing (4.5.4).

We are only interested in the perturbation $\mathbf{A}_5\mapsto\mathbf{A}_4$ of two of the three type $\mathbf{A}_5$ singular points of B, producing the set of singularities

$$\mathbf{D}_4\oplus\mathbf{A}_5\oplus2\mathbf{A}_4.$$

(Since B splits into three components, we need to perturb two points to get an irreducible curve.) Assume that the points perturbed are those over R_5. Then the extra relations for $\pi_1(\mathbb{P}^2\smallsetminus B')$ are $\alpha=\beta$ and $\bar\alpha=\bar\beta$, see Lemma 3.5.2, and it is straightforward that the resulting group is abelian.

4.7. The set of singularities $(\mathbf{A}_{11} \oplus \mathbf{A}_5) \oplus \mathbf{A}_3$. Take for $\bar{L}$ be the section given by (2.5.7), see the dotted grey line in Figure 11. Choosing the generators $(\eta_1, \eta_2, \eta_3, \eta_4) = (\alpha, \beta, \delta, \gamma)$ in a real generic fiber between R_5 and R_1 (*e.g.*, over $x = 1$), we obtain the following relations:

$$\begin{aligned}
&[\beta, \delta\gamma] = [\delta, \gamma\beta] = [\gamma, \beta\delta] = 1 && \text{(the fiber through } R_1\text{)},\\
&[\delta, \alpha\beta] = 1 && \text{(the fiber through } R_5\text{)},\\
&\delta(\alpha\beta)^3 = \beta\alpha\beta\delta\alpha\beta\alpha && \text{(the fiber through } R_5\text{)},\\
&(\beta\alpha\beta\delta)^{-1}\alpha(\beta\alpha\beta\delta) = \gamma && \text{(the vertical tangent)},\\
&(\alpha\beta\delta\gamma)^2 = 1 && \text{(the relation at infinity)}.
\end{aligned}$$

Passing to α, $\bar{\alpha}$, β, $\bar{\beta}$, γ, $\bar{\gamma}$, see Lemma 3.1.2, we obtain a presentation for the group of a reducible sextic of torus type with the set of singularities $(\mathbf{A}_{11} \oplus \mathbf{A}_5) \oplus \mathbf{A}_3$:

$$\bar{\beta}\gamma = \gamma\beta = \bar{\gamma}\bar{\beta} = \beta\bar{\gamma}, \tag{4.7.1}$$

$$\alpha\beta = \bar{\alpha}\bar{\beta}, \tag{4.7.2}$$

$$(\alpha\beta)^3 = \bar{\beta}\bar{\alpha}\bar{\beta}\alpha\beta\alpha = \beta\alpha\beta\bar{\alpha}\bar{\beta}\bar{\alpha}, \tag{4.7.3}$$

$$(\beta\alpha)^{-1}\alpha(\beta\alpha) = \gamma, \quad (\bar{\beta}\bar{\alpha})^{-1}\bar{\alpha}(\bar{\beta}\bar{\alpha}) = \bar{\gamma}, \tag{4.7.4}$$

$$\gamma\alpha\beta\bar{\gamma}\bar{\alpha}\bar{\beta} = 1. \tag{4.7.5}$$

The perturbation $\mathbf{A}_5 \mapsto \mathbf{A}_4$ produces an irreducible plane sextic with the set of singularities

$$\mathbf{A}_{11} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3,$$

while adding the relations $\alpha = \gamma^{-1}\bar{\gamma}\bar{\alpha}\bar{\gamma}^{-1}\gamma = \gamma = \gamma^{-1}\bar{\gamma}\gamma$, see Lemma 3.5.2, which imply $\gamma = \bar{\gamma} = \alpha = \bar{\alpha}$. (Observe that the standard generators in a fiber over $x \gg 0$ are $\alpha_1 = \alpha$, $\beta_1 = \gamma$, $\delta_1 = \gamma^{-1}\delta\gamma$, $\gamma_1 = \beta$, and the extra relations given by Lemma 3.5.2 are $\alpha_1 = \delta_1^{-1}\alpha_1\delta_1 = \beta_1 = \delta_1^{-1}\beta_1\delta_1$.) The perturbations $\mathbf{A}_{11} \mapsto \mathbf{A}_{10}$ or $\mathbf{A}_6 \oplus \mathbf{A}_4$ produce irreducible sextics with the sets of singularities

$$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_3, \quad \mathbf{A}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$$

and add the relations $\alpha = \bar{\alpha} = \beta = \bar{\beta}$, see Lemma 3.5.1. In both cases, it is immediate that the resulting group is abelian.

5. Digression: classification of reducible sextics

The curves considered in §4 are sextics of torus type splitting into a quartic and a conic. Here, we state and indicate the proofs of a few results concerning the classification and the fundamental groups of such curves. Details will be published elsewhere. In Section 5.3, we outline the proof of Theorem 1.1.2.

5.1. The symmetry. Theorem 5.1.1 below substantiates Conjecture 4.2.3 in [9], concerning the relation between involutive stable symmetries of plane sextics and maximal trigonal curves in Σ_2.

5.1.1. Theorem. *Let B be a plane sextic of torus type, with simple singularities only, splitting into*

- *an irreducible quartic and irreducible conic,*
- *an irreducible quartic and two lines, or*
- *three irreducible conics.*

Then B admits an involutive stable (in the sense of [9]) symmetry $c\colon \mathbb{P}^2 \to \mathbb{P}^2$, and the quotient B/c is the maximal trigonal curve $\bar{B}_2 \subset \Sigma_2$ with the set of singularities $\mathbf{A}_5 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, see Section 2.5.

Conversely, for any section $\bar{L}$ of Σ_2 not tangent to $\bar{B}_2$ at its type $\mathbf{A}_5$ singular point, the double $B = \operatorname{Dbl}_{\bar{L}} \bar{B}_2$ is a plane sextic of torus type splitting as above.

5.1.2. Remark. The condition that $\bar{L}$ should not be tangent to $\bar{B}_2$ at its type $\mathbf{A}_5$ singular point is necessary and sufficient for B to have simple singularities only.

5.1.3. Remark. For most sextics B as in Theorem 5.1.1, the group of stable symmetries of B is $\mathbb{Z}_2$. Exceptions are sextics splitting into three irreducible conics: for each such sextic B, the group of stable symmetries of B is the group $\mathbb{S}_3$ of permutations of the components of B.

Proof. The proof is similar to [7], [8], and [9]. Assume that B splits into a quartic B_4 and a conic B_2. It is clear that the inner singularities are two cusps R'_∞, R''_∞ of B_4 (which may degenerate to a single type $\mathbf{A}_5$ or $\mathbf{E}_6$ singular point R_∞ of B_4) and two type $\mathbf{A}_5$ points R'_5, R''_5 of inflection tangency of B_4 and B_2 (which may degenerate to a single type $\mathbf{A}_{11}$ point R_5 of 6-fold intersection). The outer singularities are the two points R'_1, R''_1 of simple intersection of B_4 and B_2, which may degenerate to a single type $\mathbf{A}_3$ point R_1. Besides, B_4 may have an extra node or cusp Q_4, and B_2 may have an extra node Q_2 (splitting into two lines). As a further degeneration, R_1 may merge with Q_4 or Q_2.

Consider the minimal resolution $\tilde{X}$ of the double covering $X \to \mathbb{P}^2$ ramified at B. It is a $K3$-surface. Let $L = H_2(\tilde{X})$, let $\Sigma_P \subset L$ be the resolution lattice of a singular point P of B, and let Γ_P be the Dynkin graph of P. Denote $\Sigma = \bigoplus_P \Sigma_P$ and $\Gamma = \bigcup_P \Gamma_P$, and consider the lattice $S = \Sigma \oplus \langle h \rangle \subset L$, where $h \in L$ is the class of the pull-back of a generic line in $\mathbb{P}^2$. One has $h^2 = 2$. Let $\mathcal{K}$ be the image of $L = L^*$ in the discriminant group $\operatorname{discr} S = S^*/S$. Since B is of torus type, $\mathcal{K}$ has an element of order 3, see [3]. Besides, $\mathcal{K}$ has an element of order 2 responsible for the splitting $B = B_4 + B_2$, see [5]. (For example, the element of order 2 is represented by the class $[B_2] \in H_2(X) = L$.)

Consider an involutive symmetry $c_\Gamma\colon \Gamma \to \Gamma$ acting as follows: c_Γ transposes the two points within each pair (R'_∞, R''_∞), (R'_5, R''_5), (R'_1, R''_1) and acts identically on the diagram of each point $Q_\bullet$ that does not coincide with R_1. If a pair of points $R'_\bullet$, $R''_\bullet$ merges to a single point $R_\bullet$, then c_Γ acts on $\Gamma_{R_\bullet}$ by its nontrivial symmetry. (This description determines c_Γ uniquely up to a symmetry of the diagrams of R''_∞ and R''_5 whenever these points are separate.) Let $c_S\colon S \to S$ be the extension of c_Γ identical on h. One can check that c_Γ can be chosen so that c_S preserves $\mathcal{K}$ and induces the identity on $\mathcal{K}^\perp/\mathcal{K}$; hence, c_S extends to an involutive automorphism $c_*\colon L \to L$ identical on $\Sigma^\perp$. The latter is induced by a unique involution $c\colon \mathbb{P}^2 \to \mathbb{P}^2$ preserving B, *cf.* [9]. Details are left to the reader.

The fact that the quotient B/c is the curve $\bar{B}_2$ is straightforward: the quotient must have singular points of types $\mathbf{A}_5$, $\mathbf{A}_2$, and $\mathbf{A}_1$, resulting from the (pairs of) points R_5, R_∞, and R_1, respectively, and, due to [9], such a curve is unique. The converse statement is obvious. $\square$

If B splits into three conics, it has three type $\mathbf{A}_5$ inner singular points and three outer nodes, which may merge to a single type $\mathbf{D}_4$ singular point. An order 3 symmetry of B, see Remark 5.1.3, is constructed as above, starting from the order 3 symmetry $c_\Gamma\colon \Gamma \to \Gamma$ transposing cyclically the three inner points and three nodes (or acting by an order three symmetry on the Dynkin graph $\mathbf{D}_4$).

5.2. The classification. Using the stable symmetry and the description of special sections found in 2.5 and 2.6, one immediately obtains a deformation classification of sextics splitting as in Theorem 5.1.1.

5.2.1. Theorem. *Let B be a sextic as in Theorem* 5.1.1. *Then the combinatorial type of singularities of B is one of those listed in* [19] *or* $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus 2\mathbf{A}_1$. *The equisingular moduli space of sextics as in Theorem* 5.1.1 *realizing each of these combinatorial types is rational*; *in particular, it is connected.*

5.2.2. Remark. When referring to [19], we mean the combinatorial types marked as $B_2 + B_4$, $B_1 + B_1' + B_4$, or $B_2 + B_2' + B_2''$ in Theorem 1. The set of singularities $(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus 2\mathbf{A}_1$ (an irreducible quartic with a type $\mathbf{E}_6$ singular point and a conic, see Section 4.4) is erroneously missing in [19].

Proof. As in Section 2.4, the equisingular moduli spaces $\mathcal{M}(\Sigma)$ of sextics splitting as in Theorem 5.1.1 and possessing a given set of singularities Σ (more precisely, the spaces $\tilde{\mathcal{M}}(\Sigma)$ of pairs (B, c), where B is a sextic and c is an involutive stable symmetry of B) can be identified with the spaces of sections $\bar{L}$ of Σ_2 that are in a certain prescribed special position with respect to $\bar{B}_2$. The latter are described in Sections 2.5 and 2.6; they are all rational. It remains to notice that the forgetful map $\tilde{\mathcal{M}}(\Sigma) \to \mathcal{M}(\Sigma)$ is one to one, as any two stable involutions of a sextic B are projectively equivalent, see Remark 5.1.3. $\square$

5.2.3. Theorem. *Let B be a sextic as in Theorem* 5.1.1. *Then the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ factors to the group given by* (4.3.9).

5.2.4. Remark. The groups of most maximal sextics B as in Theorem 5.1.1 are computed in Sections 4.1–4.7. Perturbing $\bar{L}$, one can easily see that, if B has exactly two components, the quartic component of B has a set of singularities other than $3\mathbf{A}_2$, and $\mu(B) < 19$, then $\pi_1(\mathbb{P}^2 \smallsetminus B)$ is the group given by (4.3.9).

Proof. Any sextic as in Theorem 5.1.1 can be perturbed to a 'simplest' one, with the set of singularities $(2\mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus 2\mathbf{A}_1$, which is the double of $\bar{B}_2$ ramified at a section $\bar{L}$ transversal to $\bar{B}_2$. The group of a simplest sextic is (4.3.9), see, *e.g.*, Section 4.3. $\square$

5.3. Proof of Theorem 1.1.2. Let P be the type $\mathbf{A}_{17}$ singular point, and let $\Sigma_P \subset \Sigma \subset S \subset L$ *etc.* be as in the proof of Theorem 5.1.1. Denote $S_P = \Sigma_P \oplus \langle h \rangle$. Since the sextic is reducible and of torus type, the intersection $\mathcal{K}' = \mathcal{K} \cap \operatorname{discr} S_P$ contains an element of order 2 and an element of order 3, see [5] and [3]. On the other hand, $|\operatorname{discr} S_P| = 36$; hence, $(\mathcal{K}')^\perp/\mathcal{K}' = 0$, *i.e.*, the primitive hull of S_P in L is unimodular and the classification of homological types reduces to the study of sublattices isomorphic to 0, $\mathbf{A}_1$, $2\mathbf{A}_1$, or $\mathbf{A}_2$ in the direct sum of two hyperbolic planes. The rest is straightforward. $\square$

5.3.1. Remark. From the proof, it follows that each sextic B as in Theorem 1.1.2 admits a stable involutive symmetry c (constructed as in the proof of Theorem 5.1.1 starting from the nontrivial symmetry of Γ_P). However, one has $O_c \in B$; hence, the quotient $B/c \subset \Sigma_2$ is not a trigonal curve but rather a hyperelliptic curve with a type $\mathbf{A}_7$ singular point at E. Thus, Conjecture 4.2.3 in [9] needs to be modified to include maximal, in some sense, hyperelliptic curves as well.

6. Other fundamental groups

We consider a triple $\bar{B}_2'$, $\bar{L}'$, $\bar{L}$ as in §4 and make $\bar{L}$ and $\bar{L}'$ trade rôles, *i.e.*, we let $\bar{B} = \bar{B}_2' + \bar{L}$ and consider the double $B = \operatorname{Dbl}_{\bar{L}'}(\bar{B}_2' + \bar{L})$ ramified at $\bar{L}'$. The groups $\pi_1(\Sigma_2 \smallsetminus (\bar{B}_2' \cup \bar{L}' \cup \bar{L} \cup E))$ are computed in §4, and we merely use an appropriately modified version of Lemma 3.1.2 (with the rôle of δ played by the generator corresponding to $\bar{L}'$) to obtain $\pi_1(\mathbb{P}^2 \smallsetminus B)$. Then, as in §4, we perturb B to an irreducible sextic B' and apply Lemmas 3.5.1–3.5.6.

6.1. The set of singularities $2\mathbf{E}_7 \oplus \mathbf{D}_5$. Take for $\bar{L}$ the section given by (2.5.4), see Section 4.4 and Figure 10 (the solid grey line). The resulting sextic B has the set of singularities $2\mathbf{E}_7 \oplus \mathbf{D}_5$, and the group $\pi_1(\mathbb{P}^2 \smallsetminus B)$ is obtained from (4.4.1) by letting $\beta^2 = 1$ and passing to the subgroup generated by α, $\bar{\alpha} = \beta\alpha\beta$, δ, and $\bar{\delta} = \beta\delta\beta$. One has $\delta = \bar{\delta}$ and hence

$$\pi_1(\mathbb{P}^2 \smallsetminus B) = \bigl\langle \alpha, \bar{\alpha}, \delta \bigm| \alpha\bar{\alpha}\alpha = \bar{\alpha}\alpha\bar{\alpha},\ [\alpha, \delta] = [\bar{\alpha}, \delta] = 1,\ \alpha^2\bar{\alpha}^2\delta^2 = 1 \bigr\rangle. \tag{6.1.1}$$

The irreducible perturbation $\mathbf{D}_5 \mapsto \mathbf{A}_4$ produces the set of singularities

$$2\mathbf{E}_7 \oplus \mathbf{A}_4$$

and adds the relation $\alpha = \bar{\alpha} = \delta$, see Lemma 3.5.4. The irreducible perturbations $\mathbf{E}_7 \mapsto \mathbf{E}_6$, $\mathbf{A}_6$, or $\mathbf{A}_4 \oplus \mathbf{A}_2$ of one of the two type $\mathbf{E}_7$ singular points of B produce the sets of singularities

$$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5, \quad \mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6, \quad \mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$$

while adding at least the relation $\alpha\delta\alpha = \delta\alpha\delta$ (or $\gamma\delta\gamma = \delta\gamma\delta$, where $\gamma = \alpha^{-1}\bar{\alpha}\alpha$), see Lemma 3.5.6 and (3.5.7). In each case, it is immediate that the resulting fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus B')$ is abelian.

6.2. The set of singularities $2\mathbf{E}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$. Take for $\bar{L}$ the section given by (2.5.5), see the dotted grey line in Figure 10. A presentation for the group $\pi_1(\Sigma_2 \smallsetminus (\bar{B} \cup \bar{L}' \cup E))$ is found in Section 4.5. To pass to $\pi_1(\mathbb{P}^2 \smallsetminus B)$, we let $\beta^2 = 1$ and consider the subgroup generated by α, $\bar{\alpha} = \beta\alpha\beta$, γ, $\bar{\gamma} = \beta\gamma\beta$, δ, and $\bar{\delta} = \beta\delta\beta$. The relations are

$$\alpha\bar{\alpha}\alpha = \bar{\alpha}\alpha\bar{\alpha}, \tag{6.2.1}$$

$$\alpha\bar{\alpha}\alpha^{-1} = \gamma, \quad \bar{\alpha}\alpha\bar{\alpha}^{-1} = \bar{\gamma}, \tag{6.2.2}$$

$$\alpha^{-1}\delta\alpha = \bar{\alpha}^{-1}\bar{\delta}\bar{\alpha}, \tag{6.2.3}$$

$$\delta\bar{\gamma} = \gamma\delta = \bar{\gamma}\bar{\delta} = \bar{\delta}\gamma, \tag{6.2.4}$$

$$\gamma\delta\alpha\bar{\gamma}\bar{\delta}\bar{\alpha} = 1. \tag{6.2.5}$$

The irreducible perturbation $\mathbf{A}_3 \mapsto \mathbf{A}_2$ produces the set of singularities

$$2\mathbf{E}_7 \oplus 2\mathbf{A}_2$$

and adds the relation $\gamma = \bar{\gamma} = \delta = \bar{\delta}$. Then, comparing (6.2.2) and (6.2.3), one concludes that $\alpha = \bar{\alpha}$ and hence $\alpha = \gamma$. Thus, the group is abelian.

Consider a maximal irreducible perturbation of one of the two type $\mathbf{E}_7$ singular points of B, producing irreducible sextics with the sets of singularities

$$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2, \quad \mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2, \quad \mathbf{E}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

see Lemma 3.5.6. A generic real fiber close to R_∞ (the type $\mathbf{E}_7$ singular point of $\bar{B}$) is over $x \gg 0$, and the standard generators in this fiber are α, $(\beta\gamma\delta)\delta(\beta\gamma\delta)^{-1} = \delta$, $\beta\gamma\beta^{-1}$, and β. Hence, the group $\pi_1(U \smallsetminus B)$ of a Milnor ball about the point perturbed is generated by α, δ, and $\bar{\gamma}$, and the perturbation adds at least the relation $\bar{\gamma}\alpha\bar{\gamma} = \delta\bar{\gamma}\alpha$ (the third relation in (3.5.7)). Using (6.2.4), the additional relation simplifies to $\alpha\bar{\gamma} = \bar{\delta}\alpha$. Hence, $\bar{\delta} = \alpha\bar{\gamma}\alpha^{-1} = \bar{\alpha}$ (substituting to (6.2.2) and using (6.2.1)), $\alpha^{-1}\delta\alpha = \bar{\alpha}$ (from (6.2.3)), $\delta = \alpha\bar{\alpha}\alpha^{-1} = \gamma$ (from (6.2.2)), $\gamma = \bar{\gamma}$ and $[\bar{\gamma}, \bar{\alpha}] = 1$ (from (6.2.4) again), and $\alpha = \bar{\gamma} = \gamma = \bar{\alpha}$ (from (6.2.2)). Thus, the group is abelian.

6.3. The set of singularities $2\mathbf{E}_7 \oplus \mathbf{A}_2 \oplus 3\mathbf{A}_1$. Consider the section $\bar{L}$ tangent to $\bar{L}'$ and tangent to $\bar{B}_2'$ at its cusp R_∞. It is given by $y = 3/4$. Choose the generators $(\eta_1, \eta_2, \eta_3, \eta_4) = (\alpha, \delta, \beta, \gamma)$ in a real fiber F over $x \gg 0$.

We are only interested in the perturbation $\mathbf{E}_7 \mapsto \mathbf{A}_6$ of *both* type $\mathbf{E}_7$ singular points of B, producing an irreducible sextic with the set of singularities

$$2\mathbf{A}_6 \oplus \mathbf{A}_2 \oplus 3\mathbf{A}_1.$$

This perturbation can be realized symmetrically, by perturbing the type $\mathbf{E}_7$ singular point of $\bar{B}$ in Σ_2. According to Lemma 3.5.6, this gives the relations $\alpha = \delta = \beta$, and the monodromy about R_1 adds the relation $[\beta, \gamma] = 1$. Hence, the resulting group $\pi_1(\mathbb{P}^2 \smallsetminus B')$ is abelian.

6.4. The set of singularities $2\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$. Let $\bar{L}$ be the section given by (2.5.6), see the solid grey line in Figure 11. As in Section 4.6, the relations for $\pi_1(\mathbb{P}^2 \smallsetminus B)$ are (6.2.1)–(6.2.3), (6.2.5), and the relations

$$\gamma\delta = \bar{\gamma}\bar{\delta}, \quad \bar{\delta}\gamma\delta\gamma = (\gamma\delta)^2, \quad \delta\bar{\gamma}\bar{\delta}\bar{\gamma} = (\bar{\gamma}\bar{\delta})^2 \tag{6.4.1}$$

replacing (6.2.4).

The irreducible perturbations $\mathbf{A}_7 \mapsto \mathbf{A}_6$ or $\mathbf{A}_4 \oplus \mathbf{A}_2$, see Lemma 3.5.3, produce the sets of singularities

$$2\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2, \quad 2\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$$

and add the relations $\gamma = \bar{\gamma} = \delta = \bar{\delta}$. As in Section 4.6, the resulting groups are abelian. The irreducible perturbation $\mathbf{D}_5 \mapsto \mathbf{A}_4$ of one of the type $\mathbf{D}_5$ singular points produces the set of singularities

$$\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2.$$

The standard generators in a generic fiber close to R_∞ (a fiber over $x \gg 0$) are α, $(\beta\gamma\delta)\gamma(\beta\gamma\delta)^{-1}$, $(\beta\gamma)\delta(\beta\gamma)^{-1}$, and β. In view of Lemma 3.5.4, the extra relations added to the group are $\alpha = (\bar{\gamma}\bar{\delta})\bar{\gamma}(\bar{\gamma}\bar{\delta})^{-1} = \bar{\gamma}\bar{\delta}\bar{\gamma}^{-1}$. This implies $\alpha = \bar{\gamma} = \bar{\delta}$, and using (6.2.1)–(6.2.3) one derives that $\alpha = \bar{\alpha} = \gamma = \delta$. Hence, the group is abelian.

6.5. The set of singularities $3\mathbf{D}_5 \oplus \mathbf{A}_3$. Let $\bar{L}$ be the section given by (2.5.7), see the dotted grey line in Figure 11. Starting from the presentation found in Section 4.7, letting $\beta^2 = 1$, and passing to the subgroup generated by α, $\bar{\alpha} = \beta\alpha\beta$, γ, $\bar{\gamma} = \beta\gamma\beta$, δ, $\bar{\delta} = \beta\delta\beta$, we obtain the following relations for $\pi_1(\mathbb{P}^2 \smallsetminus B)$:

$$\delta\gamma = \bar{\delta}\bar{\gamma} = \gamma\bar{\delta} = \bar{\gamma}\delta, \tag{6.5.1}$$

$$\delta\alpha = \alpha\bar{\delta}, \quad \bar{\delta}\bar{\alpha} = \bar{\alpha}\delta, \tag{6.5.2}$$

$$\delta\alpha\bar{\alpha}\alpha = \bar{\delta}\bar{\alpha}\alpha\bar{\alpha}, \tag{6.5.3}$$

$$\alpha^{-1}\bar{\alpha}\alpha = \gamma, \quad \bar{\alpha}^{-1}\alpha\bar{\alpha} = \bar{\gamma}, \tag{6.5.4}$$

$$\delta\gamma\alpha\bar{\delta}\bar{\gamma}\bar{\alpha} = 1. \tag{6.5.5}$$

The irreducible perturbation $\mathbf{A}_3 \mapsto \mathbf{A}_2$ produces the set of singularities

$$3\mathbf{D}_5 \oplus \mathbf{A}_2,$$

adding the relation $\gamma = \bar{\gamma} = \delta = \bar{\delta}$. The irreducible perturbation $\mathbf{D}_5 \mapsto \mathbf{A}_4$ of one of the type $\mathbf{D}_5$ singular points of B produces the set of singularities

$$2\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3,$$

adding the relation $\alpha = \bar{\alpha} = \delta = \bar{\delta}$, see Lemma 3.5.4. (We can assume that the point perturbed is over R_5.) It is straightforward that, in both cases, the new fundamental group is abelian.

6.6. Concluding remarks. The groups of all reducible curves obtained in this section are non-abelian; they all factor to the 'minimal' group G given by (6.1.1), which can also be described as a central extension

$$1 \longrightarrow \langle\delta\rangle \longrightarrow G \longrightarrow \mathbb{A}_4 \longrightarrow 1.$$

This result is quite expectable, as all curves split into a conic B_2 and a three cuspidal quartic B_4, and $\pi_1(\mathbb{P}^2 \smallsetminus B_4) = \mathbb{A}_4$.

It is worth mentioning that all curves admit regular $\mathbb{S}_3$-coverings while obviously not being of torus type. Hence, Theorem 4.1.1 in [5] does not extend to reducible curves literally. Certainly, the reason is the fact that the cyclic part of the covering is not ramified at B but rather at B_4 only.

In most cases, the trigonal curve $\bar{B} = \bar{B}'_2 + \bar{L} \subset \Sigma_2$ used in the construction is maximal, with the set of singularities $\mathbf{E}_7 \oplus \mathbf{A}_1$ or $\mathbf{D}_5 \oplus \mathbf{A}_3$. Thus, one may hope that the deck translation is a stable symmetry of the covering sextic B (*cf.* Conjecture 4.2.3 in [9]) and use this correspondence to classify sextics.

In the former case, the set of singularities $\mathbf{E}_7 \oplus \mathbf{A}_1$, the sextics are characterized by the splitting $B = B_4 + B_2$, where B_4 is a quartic with at least two cusps and B_2 is a conic (possibly reducible) tangent to B_4 at two of its cusps. Any such curve is indeed symmetric: in appropriate affine coordinates (x, y) in $\mathbb{P}^2$, the curves B_4 and B_2 can be given by $a + b(x + y) + cx^2y^2 = 0$ and $d + exy = 0$, respectively. There are 13 deformation families of such curves, of which four are maximal. Three maximal families are considered in Sections 6.1–6.3; the fourth one has the set of singularities $2\mathbf{E}_7 \oplus \mathbf{D}_4 \oplus \mathbf{A}_1$ (the conic B_2 splits and the quartic B_4 has an extra node and passes through the node of B_2).

In the latter case, the set of singularities $\mathbf{D}_5 \oplus \mathbf{A}_3$, the sextic splits into $B_4 + B_2'$, where B_4 is a quartic with at least two cusps and B_2' is a conic passing through two cusps of B_4 and tangent to B_4 at all other intersection points. It appears that this configuration, as well as most of its degenerations, is realized by several equisingular deformation families, only one of them admitting a stable symmetry. The symmetric sextics seem to be related to the sextics of torus type considered in §5: they are obtained by replacing the conic B_2 in the splitting $B = B_4 + B_2$, see Theorem 5.1.1, by the conic $B_2' = \{p = 0\}$, where $p^3 + q^2 = 0$ is the (only) torus structure on B. (From the point of view of the trigonal curve, we replace the $\bar{L}'$ component in $\bar{B}_2 = \bar{B}_2' + \bar{L}'$ with the section passing through R_∞ and tangent to $\bar{B}_2'$ at R_5.) We will treat this subject in details elsewhere.

Note that each double $B = \operatorname{Dbl}_{\bar{L}} \bar{B}$ of the trigonal curve $\bar{B}$ with the set of singularities $\mathbf{D}_5 \oplus \mathbf{A}_3$ has non-abelian fundamental group: all groups factor to the 'simplest' one, corresponding to the case when $\bar{L}$ is transversal to $\bar{B}$. Letting $\alpha = \bar{\alpha}$, $\gamma = \bar{\gamma}$, and $\delta = \bar{\delta}$ in the presentation in Section 6.4, we obtain the following presentation for the simplest group G:

$$G = \langle \gamma, \delta \,|\, (\gamma\delta)^2 = (\delta\gamma)^2,\ (\gamma\delta\gamma)^2 = 1 \rangle.$$

Introducing new generators $u = \delta\gamma$, $v = \gamma\delta\gamma$, we can simplify the presentation to $G = \langle u, v \,|\, v^2 = [v, u^2] = 1 \rangle$. It is clear that the commutant $[G, G]$ equals $\mathbb{Z}$, both u and v acting on $[G, G]$ by the multiplication by (-1). In particular, all curves admit regular $\mathbb{D}_{2n}$-coverings for any $n \geqslant 3$; however, they are *not* $\mathbb{D}_{2n}$-sextics.

7. SUMMARY

We summarize the results on the fundamental group of an irreducible plane sextic obtained in this paper and combine them with [8] and [10]. We confine ourselves to the case of simple singularities only; the groups of sextics with a non-simple singular point are essentially found in [5] and [6] (see also Oka, Pho [21]).

7.1. Sextic of torus type. According to Oka, Pho [20], there are 19 tame and 109 non-tame sets of simple singularities realized by irreducible plane sextics of torus type. At present, the fundamental group is known for 113 sets of singularities, including all tame ones. The result is summarized in Table 1, where 'nt#' is the notation introduced in [20] and the last column indicates a proof, either by referring to the appropriate paper/section or by suggesting a degeneration (in the form '→ nt#') to a set of singularities with known group. We only list the sets of singularities for which the degenerations suggested in [20] lead to sextics whose groups are still unknown.

With few exceptions, the fundamental group of an irreducible sextic of torus type is Zariski's group $\mathbb{B}_3 \cong \mathbb{Z}_2 * \mathbb{Z}_3$. The known exceptions are

- sextics of weight 8 and 9 in the sense of [5], see [8];
- sextics marked with a * in Table 1, see references in the table;
- the set of singularities $2\mathbf{E}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$, see [10].

Various perturbations of the exceptional sextics are studied explicitly in [8] and [10]; all other groups are given by Corollary 3.6.2.

7.1.1. Remark. For most non-maximal sets of singularities, the connectedness of the equisingular deformation family is still unknown, although expected, see

TABLE 1. Sextics of torus type: known groups

No.	The set of singularities	Where?
nt23	${}^*(6\mathbf{A}_2) \oplus 3\mathbf{A}_2$	see [8]
nt32	${}^*(\mathbf{A}_5 \oplus 4\mathbf{A}_2) \oplus \mathbf{E}_6$	see [8]
nt36	$(\mathbf{A}_5 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	→ nt32
nt47	${}^*(\mathbf{E}_6 \oplus 4\mathbf{A}_2) \oplus \mathbf{A}_5$	same as nt32
nt54	$(2\mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_1$	→ nt55
nt57	$(2\mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$	see 4.2
nt63	$(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$	→ nt70
nt67	${}^*(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus 2\mathbf{A}_2$	same as nt32
nt70	${}^*(2\mathbf{E}_6 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$	see [10]
nt74	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_3$	→ nt128
nt77	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{D}_4$	see 4.3
nt81	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	→ nt88
nt88	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$	see 4.1
nt99	${}^*(2\mathbf{E}_6 \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$	see [10], [13]
nt100	${}^*(3\mathbf{E}_6) \oplus \mathbf{A}_1$	see [10], [21]
nt103	$(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$	→ nt128
nt105	$(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus 2\mathbf{A}_1$	→ nt103
nt108	$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_1$	→ nt112
nt112	$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus 2\mathbf{A}_1$	see 4.4
nt117	$(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_3$	see 4.3
nt120	$(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	see 4.1
nt128	$(2\mathbf{A}_8) \oplus \mathbf{A}_3$	see 3.4
nt134	$(\mathbf{A}_{11} \oplus \mathbf{A}_5) \oplus \mathbf{A}_2$	→ nt145
nt135	$(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_1$	see 4.4
nt138	$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2$	→ nt145
nt145	$(\mathbf{A}_{17}) \oplus \mathbf{A}_2$	see 3.3

Conjecture 1.2.1. For these sets of singularities, we can only state the result in the form of existence, *i.e.*, to assert that there is a sextic B of torus type realizing a given set of singularities and such that $\pi_1(\mathbb{P}^2 \smallsetminus B) = \bar{\mathbb{B}}_3$. To my knowledge, the sets of singularities for which the classification is completed are:

- sextics admitting a stable involutive symmetry, see [9] for the list and [8], [10], and Theorem 1.1.1 for the classification;
- the sets of singularities of the form (inner points) $\oplus\, k\mathbf{A}_1$, see [3] and [22].

The fifteen remaining sets of singularities, for which the fundamental group is still unknown, are listed in Table 2 (with a reference to the notation of [20]).

7.2. Sextics with abelian fundamental groups. In Table 3, we list the sets of singularities realized by irreducible plane sextics with abelian fundamental group, together with the references to the sections where these curves are constructed. Combining these results with [8] and [10] and considering all perturbations, we obtain 768 sets of singularities not covered by Nori's theorem [17].

7.3. Classical Zariski pairs. The list resulting from Table 1 contains a number of sextics of weight 7 with at least two cusps. Perturbing a cusp of such a sextic, we

TABLE 2. Sextics of torus type: unknown groups

No.	The set of singularities
nt64	$(\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$
nt75	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4$
nt78	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{D}_5$
nt82	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$
nt83	$(\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$
nt104	$(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$
nt106	$(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$
nt109	$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2$
nt110	$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$
nt113	$(\mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$
nt118	$(\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4$
nt136	$(\mathbf{E}_6 \oplus \mathbf{A}_{11}) \oplus \mathbf{A}_2$
nt139	$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3$
nt141	$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus 2\mathbf{A}_1$
nt142	$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$

TABLE 3. Extremal sextics with abelian fundamental groups

The set of singularities	Where?
$2\mathbf{E}_7 \oplus \mathbf{A}_4$	6.1
$2\mathbf{E}_7 \oplus 2\mathbf{A}_2$	6.2
$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$	6.1
$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	6.2
$\mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6$	6.1
$\mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	6.1
$\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	6.2
$\mathbf{E}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	6.2
$\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus 2\mathbf{A}_1$	4.4
$\mathbf{E}_6 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_1$	4.4
$\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	4.5
$3\mathbf{D}_5 \oplus \mathbf{A}_2$	6.5
$2\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	6.4
$2\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	6.5
$2\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$	6.4
$\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	6.4
$\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$	4.2
$\mathbf{D}_4 \oplus \mathbf{A}_{10} \oplus 2\mathbf{A}_2$	4.3
$\mathbf{D}_4 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$	4.3
$\mathbf{D}_4 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$	4.6
$\mathbf{A}_{16} \oplus \mathbf{A}_2$	3.6
$\mathbf{A}_{15} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	3.6
$\mathbf{A}_{13} \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	3.6
$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	3.6
$\mathbf{A}_{11} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	4.7
$\mathbf{A}_{11} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	4.3
$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	3.6
$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$	4.7
$\mathbf{A}_{10} \oplus 3\mathbf{A}_2 \oplus 2\mathbf{A}_1$	4.1
$\mathbf{A}_9 \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$	3.6
$\mathbf{A}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3$	3.6
$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$	3.6
$\mathbf{A}_8 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_3$	3.6
$2\mathbf{A}_6 \oplus \mathbf{A}_2 \oplus 3\mathbf{A}_1$	6.3
$\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	4.7
$\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus 3\mathbf{A}_2 \oplus 2\mathbf{A}_1$	4.1

obtain 30 so called *classical Zariski pairs*, *i.e.*, pairs of irreducible sextics that share the same set of singularities but differ by their Alexander polynomials (see [3] for details and further references; the sextic is/is not of torus type if the cusp perturbed is, respectively, outer/inner). One can add to this list the sets of singularities

$$\mathbf{E}_6 \oplus \mathbf{A}_{11}, \quad \mathbf{E}_6 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2, \quad \mathbf{A}_{17}, \quad \mathbf{A}_{11} \oplus \mathbf{A}_5, \quad 2\mathbf{A}_8, \tag{7.3.1}$$

which are realized by sextics with abelian fundamental groups in Eyral, Oka [12], thus obtaining 35 classical Zariski pairs. (Sextics of torus type realizing (7.3.1) are constructed in this paper. The two other sets of singularities discovered in [12] are already on the list.) In each pair, the groups of the two curves are $\bar{\mathbb{B}}_3$ and $\mathbb{Z}_6$.

According to [3] and [22], each of the 35 sets of singularities obtained above is realized by exactly two equisingular deformation families of irreducible sextics, one of torus type and one not. Altogether, there are 51 classical Zariski pairs of irreducible sextics (one of them being, in fact, a triple). For all but one of them (the set of singularities $(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus 2\mathbf{A}_1$), the group of the curve of torus type is known; it equals $\bar{\mathbb{B}}_3$.

REFERENCES

1. A. Degtyarev, *Isotopy classification of complex plane projective curves of degree* 5, Algebra i Analis **1** (1989), no. 4, 78–101 (Russian); English transl. in Leningrad Math. J. **1** (1990), no. 4, 881–904.
2. A. Degtyarev, *Quintics in* $\mathbb{C}\mathrm{p}^2$ *with nonabelian fundamental group*, Algebra i Analis **11** (1999), no. 5, 130–151 (Russian); English transl. in Leningrad Math. J. **11** (2000), no. 5, 809–826.
3. A. Degtyarev, *On deformations of singular plane sextics*, J. Algebraic Geom. **17** (2008), 101–135.
4. A. Degtyarev, *Zariski k-plets via dessins d'enfants*, Comment. Math. Helv., `arXiv:0710.0279` (to appear).
5. A. Degtyarev, *Oka's conjecture on irreducible plane sextics*, J. London Math. Soc., `arXiv:math.AG/0701671` (to appear).
6. A. Degtyarev, *Oka's conjecture on irreducible plane sextics.* II, `arXiv:math.AG/0702546`.
7. A. Degtyarev, *On irreducible sextics with non-abelian fundamental group*, Fourth Franco-Japanese Symposium on Singularities (Toyama, 2007), `arXiv:0711.3070` (to appear).
8. A. Degtyarev, *Irreducible plane sextics with large fundamental groups*, `arXiv:0712.2290`.
9. A. Degtyarev, *Stable symmetries of plane sextics*, `arXiv:0802.2336`.
10. A. Degtyarev, *Fundamental groups of symmetric sextics*, `arXiv:0803.3219`.
11. A. Degtyarev, M. Oka, *A plane sextic with finite fundamental group*, Fourth Franco-Japanese Symposium on Singularities (Toyama, 2007), `arXiv:0711.3067` (to appear).
12. C. Eyral, M. Oka, *On the fundamental groups of the complements of plane singular sextics*, J. Math. Soc. Japan **57** (2005), no. 1, 37–54.
13. C. Eyral, M. Oka, *Fundamental groups of the complements of certain plane non-tame torus sextics*, Topology Appl. **153** (2006), no. 11, 1705–1721.
14. C. Eyral, M. Oka, *On a conjecture of Degtyarev on non-torus plane curves*, Fourth Franco-Japanese Symposium on Singularities (Toyama, 2007) (to appear).
15. E. R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), 255–260.
16. E. Looijenga, *The complement of the bifurcation variety of a simple singularity*, Invent. Math. **23** (1974), 105–116.
17. M. V. Nori, *Zariski conjecture and related problems*, Ann. Sci. Éc. Norm. Sup., 4 série **16** (1983), 305–344.
18. M. Oka, *Symmetric plane curves with nodes and cusps.*, J. Math. Soc. Japan **44** (1992), no. 3, 375–414.
19. M. Oka, *Geometry of reduced sextics of torus type*, Tokyo J. Math. **26** (2003), no. 2, 301–327.
20. M. Oka, D. T. Pho, *Classification of sextics of torus type*, Tokyo J. Math. **25** (2002), no. 2, 399–433.
21. M. Oka, D. T. Pho, *Fundamental group of sextics of torus type*, Trends in singularities, Trends Math., Birkhäuser, Basel, 2002, pp. 151–180.
22. A. Özgüner, *Classical Zariski pairs with nodes*, M.Sc. thesis, Bilkent University, 2007.
23. H. Tokunaga, $(2,3)$*-torus sextics and the Albanese images of* 6*-fold cyclic multiple planes*, Kodai Math. J. **22** (1999), no. 2, 222–242.
24. H. Tokunaga, *A note on triple covers of* $\mathbb{P}^2$ (to appear).
25. O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.

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