Lines generate the Picard groups of certain Fermat surfaces

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ABSTRACT

We answer a question of T. Shioda and show that, for any positive integer \( m \) prime to 6, the Picard group of the Fermat surface \( \Phi_m \) is generated by the classes of lines contained in \( \Phi_m \). A few other classes of surfaces are also considered.

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1. Introduction

1.1. Principal results

All algebraic varieties in the paper are over \( \mathbb{C} \). Let \( m \) be a positive integer, and let

\[ \Phi_m := \{ z_0^m + z_1^m + z_2^m + z_3^m = 0 \} \subset \mathbb{P}^3 \]
be the Fermat surface. If \( m = 1 \) (plane) or \( m = 2 \) (quadric), then \( \Phi_m \) contains infinitely many lines (meaning true straight lines in \( \mathbb{P}^3 \)); otherwise, \( \Phi_m \) is known to contain exactly \( 3m^2 \) lines.

Since \( \Phi_m \) is simply connected, one can identify its Picard group \( \text{Pic}\Phi_m \) and its Néron–Severi lattice \( \text{NS}(\Phi_m) \). Citing [1], the Néron–Severi group “... is a rather delicate invariant of arithmetic nature. Perhaps for this reason it usually requires some nontrivial work before one can determine the Picard number of a given variety, let alone the full structure of its Néron–Severi group.” The Picard groups of Fermat surfaces are related to those of the more general Delsarte surfaces (see [15]; they fit into the framework outlined in Section 2.4). Furthermore, continuing the citation, “Combined with the method based on the inductive structure of Fermat varieties, this might lead to the verification of the Hodge conjecture for all Fermat varieties.”

Let \( S_m \subset \text{Pic}\Phi_m \) be the subgroup generated by the classes of the lines contained in \( \Phi_m \). Then, according to [14], one has

\[
S_m \otimes \mathbb{Q} = (\text{Pic}\Phi_m) \otimes \mathbb{Q} \quad \text{if and only if} \quad m \leq 4 \text{ or } \gcd(m, 6) = 1. \tag{1.1}
\]

This statement is proved by comparing the dimensions of the two spaces, which are computed independently. In other words, the classes of lines generate \( \text{Pic}\Phi_m \) rationally, and a natural question, raised in [1], is whether they also generate the Picard group over the integers. A partial answer to this question was given in [12], almost 30 years later: the equality \( \text{Pic}\Phi_m = S_m \) holds for all integers \( m \) prime to 6 in the range \( 5 \leq m \leq 100 \). This fact is proved by supersingular reduction and a computer aided computation of the discriminants of the lattices involved. (The case \( m = 3 \) is classical: any nonsingular cubic contains 27 lines, which generate its Picard group. The case \( m = 4 \), i.e., that of \( K'3 \)-surfaces, was settled in [10], see also [3] for a slight generalization. The proof suggested below works for both cases.)

The principal result of the present paper is the following theorem, answering the above question in the affirmative in the general case.

**Theorem 1.2.** Let \( m \geq 1 \) be an integer such that either \( m \leq 4 \) or \( \gcd(m, 6) = 1 \). Then \( \text{Pic}\Phi_m = S_m \), i.e., \( \text{Pic}\Phi_m \) is generated by the classes of lines.

Since the \( 3m^2 \) lines in \( \Phi_m \) admit a very explicit description (cf. Section 2.4) and one can easily see how they intersect (see, e.g., Eq. (6) in [12]), Theorem 1.2 gives us a complete description of \( \text{Pic}\Phi_m = \text{NS}(\Phi_m) \), including the intersection form and the action of the automorphism group of \( \Phi_m \).

In view of (1.1), Theorem 1.2 is an immediate consequence of the following statement, which is actually proved in the paper, see Section 4.2. (Throughout the paper, we use Tors \( A \) for the \( \mathbb{Z} \)-torsion of an abelian group/module \( A \).)

**Theorem 1.3.** For any integer \( m \geq 1 \), one has \( \text{Tors}(\text{Pic}\Phi_m/S_m) = 0 \).
In the mean time, an interesting generalization, approaching the problem from a different angle, was suggested in [13]. Briefly, $\Phi_m$ can be represented as an $m^3$-fold ramified covering of the plane, and one can try to study other multiple planes with the same ramification locus (see Section 2.4 and Problem 2.6 for details). Considered in [13] are cyclic coverings of degree at most 50, and, similar to [12], the proof is also based on comparing the discriminants of the two lattices.

The approach developed in the present paper, including the computation of the Alexander module $A[\alpha]$ (see Section 3.3), which is crucial for the proof, applies to Delsarte surfaces as well. Here, we make a few first steps towards this generalized problem and work out another special case, see Theorem 4.18. In the forthcoming paper [5], we close the case of cyclic Delsarte surfaces started in [13] and modify part of the proof of Theorem 1.3 (see Section 4.2) to adapt it to slightly more general diagonal Delsarte surfaces. On the other hand, numeric experiments show that Theorem 1.3 does not extend literally to all Delsarte surfaces: sometimes, the quotient does have torsion. Next special classes to be studied in more details would probably be the nonsingular Delsarte surfaces and those with $A$–$D$–$E$ singularities.

As yet another application, we consider another class of surfaces whose Picard group is rationally generated by lines, see [3]. Let $p$ and $q$ be two square free bivariate homogeneous polynomials of degree $m$, and denote

$$\Sigma_{p,q} := \{p(z_0, z_1) = q(z_2, z_3)\} \subset \mathbb{P}^3.$$ 

This nonsingular surface contains an obvious set of $m^2$ lines, viz. those connecting the points $[z_0 : z_1 : 0 : 0]$ and $[0 : 0 : z_2 : z_3]$, where $p(z_0, z_1) = q(z_2, z_3) = 0$, and we denote by $S_{p,q} \subset \text{Pic} \Sigma_{p,q}$ the subgroup generated by the classes of these lines.

**Theorem 1.4.** (See Section 4.4.) For any pair $p, q$ as above, $\text{Tors}(\text{Pic} \Sigma_{p,q}/S_{p,q}) = 0$.

**Corollary 1.5.** (See Section 4.4.) If $m$ is prime and $p, q$ as above are sufficiently generic, then $\text{Pic} \Sigma_{p,q}$ is generated by the classes of the $m^2$ lines contained in $\Sigma_{p,q}$.

1.2. An outline of the proof

In Section 2, we reduce the question to the computation of the torsion of the 1-homology of a certain space, see Theorem 2.2. We also recall the classical description of the lines in $\Phi_m$ by means of a ramified covering of the plane and, following [13], describe a generalization of the problem to a wider class of surfaces. In Section 3, we compute the so-called Alexander module (or rather Alexander complex) $A[m]$ of the above covering and its reduced version $\overline{A}[m]$. The heart of the proof is a tedious computation of the length $\ell(\overline{A}[m])$, see Lemma 4.4 in Section 4; then, Theorem 1.3 follows from comparing the result to the expected value given by [1, 14], see Section 4.2. In Section 4.3, we work out a toy example, illustrating the suggested line of attack to the generalized problem.
2. Preliminaries

2.1. Prerequisites

For the reader’s convenience, we recall, with references to [7], a few necessary facts
from algebraic topology. An ultimate reference would be [8]; unfortunately it is only
available in Russian.

By definition, for any topological pair \((X, A)\) we have the following short exact
sequence of singular chain complexes:

\[
0 \rightarrow S_*(A) \rightarrow S_*(X) \rightarrow S_*(X, A) \rightarrow 0.
\]

All complexes are free; hence, applying \(\otimes G\) or \(\text{Hom}(\cdot, G)\), we also have short exact
sequences of (co-)chain complexes with any coefficient group \(G\). The associated long
exact sequences in (co-)homology are called the (co-)homology exact sequences of pair
\((X, A)\), cf. (3.2) in [7, Chapter III].

Unless specified otherwise, all (co-)homology are with coefficients in \(\mathbb{Z}\). The other
groups can be computed using the so-called universal coefficient formulas (see, e.g.,
(7.9) and (7.10) in [7, Chapter VI]): for any topological space \(X\), abelian group \(G\), and
integer \(n\), there are natural split (not naturally) exact sequences

\[
0 \rightarrow H_n(X) \otimes G \rightarrow H_n(X; G) \rightarrow \text{Tor}(H_{n-1}(X), G) \rightarrow 0,
\]

\[
0 \rightarrow \text{Ext}(H_{n-1}(X), G) \rightarrow H^n(X; G) \rightarrow \text{Hom}(H_n(X), G) \rightarrow 0.
\]

(Here, \(\text{Tor} = \text{Tor}_1\) and \(\text{Ext} = \text{Ext}^1\) are the derived functors in the category of \(\mathbb{Z}\)-modules.)

Similar statements hold for the relative groups of pairs \((X, A)\). Assuming all groups
finitely generated (e.g., \(X\) is a finite CW-complex), a consequence of the second exact
sequence is the assertion that \(H^n(X)\) is free if and only if so is \(H_{n-1}(X)\); in this case,
\(H^n(X) = \text{Hom}(H_n(X), \mathbb{Z})\).

We use the following terminology for various duality isomorphisms in topology of
manifolds. Let \(M\) be an oriented compact manifold, possibly with boundary, \(\dim M = n\),
and \(A \subset M\) a ‘sufficiently good’ (see the end of this paragraph) closed subset. If \(\partial M = \emptyset\),
the multiplication by the fundamental class \([M]\) establishes canonical isomorphisms

- \(H^p(M) = H_{n-p}(M)\) (Poincaré duality, [8, Theorem 4 in §17.3]) and
- \(H^p(M, A) = H_{n-p}(M \setminus A)\) (Poincaré–Lefschetz duality, [8, Theorem 14, Exercise 44,
  and Corollary thereof in §17.9]).

In general, the multiplication by \([M, \partial M]\) is an isomorphism

- \(H^p(M) = H_{n-p}(M, \partial M)\) (Lefschetz duality, [8, Part B of §17.9]).
All statements are classical and well known. For example, they can be derived as special cases of Proposition 7.2 in [7, Chapter VIII], with an extra observation that, in all cases considered in the paper, $M$ and $A$ are at worst compact semialgebraic sets, thus admitting finite triangulations (see, e.g., [9]); hence, they are absolute neighborhood retracts and the Čech cohomology groups in [7] can be replaced with the singular ones. As another consequence of [9], all (co-)homology groups involved are finitely generated.

### 2.2. Divisors

Consider a smooth projective algebraic surface $X$. By Poincaré duality $H^2(X) = H_2(X)$, we can regard the Néron–Severi lattice $NS(X)$ as a subgroup of the intersection index lattice $H_2(X)/\text{Tors}$, representing a divisor $D \subset X$ by its fundamental class $[D]$, see Section 2.3 below. (The Néron–Severi lattice is the group of divisors modulo numeric equivalence; thus, we ignore the torsion.) Since $\text{Pic} \, X = H^1(X; O_X^*)$ and $H^2(X; O_X)$ is a $\mathbb{C}$-vector space, the exponential exact sequence

$$H^1(X; O_X) \longrightarrow H^1(X; O_X^*) \longrightarrow H^2(X) \longrightarrow H^2(X; O_X)$$

(2.1)

implies that $NS(X)$ is a primitive subgroup in $H_2(X)/\text{Tors}$.

If $H_1(X) = 0$, then $H^2(X) = \text{Hom}(H_2(X), \mathbb{Z})$ is torsion free (the universal coefficient formula), and so is $H_2(X) = H^2(X)$. Since also $H^1(X; O_X) = H^{0,1}(X)$ is trivial in this case, from (2.1) we have $\text{Pic} \, X = NS(X)$, i.e., we do not need to distinguish between linear, algebraic, or numeric equivalence of divisors.

Consider a reduced curve $D \subset X$. Topologically, the normalization $\tilde{D}$ of $D$ is a closed surface, and the projection $\sigma: \tilde{D} \to D$ is a homeomorphism outside a finite subset $S \subset \tilde{D}$. We have isomorphisms

$$H_2(D) \xrightarrow{\partial} H_2(D, \sigma(S)) \xleftarrow{\sigma^*} H_2(\tilde{D}, S) \xleftarrow{\tilde{\partial}} H_2(\tilde{D}) = \bigoplus \mathbb{Z} \cdot [D_i],$$

where $\partial, \tilde{\partial}$ are the connecting isomorphisms in the respective exact sequences of pairs and $\sigma^*$ is induced by the relative homeomorphism $(\tilde{D}, \tilde{S}) \to (D, S)$. (For the latter, one can choose a triangulation of $\tilde{D}$ with respect to which the points of $\tilde{S}$ are vertices and project this triangulation to $D$; then, $\sigma$ would induce an isomorphism of the relative cellular chain complexes.) Thus, $H_2(D) = H_2(\tilde{D})$ is the free abelian group generated by the fundamental classes $[D_i]$ of the irreducible components $D_i$ of $D$ (or, equivalently, the fundamental classes $[\tilde{D}_i]$ of the connected components $\tilde{D}_i$ of $\tilde{D}$; in view of the canonical isomorphism, we do not distinguish $[\tilde{D}_i]$ from $[D_i]$). A similar computation in cohomology (the above sequence with all indices lifted and all arrows reversed) gives us an isomorphism $H^2(D) = H^2(\tilde{D})$. Since the group $H^2(\tilde{D}) = H_0(\tilde{D})$ (Poincaré duality) is torsion free, from the universal coefficient formula we have further

$$H^2(D) = H^2(\tilde{D}) = \text{Hom}(H_2(\tilde{D}), \mathbb{Z}) = \bigoplus \mathbb{Z} \cdot [D_i]^*. $$
(The last identification uses the canonical basis \{[D_i]\}.) Another application of the universal coefficient formula shows that \(H_1(D)\) is also free. (Essentially, we only use the fact that the singular locus has real codimension at least two.)

2.3. Imprimitivity via homology

As above, let \(D \subset X\) be a reduced curve in a smooth projective surface \(X\). Denoting by \(ι: D \hookrightarrow X\) the inclusion, let

\[
S(D) = \text{Im}[ι_*: H_2(D) \to H_2(X)/\text{Tors}].
\]

As explained in Section 2.2, \(S(D) \subset NS(X)\) is the subgroup generated by the irreducible components of \(D\). For convenience, we retain the notation \(ι: D \hookrightarrow X\) and \(S(D)\) in the case when \(D = \sum n_iD_i, n_i \neq 0\), is a divisor in \(X\) (thus identifying \(D\) with its support \(\cup D_i\)). The fundamental class of a divisor \(D\) is \([D] := \sum n_i[D_i]\).

**Theorem 2.2.** Let \(ι: D \hookrightarrow X\) be as above, and assume that \(H_1(X) = 0\). Then there are canonical isomorphisms

\[
\text{Tors } H_1(X \setminus D) = \text{Hom}(T(D), \mathbb{Q}/\mathbb{Z}), \quad H_1(X \setminus D)/\text{Tors} = \text{Hom}(K(D), \mathbb{Z}),
\]

where \(T(D) := \text{Tors}(NS(X)/S(D))\) and \(K(D) := \text{Ker}[ι_*: H_2(D) \to H_2(X)]\).

**Proof.** By Poincaré–Lefschetz duality, we have \(H_1(X \setminus D) = H^3(X, D)\). Consider the following fragment of the cohomology exact sequence of pair \((X, D)\):

\[
H^2(X) \xrightarrow{ι^*} H^2(D) \xrightarrow{δ} H^3(X, D) \to H^3(X).
\]

Since \(H^3(X) = H_1(X) = 0\) (Poincaré duality), we have a canonical isomorphism

\[
H_1(X \setminus D) = \text{Coker } ι^*.
\]  

(2.3)

As explained above, both \(H^2(X)\) and \(H^2(D)\) are free abelian groups and, for both spaces, we have \(H^2(\cdot) = \text{Hom}(H_2(\cdot), \mathbb{Z})\); hence, \(ι^* = \text{Hom}(ι_* , \text{id}_\mathbb{Z})\). The exact sequence

\[
0 \to K(D) \xrightarrow{\text{in}} H_2(D) \xrightarrow{ι_*} H_2(X)
\]

can be regarded as a free resolution of \(Q := H_2(X)/S(D)\). Applying \(\text{Hom}(\cdot, \mathbb{Z})\), we obtain a cochain complex

\[
0 \to H^2(X) \xrightarrow{ι^*} H^2(D) \xrightarrow{\text{in}*} H_2(K(D), \mathbb{Z}) \to 0
\]
computing the derived functors: $H^0 = \text{Hom}(\mathbb{Q}, \mathbb{Z})$, $H^1 = \text{Ext}(\mathbb{Q}, \mathbb{Z})$, $H^i = 0$ for $i \geq 2$. By the definition of $H^1$ and $H^2$, this gives us a short exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Q}, \mathbb{Z}) \rightarrow \text{Coker } i^* \rightarrow \text{Hom}(\mathbb{K}(D), \mathbb{Z}) \rightarrow 0.$$  

Here, the first group is finite and the last one is free. Hence,

$$\text{Ext}(\mathbb{Q}, \mathbb{Z}) = \text{Tors Coker } i^* \quad \text{and} \quad \text{Hom}(\mathbb{K}(D), \mathbb{Z}) = \text{Coker } i^*/\text{Tors}.$$  

In view of (2.3), these two isomorphisms prove the two statements of the theorem. For the first statement, one should also observe that $\text{Ext}(\mathbb{Q}, \mathbb{Z}) = \text{Ext}(\text{Tors } \mathbb{Q}, \mathbb{Z})$ (a property of finitely generated abelian groups), $\text{Tors } \mathbb{Q} = T(D)$ (using the fact that $NS(X)$ is primitive in $H_2(S)$), and $\text{Ext}(T(D), \mathbb{Z}) = \text{Hom}(T(D), \mathbb{Q}/\mathbb{Z})$ (apply $\text{Hom}(T(D), \cdot)$ to the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$).  

2.4. The covering $\Phi_m \rightarrow \Phi$

We make extensive use of the ramified covering $\text{pr}_m: \Phi_m \rightarrow \Phi := \Phi_1$ given by

$$\text{pr}_m: [z_0 : z_1 : z_2 : z_3] \mapsto [z_0^m : z_1^m : z_2^m : z_3^m].$$

Clearly, $\Phi$ is the plane $\{z_0 + z_1 + z_2 + z_3 = 0\}$, and $\text{pr}_m$ is ramified over the union of four lines $R_i := \Phi \cap \{z_i = 0\}$, $i = 0, 1, 2, 3$. The Galois group of $\text{pr}_m$ is $(\mathbb{Z}/m)^3$. Assuming that $m \geq 3$, the $3m^2$ lines in $\Phi_m$ are the irreducible components of the preimage of the three lines $L_i := \Phi \cap \{z_0 + z_i = 0\}$, $i = 1, 2, 3$. Introduce the divisors $L := L_1 + L_2 + L_3$, $R := R_0 + R_1 + R_2 + R_3$, and $V := L + R$ in $\Phi$.

With a further generalization in mind, redenote $\Phi[m] := \Phi_m$ and consider the pullbacks $L_m := \text{pr}_m^{-1}(L_0), R_m := \text{pr}_m^{-1}(R_0)$, and $V[m] := \text{pr}_m^{-1}(V)$, where $*$ is an appropriate subscript, possibly empty. Each $R_j[m]$ is a plane section of $\Phi[m]$, irreducible and reduced: it is the Fermat curve cut off $\Phi[m]$ by the plane $\{z_j = 0\}$. On the other hand, $L[m]$ also contains a number of plane sections, e.g., those cut off by $\{z_i = \xi z_j\}$, $i \neq j, \xi^m = -1$. Thus, for any subset $J \subset \{0, 1, 2, 3\}$, one has

$$S(V[m]) = S(L[m] + R_j[m]) = S(L[m]) = S_m,$$

where $R_j[m] := \sum_{j \in J} R_j[m]$.

Since $R$ is a generic configuration in the plane $\Phi$, the fundamental group $\mathbb{G} := \pi_1(\Phi \setminus R)$ equals $\mathbb{Z}^3$, see [11, Lemma in the proof of Theorem 8]. Since $\mathbb{G}$ is abelian, from the Hurewicz theorem we have $\mathbb{G} = H_1(\Phi \setminus R) = \text{Hom}(\mathbb{K}(R), \mathbb{Z})$, see Theorem 2.2. This group has four canonical generators $g_j$, $j = 0, 1, 2, 3$, viz. the restrictions to $\mathbb{K}(R)$ of the four generators of the group $H^2(R) = \bigoplus_j \mathbb{Z} \cdot [R_j]^*$. We have $g_0 + g_1 + g_2 + g_3 = 0$, and $\mathbb{G}$ is freely generated by $g_1, g_2, g_3$, cf., e.g., (2.3).
An interesting generalization of the original question was suggested in [13]. Given an epimorphism \( \alpha: G \rightarrow G \) to a finite abelian group \( G \), denote by \( \text{pr}: \Phi[\alpha] \rightarrow \Phi \) the minimal resolution of singularities of the ramified covering of \( \Phi \) defined by \( \alpha \). Let \( L_\alpha, R_\alpha, \) and \( V[\alpha] \) be the pull-backs in \( \Phi[\alpha] \) of \( L, R, \) and \( V \), respectively. To be consistent with the previous notation, we regard an integer \( m \) as the quotient projection \( m: \mathbb{G} \rightarrow \mathbb{G}/m\mathbb{G} \). The components of \( V[\alpha] \) (including the exceptional divisors) represent some ‘obvious’ elements of \( \text{NS}(\Phi[\alpha]) \). Using (1.1) and the finite degree map \( \Phi[m] \rightarrow \Phi[\alpha] \) defined by the inclusion \( \text{Ker} \alpha \subset m\mathbb{G}, m := |G| \), one has

\[
\mathbf{S}(V[\alpha]) \otimes \mathbb{Q} = (\text{Pic} \Phi[\alpha]) \otimes \mathbb{Q} \quad \text{whenever g.c.d.} (|G|, 6) = 1.
\]

Thus, it is natural to ask whether \( \mathbf{S}(V[\alpha]) = \text{Pic} \Phi[\alpha] \), or, not assuming that \( |G| \) is prime to 6, whether \( \mathbf{S}(V[\alpha]) \subset \text{Pic} \Phi[\alpha] \) is a primitive subgroup.

**Problem 2.6.** (See Shimada and Takahashi [13].) When does one have \( \mathbf{T}(V[\alpha]) = 0? \)

According to [13], the answer to this question is in the affirmative if the image \( G \) of \( \alpha \) is a cyclic group of order \( |G| \leq 50 \). Another example is worked out in Section 4.3, see Theorem 4.18: the answer is also in the affirmative if \( \alpha(g_i) = 0 \) for at least one of the standard generators \( g_i, i = 0, 1, 2, 3 \).

3. The Alexander module

3.1. The fundamental group

The line arrangement \( L + R \subset \mathbb{P}^2 \) is well known; sometimes it is referred to as Ceva-7. Its fundamental group has been computed in many ways and in many places; however, since we will work with a particular presentation of this group, we repeat the computation here.

We will use the affine coordinates \( x := -z_1/z_0, y := -z_3/z_0 \) in the plane \( \Phi \). In these coordinates, \( R_0 \) becomes the line at infinity, and the other components of \( V \) are the lines of the form \( \{r_x x + r_y y = r\} \) with \( r_x, r_y, r \in \{0, 1\} \), see Fig. 1. The fundamental group \( \pi_1 := \pi_1(\Phi \setminus V) \) is easily computed by the Zariski–van Kampen method [11,16]. Since we use a modified (or rather intermediate) version of this approach, we outline briefly its proof, using \( V \) as a model. (In full detail, the computation using the projection from a singular point is explained, e.g., in [4].) Consider the projection \( p: \Phi \twoheadrightarrow \mathbb{P}^1, (x, y) \mapsto x \). This projection has four special fibers \( F_a, \) viz. those over the points \( a \in \Delta := \{-1, 0, 1, \infty\} \). (Three of these fibers are components of \( V \), but this fact is irrelevant for the moment.) Let \( \Phi = \bigcup F_a, a \in \Delta \). Then the restriction \( p: \Phi \setminus (V \cup \Phi) \rightarrow \mathbb{P}^1 \setminus \Delta \) is a locally trivial fibration and, since \( \pi_2(\mathbb{P}^1 \setminus \Delta) = 0 \) and the fiber is connected, Serre’s exact sequence (aka long exact sequence of a fibration) gives us a short exact sequence of fundamental groups.
Fig. 1. The divisor $V := L + R \subset \Phi$.

$$\{1\} \rightarrow \pi_1(F \setminus V) \rightarrow \pi_1(\Phi \setminus (V \cup F_*)) \rightarrow \pi_1(\mathbb{P}^1 \setminus \Delta) \rightarrow \{1\},$$

where $F$ is a typical fiber of $p$, e.g., the one over $x = \frac{1}{2}$. Choosing $(\frac{1}{2}, -\frac{3}{2})$ for the basepoint, we have $\pi_1(F \setminus V) = \langle v_1, v_2, v_3, v_4 \rangle$, see Fig. 1. The group $\pi_1(\mathbb{P}^1 \setminus \Delta)$ is free, and the exact sequence splits. A splitting can be constructed geometrically, identifying $\pi_1(\mathbb{P}^1 \setminus \Delta)$ with $\pi_1(\mathbb{S} \setminus F_*) = \langle h_1, h_2, h_3 \rangle$, where $\mathbb{S} \subset \Phi$ is the section $y = -\frac{3}{2}$, the generators $h_1, h_2$ are as shown in Fig. 1, and $h_3$ is a similar loop about the fiber $F_{-1}$, not shown in the figure. (As long as $h_1, h_2, h_3$ generate $\pi_1(\mathbb{P}^1 \setminus \Delta)$, the particular choice of $h_3$ is irrelevant. For the reader’s convenience, we fix it to be $lcl^{-1}$, where $c$ is the circle $t \mapsto -1 + \frac{1}{2} \exp(2\pi t)$ about $-1$ and $l$ is the semicircle $t \mapsto \frac{1}{2} \exp(\pi t)$ circumventing the origin.) Thus, one arrives at the presentation

$$\pi_1(\Phi \setminus (V \cup F_*)) = \langle v_1, v_2, v_3, v_4, h_1, h_2, h_3 \mid h_i^{-1} v_j h_i = \beta_i(v_j) \rangle,$$

where $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and $\beta_i \in \text{Aut}(v_1, v_2, v_3, v_4)$ is the so-called braid monodromy, i.e., the automorphism of the fundamental group obtained by dragging the fiber along $h_i$ while keeping the basepoint in $S$. (The formal definition is in terms of a trivialization of the induced fibration $(p \circ h_i)^*p$ over the segment $[0, 1]$, where $p \circ h_i$ is regarded as a map $[0, 1] \to \mathbb{P}^1 \setminus \Delta$; for all details, see [11,16].)

Now, in order to pass to $\pi_1(\Phi \setminus V)$, one needs to patch in the only special fiber $F_{-1}$ that is not a component of $V$. This is done using the Seifert–van Kampen theorem [16].
fact, the principal application of the theorem in [16] is the following simple observation, which we state in a slightly generalized form.

**Lemma 3.1.** Let \( X \) be a smooth quasi-projective surface and \( D \subset X \) a closed smooth irreducible curve. Then the inclusion homomorphism \( \pi_1(X \setminus D) \to \pi_1(X) \) is an epimorphism; its kernel is normally generated by the class \( [\partial \Gamma] \), where \( \Gamma \) is an analytic disc transversal to \( D \) at its center and disjoint from \( D \) otherwise. \( \square \)

Since \( D \) is assumed irreducible, the conjugacy class of \( [\partial \Gamma] \) in the statement does not depend on the choice of \( \Gamma \) or path connecting \( \partial \Gamma \) to the basepoint. The proof of the lemma is literally the same as in [16], using a tubular neighborhood of \( D \).

Applying Lemma 3.1 to the curve \( F_{-1} \setminus V \) in \( \Phi \setminus V \), we obtain an extra relation \( h_3 = 1 \). In other words, we disregard the generator \( h_3 \) and convert the four relations \( h_3^{-1}v_jh_3 = \beta_3(v_j) \) into \( v_j = \beta_3(v_j) \), \( j = 1, 2, 3, 4 \).

The computation of the braid monodromy is straightforward and well known, \textit{e.g.}, using equations of the lines; it is left to the reader. (Essentially, it is the braid monodromy of the nodal arrangement \( L_1 + L_2 + R_2 + R_3 \) of four lines.) Denoting by \( \sigma_1, \sigma_2, \sigma_3 \) the Artin generators [2] of the braid group \( \mathbb{B}_4 \) acting on \( \langle v_1, v_2, v_3, v_4 \rangle \), we have \( \beta_1 = \sigma_1^2 \sigma_3^2, \beta_2 = \sigma_2^2, \) and \( \beta_3 = \sigma_1^{-1} \sigma_3^{-1} \sigma_2^2 \sigma_3 \sigma_1 \) (It is worth recalling that, assuming the left action of the automorphism group, the braid monodromy is an \textit{anti}-homomorphism \( \pi_1(\mathbb{P}^1 \setminus \Delta) \to \mathbb{B}_4 \)). Indeed, \( \beta_1 \) and \( \beta_2 \) are essentially computed in the very first paper on the subject, \textit{viz.} [11]: each is the local monodromy about a simple node (one full twist of a pair of points about their barycenter) or a pair of disjoint nodes. The remaining braid \( \beta_3 \) is the local monodromy \( \sigma_2^3 \) about \(-1\) translated \textit{via} \( l \) (see the definition of \( h_3 \) above) to the common reference fiber; the translation conjugates the local monodromy by ‘one half’ of \( \beta_1 \), which is \( \sigma_1 \sigma_3 \).

Putting everything together, after a slight simplification the nontrivial relations for the fundamental group \( \pi_1(\Phi \setminus V) \) take the form

\[
[h_2, v_1] = [h_2, v_4] = 1, \tag{3.2}
\]

\[
h_2 v_2 v_3 = v_2 v_3 h_2 = v_3 h_2 v_2 \tag{3.3}
\]

(the relations \( h_2^{-1}v_jh_2 = \beta_2(v_j) \) from the fiber \( x = 1 \)),

\[
h_1 v_1 v_2 = v_1 v_2 h_1 = v_2 h_1 v_1, \tag{3.4}
\]

\[
h_1 v_3 v_4 = v_3 v_4 h_1 = v_4 h_1 v_3 \tag{3.5}
\]

(the relations \( h_1^{-1}v_jh_1 = \beta_1(v_j) \) from the fiber \( x = 0 \)), and

\[
[v_2^{-1}v_1 v_2, v_4] = 1 \tag{3.6}
\]

(the relations \( v_j = \beta_3(v_j) \) from the fiber \( x = -1 \)). For the last relation (3.6), one can consider a local geometric basis \( v'_i := \sigma_1^{-1} \sigma_3^{-1}(v_i), i = 1, \ldots, 4 \), over \( x = -\frac{1}{2} \); in this
basis, the relation is \([v'_2, v'_3] = 1\) and, on the other hand, one has \(v'_2 = v_{-1}v_2\) and \(v'_3 = v_4\) (the result of circumventing the origin).

By Lemma 3.1, the inclusion \(\in: \Phi \setminus V \hookrightarrow \Phi \setminus R\) induces the map

\[
\text{in}_*: \pi_1 \hookrightarrow \mathbb{G}: \quad h_1 \mapsto g_1, \quad v_2 \mapsto g_2, \quad v_3 \mapsto g_3, \quad h_2, v_1, v_4 \mapsto 0.
\]

3.2. The ‘universal’ covering

Throughout the paper we use freely the following well-known fact, often referred to as theory of covering spaces: for any connected, locally path connected, and microsimply connected topological space \(X\) (e.g., for any connected simplicial complex) with a basepoint \(x_0 \in X\), there is a natural equivalence between the category of coverings \((\tilde{X}, \tilde{x}_0) \to (X, x_0)\) and covering maps (identical on \(X\)) and that of subgroups of \(\pi_1(X, x_0)\) and inclusions. If the subgroup is normal (regular, or Galois coverings), it can be described as the kernel of an epimorphism \(\alpha: \pi_1(X, x_0) \to G\); the image \(G\) serves then as the group of the deck translations of the covering.

Consider an epimorphism \(\alpha: \mathbb{G} \twoheadrightarrow G\). In this section, we do not assume \(G\) finite; in fact, we start with a study of the ‘universal’ \(G\)-covering, corresponding to the identity map \(0: \mathbb{G} \to \mathbb{G}/\mathbb{G} = \mathbb{G}\). (Admittedly awkward, this notation is in perfect agreement with \(m: \mathbb{G} \to \mathbb{G}/m\mathbb{G}\) introduced earlier.)

Consider the composition

\[
\tilde{\alpha}: \pi_1 \xrightarrow{\text{in}_*} \pi_1(\Phi \setminus R) = \mathbb{G} \xrightarrow{\alpha} G
\]

and denote by \(\Phi^\circ[\alpha]\) the \(G\)-covering of \(\Phi \setminus V\) defined by \(\tilde{\alpha}\). By the Hurewicz theorem, \(H_1(\Phi^\circ[\alpha])\) is the abelianization of \(\pi_1(\Phi^\circ[\alpha]) = \text{Ker} \tilde{\alpha}\). The action of the deck translations of the covering makes this group a \(\mathbb{Z}[G]\)-module; regarded as such, it is often referred to as the Alexander module of \(\tilde{\alpha}\).

The construction of the Alexander module fits into a more general framework and admits a purely algebraic description. Consider a group \(\pi\) and an epimorphism \(\tilde{\alpha}: \pi \to G\) to an abelian group \(G\). Then the Alexander module of \(\tilde{\alpha}\) is the abelian group \(A := \text{Ker} \tilde{\alpha}/\text{Ker} \tilde{\alpha}, \text{Ker} \tilde{\alpha}\) regarded as a \(\mathbb{Z}[G]\)-module via the \(G\)-action defined as follows: given \(a \in A\) and \(g \in G\), the image \(g(a)\) is the class in \(A\) of the element \(\tilde{\alpha}g\tilde{\alpha}^{-1}\) in \(\text{Ker} \tilde{\alpha}\), where \(\tilde{a}, \tilde{g} \in \pi\) are some lifts of \(a, g\), respectively. This class does not depend on the choice of the lifts, and the action is well defined.

Crucial is the fact that \(H_1(\Phi^\circ[\alpha])\) depends on the epimorphism \(\tilde{\alpha}: \pi_1 \to G\). Hence, we can replace \(\Phi \setminus V\) with any CW-complex \(X\) with \(\pi_1(X) = \pi_1\). We take for \(X\) a space with a single 0-cell \(e^0\), one 1-cell \(e_1 \in \{a_1, a_2, a_3, c_1, c_2, c_3\}\) for each of the six generators \(h_1, v_2, v_3, h_2, v_4, v_1\) of \(\pi_1\) (in the order listed), and one 2-cell \(e_2^j\) for each relation (3.2)–(3.6). In the \(G\)-covering \(X[0]\), each cell \(e\) gives rise to a whole \(G\)-orbit \(\{g \otimes e \mid g \in \mathbb{G}\}\). (For the moment, the symbols \(g \otimes e\) are merely cell labels; we only
assume that the labelling is compatible with the \( G \)-action, \( i.e. \), for any cell \( e \) in \( X \) and pair \( h, g \in G \) we have \( h(g \otimes e) = (h + g) \otimes e \).

Following the tradition, let us identify \( \mathbb{Z}[G] \) with the ring

\[
A := \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]
\]

of Laurent polynomials, where the variables \( t_1, t_2, t_3 \) correspond to the generators \( h_1 \mapsto g_1, v_2 \mapsto g_2, v_3 \mapsto g_3 \) about \( R_1, R_2, R_3 \), respectively. In other words, we identify \( G \) with the multiplicative abelian group generated by \( t_1, t_2, t_3 \); we shall also use this multiplicative notation in the cell labels. We can assume, in addition, that the labelling is chosen so that the left end of each ‘initial’ 1-cell \( 1 \otimes e \) is attached to \( 1 \otimes e^0 \), \( i.e., \) \((1 \otimes e)(0) = 1 \otimes e^0 \). (Here, we regard an oriented 1-cell as a path \([0, 1] \to X[0]\).) Then, from the definition of the covering it follows that the right ends are attached as follows:

\[
(1 \otimes a_i)(1) = t_i \otimes e^0, \quad (1 \otimes c_j)(1) = 1 \otimes e^0, \quad i, j = 1, 2, 3, \tag{3.7}
\]

\( i.e., \) the generators \( h_1, v_2, v_3 \) are ‘unwrapped’, whereas \( h_2, v_1, v_4 \) remain ‘latent’. The other ends are determined by the \( G \)-action: for a 1-cell \( e \) in \( X \), a monomial \( t \) in \( t_1, t_2, t_3 \), and \( \epsilon = 0, 1 \) we have \( (t \otimes e)(\epsilon) = t(1 \otimes e)(\epsilon) \).

Recall that the member \( C_n \) of the cellular chain complex associated to a \( CW \)-complex \( Y \) is the free abelian group generated by the \( n \)-cells of \( Y \). Thus, each cell \( e \) of \( X \) gives rise to a direct summand \( \bigoplus \mathbb{Z}(g \otimes e), g \in G \), in the complex of \( X[0] \); this summand is naturally identified with the free \( A \)-module \( Ae \). (It is this identification that explains the usage of \( \otimes \) in the labels.) Furthermore, since the \( CW \)-structure on \( X[0] \) is \( G \)-invariant, the boundary homomorphisms are \( A \)-linear. Thus, the chain complex \( C_* := C_*[0] \) of \( X[0] \)

is a complex of free \( A \)-modules of the form

\[
0 \longrightarrow C_2 \overset{\partial_2}{\longrightarrow} C_1 = \Lambda a_1 \oplus \Lambda a_2 \oplus \Lambda a_3 \oplus \Lambda c_1 \oplus \Lambda c_2 \oplus \Lambda c_3 \overset{\partial_1}{\longrightarrow} C_0 = \Lambda \longrightarrow 0
\]

(we omit the generator \( e^0 \) of \( C_0 \)), where \( \partial_1 \) is given by (3.7):

\[
\partial_1 a_i = (t_i - 1), \quad \partial_1 c_j = 0, \quad i, j = 1, 2, 3. \tag{3.8}
\]

The module \( C_2 \) has nine generators, of which six have non-trivial images under \( \partial_2 \):

\[
(t_2 t_3 - 1)c_1, \tag{3.9}
\]

\[
(t_3 - 1)c_1 + (t_3 - 1)a_2 - (t_2 - 1)a_3 \tag{3.10}
\]

from (3.3),

\[
(t_1 t_3 - 1)c_2, \tag{3.11}
\]

\[
(t_3 - 1)c_2 + (t_3 - 1)a_1 - (t_1 - 1)a_3 \tag{3.12}
\]
from (3.5), and
\[
(t_1 t_2 - 1)c_3, \quad (t_1 - 1)c_3 + (t_1 - 1)a_2 - (t_2 - 1)a_1
\]
from (3.4). Relations (3.2) and (3.6) contribute 0 to \( \text{Im} \partial_2 \).

Example 3.15. The proof of (3.9)–(3.14) is a straightforward computation. As an example, consider (3.3), which can be written in the form of two relations
\[
h_2 v_2 v_3 h_2^{-1} v_2^{-1} = 1, \quad h_2 v_2 v_3 v_2^{-1} h_2^{-1} v_3^{-1} = 1.
\]
The word in the left hand side of the first relation corresponds to the sequence \( c_1, a_2, a_3, c_1^{-1}, a_3^{-1}, a_2^{-1} \) of 1-cells in \( X \) along which a 2-cell \( e_1^2 \) is attached. (The inverse for a 1-cell means the reversion of the orientation.) Lift this sequence to \( X[0] \), starting each cell at the end of the previous one, see (3.7):
\[
1 \otimes c_1, \quad 1 \otimes a_2, \quad t_2 \otimes a_3, \quad (t_2 t_3 \otimes c_1)^{-1}, \quad (t_2 \otimes a_3)^{-1}, \quad (1 \otimes a_2)^{-1}.
\]
(Observe that, for example, \( t_2 \otimes a_3 \) connects \( t_2 \otimes e^0 \) to \( t_2 t_3 \otimes e^0 \), see (3.7); hence, the lift of \( a_3^{-1} \) starting at \( t_2 t_3 \otimes e^0 \) is \( (t_2 \otimes a_3)^{-1} \); it ends at \( t_2 \otimes e^0 \). Note also that \( (1 \otimes a_2)^{-1} \) ends at \( 1 \otimes e^0 \), i.e., the lift is a loop, as expected.) We obtain a sequence of 1-cells along which a 2-cell in \( X[0] \), viz. one of the lifts of \( e_1^2 \), is attached; writing this sequence as a chain, we get \( \partial_2 e_1^2 = (1 - t_2 t_3)c_1 \in C_1 \), which is (3.9) up to sign. Similarly, the second relation lifts to the sequence
\[
1 \otimes c_1, \quad 1 \otimes a_2, \quad t_2 \otimes a_3, \quad (t_3 \otimes a_2)^{-1}, \quad (t_3 \otimes c_1)^{-1}, \quad (1 \otimes a_3)^{-1},
\]
which gives us (3.10).

3.3. Other coverings

Now, given an epimorphism \( \alpha: \mathbb{G} \twoheadrightarrow G \), it induces a ring homomorphism \( \alpha*: \Lambda \rightarrow \mathbb{Z}[G] \), making \( \mathbb{Z}[G] \) a \( \Lambda \)-module. Clearly, the \( G \)-covering \( X[\alpha] \) is the quotient space \( X[0]/\text{Ker} \alpha \), the cells in \( X[\alpha] \) being the Ker-\( \alpha \)-orbits of those in \( X[0] \). The chain homomorphism \( C_* \rightarrow C_*(X[\alpha]) \) induced by the quotient projection merely identifies the basis elements (which are the cells) within each orbit of \( \text{Ker} \alpha \); algebraically, it can be expressed as the tensor product
\[
id \otimes \alpha*: C_* = C_* \otimes_\Lambda \Lambda \rightarrow C_* \otimes_\Lambda \mathbb{Z}[G] = C_*(X[\alpha]).
\]
Recall, see the beginning of Section 3.2, that the 1-homology of the covering spaces depend only on the homomorphism \( \tilde{\alpha}: \pi_1 \twoheadrightarrow G \). Hence, the group \( H_1(\Phi^\alpha[\alpha]) = H_1(X[\alpha]) \) is computed by the complex \( C_*[\alpha] := C_* \otimes_\Lambda \mathbb{Z}[G] \). In view of the right exactness
Coker(∂_2 ⊗_A α_*) = (Coker ∂_2) ⊗_A \mathbb{Z}[G]$, our primary interest is the quotient $A[α] := C_1[α]/\text{Im} \partial_2$. Explicitly, $A[α]$ can be described as the $A$-module generated by the six elements $a_1, a_2, a_3, c_1, c_2, c_3$ that are subject to relations (3.9)–(3.14) and the extra relation

$$t_1^1 t_2^2 t_3^3 = 1 \text{ whenever } \alpha(r_1 g_1 + r_2 g_2 + r_3 g_3) = 0. \quad (3.16)$$

Summarizing, after the identification $C_0[α] = \mathbb{Z}[G]$ and $H_0(X[α]) = \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow H_1(\Phi^0[α]) \longrightarrow A[α] \overset{∂_1}{\longrightarrow} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0, \quad (3.17)$$

where the last homomorphism is the augmentation $g ↦ 1, g ∈ G$.

Recall that the rank $\text{rk} A$ of a finitely generated abelian group $A$ is the maximal number of linearly independent elements of $A$, whereas its length $ℓ(A)$ is the minimal number of elements generating $A$. One has $\text{rk} A = ℓ(A)$ if and only if $A$ is free.

**Lemma 3.18.** For any epimorphism $α: \mathbb{G} ↪ G$, there is a natural isomorphism $\text{Tors } H_1(\Phi^0[α]) = \text{Tors } A[α]$. If $G$ is finite, then $ℓ(H_1(\Phi^0[α])) = ℓ(A[α]) - |G| + 1$ and $\text{rk } H_1(\Phi^0[α]) = \text{rk } A[α] - |G| + 1$.

**Proof.** Since $\text{Im } ∂_1 ⊂ \mathbb{Z}[G]$ is a free abelian group, the inclusion in (3.17) induces an isomorphism of the torsion parts. This isomorphism and the obvious fact that $ℓ(A) = \text{rk } A + ℓ(\text{Tors } A)$ for any finitely generated abelian group $A$ imply that the length and rank identities in the statement are equivalent to each other. The rank identity follows from the additivity of rank in (3.17) and the observation that $\text{rk } \mathbb{Z}[G] = |G|$.

### 3.4. Fermat surfaces

If the image $G$ of $α: \mathbb{G} ↪ G$ is finite, one obviously has $\Phi^0[α] = \Phi[α] \setminus V[α]$. If $α = m ∈ \mathbb{Z}$, i.e., in the case of a classical Fermat surface $\Phi[m]$, it is more convenient to consider a smaller divisor $\bar{L}[m] := L[m] + R_0[m]$, see (2.4). The fundamental group $π_1(\Phi[m] \setminus \bar{L}[m])$ is given by **Lemma 3.1**: it is the quotient of $\text{Ker } \bar{α} = π_1(\Phi^0[α])$ by the extra relations $h_1^m = v_2^m = v_3^m = 1$ (as the ramification index at each component of $R[m]$ is obviously $m$). Hence, the homology group $H_1(\Phi[m] \setminus \bar{L}[m])$ can be computed using the complex $C_∗[m]$ with three extra 2-cells $e_i^2$, $i = 1, 2, 3$, mapped by $∂_2$ to $φ_m(t_i)a_i$, where

$$φ_n(t) := (t^n - 1)/(t - 1), \quad n ∈ \mathbb{Z}.$$  

This computation is similar to **Example 3.15**: for example, the loop $h_1^m$ lifts to the sequence $1 ⊗ a_1, t_1 ⊗ a_1, t_1^2 ⊗ a_1, \ldots, t_1^{m-1} ⊗ a_1$ of 1-cells, which results in the chain $(1 + t_1 + t_1^2 + \ldots + t_1^{m-1})a_1 = φ_m(t_1)a_1 ∈ C_1[m]$. Note that this chain is a cycle, as in $C_1[m]$ we have the relation $t_1^m = 1$. 


Remark 3.19. Strictly speaking, the new complex is that of abelian groups rather than $\Lambda$-modules, as we add three 2-cells only, i.e., three summands $\mathbb{Z}e_i^2$ in $C_2[m]$. However, in the presence of the relations $t_i^m = 1$, $i = 1, 2, 3$, cf. (3.16), one can use (3.9)–(3.14) to show that all three images $\varphi_m(t_i)a_i$ are $\mathbb{G}$-invariant. Hence, without changing the 1-homology of the complex, we can formally replace each summand $\mathbb{Z}e_i^2$ with $Ae_i^2$, extending $\partial_2$ by $A$-linearity. Geometrically, we replace a single disk $\Gamma$ as in Lemma 3.1 with a $G$-orbit consisting of $m^3$ disks. Since the curve $R_i[m]$ patched in is irreducible (all disks intersecting the same component), this change does not affect the fundamental group.

Now, as in Section 3.3, instead of extending the $C_2$-term of the complex, we can add extra relations to $C_1$. Summarizing, we have

$$H_1(\Phi[m] \smallsetminus \bar{L}[m]) = \text{Ker}[\partial_1; \bar{A}[m] \to C_0[m]],$$

where $\bar{A}[m]$ is the quotient of $A[0]$ by the extra relations

$$t_i^m = 1, \quad \varphi_m(t_i)a_i = 0, \quad i = 1, 2, 3. \quad (3.20)$$

Arguing as in the proof of Lemma 3.18, we obtain the identity

$$\ell(H_1(\Phi[m] \smallsetminus \bar{L}[m])) = \ell(\bar{A}[m]) - m^3 + 1. \quad (3.21)$$

3.5. Other Delsarte surfaces

In the generalized case, the first question that arises is whether Theorem 2.2 is applicable, i.e., whether $H_1(\Phi[\alpha]) = 0$. To state the result, introduce the following notation: given a pair of integers $0 \leq i, j \leq 3$, let $G_{ij} := \mathbb{Z}g_i \oplus \mathbb{Z}g_j \subset G$, where $g_i \in G$ are the canonical generators, see Section 2.4.

Recall that the blow-up $\sigma: \tilde{X} \to X$ of a smooth point of a surface $X$ induces an isomorphism of both the fundamental group $\pi_1$ and first homology group $H_1$ of the surface. Hence, up to canonical isomorphism, the groups $\pi_1$ and $H_1$ do not depend on the resolution of singularities.

Proposition 3.22. For an epimorphism $\alpha: G \to G$, $|G| < \infty$, one has

$$\pi_1(\Phi[\alpha]) = H_1(\Phi[\alpha]) = \text{Ker} \alpha / \sum G_{ij} \cap \text{Ker} \alpha,$$

the summation running over all pairs $0 \leq i, j \leq 3$ of integers.

Proof. We start with the abelian group $\pi_1(\Phi \setminus R) = \mathbb{G}$ generated by $h_1 \mapsto g_1$, $v_2 \mapsto g_2$, $v_3 \mapsto g_3$, see Section 3.1. Clearly, $\pi_1(\Phi[\alpha] \setminus R[\alpha]) = H_1(\Phi[\alpha] \setminus R[\alpha]) = \text{Ker} \alpha$. (This group can also be regarded as an $\Lambda$-module, but the module structure is trivial: $t_1 = t_2 = t_3 = 1$.)
For the rest of the proof, we use the additive notation for the fundamental group (as the groups of interest are subquotients of \( G \)).

Let \( \Phi'[\alpha] \) be the manifold obtained from \( \Phi[\alpha] \setminus R[\alpha] \) by patching the components of the proper transform of \( R[\alpha] \) away from the exceptional divisor. At a generic point of \( R_i \), the ramification index \( m_i \) of the ramified covering \( \Phi[\alpha] \to \Phi \) equals the index \( [G_{ii} : G_{ii} \cap \text{Ker} \alpha] \), \( i = 0, 1, 2, 3 \). Hence, by Lemma 3.1, the inclusion induces an epimorphism \( \text{Ker} \alpha \to \pi_1(\Phi'[\alpha]) \) whose kernel is generated by the elements \( m_i g_i \). Thus, we have an isomorphism

\[
\pi_1(\Phi'[\alpha]) = \text{Ker} \alpha / \sum_i G_{ii} \cap \text{Ker} \alpha, \quad i = 0, 1, 2, 3. \tag{3.23}
\]

(Strictly speaking, unlike the case of the Fermat surfaces, the curve \( R_i[\alpha] \) may be reducible, so that we need to attach a separate disk \( \Gamma \) as in Lemma 3.1 for each component of this curve. However, since the \( G \)-action is trivial in the 1-homology \( H_1 = \pi_1 \), all disks result in the same relation \( m_i g_i = 0 \), cf. Remark 3.19.)

What remains is patching the exceptional divisors. Fix a pair \( 0 \leq i < j \leq 3 \) and let \( \tilde{S} \) be a singular point of the normalized, but yet unresolved ramified covering over the point \( S := R_i \cap R_j \). Fix a resolution of singularities and let \( E \) be the exceptional divisor over \( \tilde{S} \). Pick a sufficiently small ball \( U \subset \Phi \) about \( S \) and denote by \( \tilde{U} \) the connected component of the preimage of \( U \) containing \( E \). With respect to an appropriate smooth triangulation, \( \tilde{U} \) is a regular neighborhood of \( E \); hence, \( E \) is a strict deformation retract of \( \tilde{U}, \tilde{U} \sim E \). On the other hand, \( \tilde{U} \) is a 4-manifold with boundary \( \partial \tilde{U} \), and the latter is a covering of the 3-sphere \( \partial U \) ramified over the Hopf link \( R \cap \partial U \).

Note also that the contraction of \( E \) gives us the space \( \tilde{U}/E \) which is the cone over \( \partial \tilde{U} \) (with the vertex \( \tilde{S} = E/E \)); hence, we have a homotopy equivalence (strict deformation retraction) \( \tilde{U} \setminus E = (\tilde{U}/E) \setminus \tilde{S} \sim \partial \tilde{U} \).

We have \( \pi_1(\partial U \setminus R) = G_{ij} \) and, hence, \( \pi_1(\partial \tilde{U} \setminus R[\alpha]) = G_{ij} \cap \text{Ker} \alpha \). As above, similar to Lemma 3.1, patching the union of circles \( \partial U \cap R[\alpha] \) results in the pair of relations \( m_i g_i = m_j g_j = 0 \). Thus,

\[
\pi_1(\partial \tilde{U}) = (G_{ij} \cap \text{Ker} \alpha)/(G_{ii} \cap \text{Ker} \alpha + G_{jj} \cap \text{Ker} \alpha) \tag{3.24}
\]

is a finite group. Then \( H_1(\partial \tilde{U}; \mathbb{Q}) = 0 \), i.e., \( \partial \tilde{U} \) is a rational homology sphere and \( \tilde{S} \) is a rational singular point. For us, important is the fact that \( \pi_1(\tilde{U}) = \pi_1(E) = 0 \), which can easily be proved directly. Indeed, since \( \tilde{U} \sim E \) and \( \dim \mathbb{R} E = 2 \), we have \( H^3(\tilde{U}; \mathbb{Q}) = 0 \); then also \( H_1(\tilde{U}, \partial \tilde{U}; \mathbb{Q}) = 0 \) (Lefschetz duality), and the fragment

\[
H_1(\partial \tilde{U}; \mathbb{Q}) \to H_1(\tilde{U}; \mathbb{Q}) \to H_1(\tilde{U}, \partial \tilde{U}; \mathbb{Q})
\]

of the homology exact sequence of pair \( (\tilde{U}, \partial \tilde{U}) \) implies \( H_1(\tilde{U}; \mathbb{Q}) = H_1(E; \mathbb{Q}) = 0 \). On the other hand, \( E \) is a connected projective algebraic curve, and it is easily seen that \( E \) is homotopy equivalent to the wedge of closed topological surfaces (the components of
the normalization of $E$) and a number of circles. (Roughly, we can ‘blow-up’ the locally reducible singular points of $E$ to line segments, separating the analytic branches and replacing $E$ with a disjoint union of topologically nonsingular closed surfaces with a number of segments attached. Then, within each surface, move the ends of the segments to a single point. Finally, contract several segments to make the surfaces share a common point; the result is a wedge as stated.) For such a wedge $E \sim \bigvee_{i} E_{i}$, all groups are easily computed (e.g., using iteratedly the Mayer–Vietoris exact sequence (8.8) in [7, Chapter III] and Seifert–van Kampen theorem [16], or just decomposing the wedge into cells):

$$H_{1}(E; \mathbb{Q}) = \bigoplus_{i} H_{1}(E_{i}; \mathbb{Q}), \quad \pi_{1}(E) = \ast_{i} \pi_{1}(E_{i}).$$

Clearly, $H_{1}(E; \mathbb{Q}) = 0$ if and only if all surface components are 2-spheres and there are no circles present. Then obviously $\pi_{1}(E) = 0$.

Now, start with $\Phi'[\alpha]$ and proceed patching the exceptional divisors one by one. Let $\Phi''$ be an intermediate space, not yet containing $E$. Applying the Seifert–van Kampen theorem [16] to the union $\Phi'' \cup \tilde{U}$ and using the homotopy equivalence $\Phi'' \cap \tilde{U} = \tilde{U} \setminus E \sim \partial \tilde{U}$, we obtain the amalgamated free product

$$\pi_{1}(\Phi'' \cup \tilde{U}) = (\pi_{1}(\Phi'') \ast \pi_{1}(\tilde{U}))/\pi_{1}(\partial \tilde{U}) = \pi_{1}(\Phi'')/(G_{ij} \cap \text{Ker } \alpha).$$

(For the last isomorphism, we use (3.24) and the identity $\pi_{1}(\tilde{U}) = 0$.) The group $\pi_{1}(\Phi'[\alpha])$ is given by (3.23) and, after all the exceptional divisors have been patched, we arrive at the expression in the statement. \[\square\]

If $H_{1}(\Phi[\alpha]) = 0$, Theorem 2.2 and Lemma 3.18 imply that

$$T\langle V[\alpha] \rangle \cong \text{Tors } A[\alpha]. \quad (3.25)$$

Unfortunately, as a $\mathbb{Z}[G]$-module, $A[\alpha]$ is far from free and it is difficult to control its $\mathbb{Z}$-torsion. (Experiments show that, at least, the intermediate quotients similar to those considered in Lemma 4.4 do often have torsion.) An attempt of a direct computation is made in Section 4.3, whereas in the case of the classical Fermat surfaces we have to take a detour and estimate the length instead. The following two exact sequences may prove useful:

$$A[\alpha] \xrightarrow{\partial_{1}} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0,$$

where $\epsilon$ is the augmentation, see (3.17), and

$$0 \rightarrow A^{\circ}[\alpha] \rightarrow \text{Ker } \partial_{1} \rightarrow \text{Ker } \alpha \rightarrow 0,$$
where \( A^\circ[\alpha] \subset A[\alpha] \) is the submodule generated by \( c_1, c_2, c_3 \). The former sequence merely states that \( H_0(C_\ast[\alpha]) = H_0(\Phi^\circ[\alpha]) = Z \). For the latter, we patch \( L[\alpha] \) (by using Lemma 3.1 or merely forgetting the generators \( h_2, v_1, v_4 \), hence \( c_1, c_2, c_3 \) in the first place) to compute the group \( H_1(\Phi[\alpha] \setminus R[\alpha]) = \pi_1(\Phi[\alpha] \setminus R[\alpha]) = \text{Ker} \, \alpha \); the resulting complex is \( 0 \to A[\alpha]/A^\circ[\alpha] \to \mathbb{Z}[G] \to 0 \). Both sequences split, and we can extend (3.25) to

\[
\mathbf{T}(V[\alpha]) \cong \text{Tors } A^\circ[\alpha] = \text{Tors } A[\alpha],
\]

(3.26)

still under the assumption that \( H_1(\Phi[\alpha]) = 0 \).

4. Proof of the main theorem

4.1. The length of \( \bar{A}[m] \)

Fix an integer \( m \geq 1 \) and consider the \( \Lambda \)-module \( \bar{A}[m] \) introduced in Section 3.4. Recall that \( \bar{A}[m] \) is generated by six elements \( a_i, c_j, i, j = 1, 2, 3 \), subject to the relations (3.9)–(3.14) and (3.20). Observe that relations (3.9), (3.11), and (3.13) can be recast in the form

\[
t_i c_k = t_j^{-1} c_k \quad \text{whenever } \{i, j, k\} = \{1, 2, 3\}.
\]

(4.1)

We introduce a few \textit{ad hoc} notations. Given \( i = 1, 2, 3 \), let

\[
A_i := \mathbb{Z}[t_i]/(t_i^m - 1), \quad \bar{A}_i := \mathbb{Z}[t_i]/\varphi_m(t_i).
\]

These rings can be regarded as \( \Lambda \)-modules, but we usually do not specify the action of the other two variables: it varies from case to case. In fact, we repeatedly use the following simple observation, which is an immediate consequence of (4.1).

\textbf{Lemma 4.2.} Let \( i, j, k \in \{1, 2, 3\}, \, k \neq i, \), and \( p \in \Lambda \), and let \( A \) be a subquotient of \( \bar{A}[m] \) generated by a single element \( x := pc_i \). Assume that either \( t_j = 1 \) or \( t_i = t_k^{\pm 1} \) on \( A \). Then \( A \) is a quotient of \( \Lambda_s x \) for an appropriate index \( s \in \{1, 2, 3\} \).

If \( x \) is also annihilated by \( \varphi_m(t_s) \), then \( A \) is a quotient of \( \bar{A}_s x \). \( \square \)

The precise description of the ‘appropriate’ index \( s \) (not necessarily unique) is left to the reader. Clearly, \( \ell(\Lambda_s) = m \) and \( \ell(\bar{A}_s) = m - 1 \).

For a generator \( x \in \{a_1, a_2, a_3, c_1, c_2, c_3\} \), let

\[
x' := (t_1 - 1)x, \quad \bar{x} := (t_3 - 1)x, \quad \bar{x}' := (t_1 - 1)\bar{x}.
\]

Observe that always

\[
\varphi_m(t_1)x' = \varphi_m(t_3)\bar{x} = \varphi_m(t_1)\bar{x}' = \varphi_m(t_3)\bar{x}' = 0.
\]

(4.3)
We will use a filtration \(0 = A_0 \subset A_1 \subset \ldots \subset A_7 = \bar{A}[m]\), where \(A_k \subset \bar{A}[m]\) are the submodules defined in Lemma 4.4 below.

Let \(\delta_m := 1\) if \(m\) is even and \(\delta_m := 0\) if \(m\) is odd.

**Lemma 4.4.** One has the following equations and inequalities:

1. \(\ell(A_1/A_0) = m^3 - m^2\), where \(A_1\) is the submodule generated by \(a_3\);
2. \(\ell(A_2/A_1) \leq 3(m - 1) - \delta_m\), where \(A_2 := A_1 + \Lambda \bar{a}_2 + \Lambda \bar{c}_3\);
3. \(\ell(A_3/A_2) \leq 3(m - 1)\), where \(A_3 := A_1 + (t_3 - 1)A[m]\);
4. \(\ell(A_4/A_3) = m^2 - m\), where \(A_4 := A_3 + \Lambda a_1\);
5. \(\ell(A_5/A_4) \leq m - 1\), where \(A_5 := A_4 + \Lambda a_2 + \Lambda c_3\);
6. \(\ell(A_6/A_5) = m - 1\), where \(A_6 := A_5 + \Lambda a_2\);
7. \(\ell(A_7/A_6) \leq 2m + 1\), where \(A_7 := \bar{A}[m]\).

Hence, \(\ell(A) \leq m^3 + 9m - 7 - \delta_m\).

**Proof.** One has \(\ell(A_1) \leq m^2(m - 1)\) due to (3.20). On the other hand, the boundary homomorphism \(\partial_1\) maps \(A_1\) onto \((t_3 - 1)C_0[m]\). Hence, there are no other relations in \(A_1\), and statement (1) holds. Furthermore, \(\partial_1\) factors to a homomorphism

\[\bar{A}[m]/A_3 \to C'_0 := C_0[m]/(t_3 - 1)\]

which maps \(A_4/A_3\) isomorphically onto \((t_1 - 1)C'_0\), proving statement (4). Then, \(\partial_1\) factors to

\[\bar{A}[m]/A_5 \to C''_0 := C'_0/(t_1 - 1) = A_2.\]

Since \(A_6/A_5\) is (a priori a quotient of) the cyclic \(\bar{A}_2\)-module \(\bar{A}_2 a_2\), the restriction of \(\partial_1\) maps it isomorphically onto \((t_2 - 1)C''_0 = \bar{A}_2\), proving statement (6).

For the other statements, it suffices to estimate the number of generators. With possible future applications in mind, we describe the structure of the intermediate quotients in the form (known module) \(\mapsto A_k/A_{k-1}\). In fact, all these epimorphisms are isomorphisms, see Remark 4.14 below.

In \(\bar{A}[m]/A_4\), one has

\[t_3 = 1, \quad a_1 = a_3 = 0, \quad a'_2 = -c'_3;\]

the last relation follows from (3.14). Thus, \(A_5/A_4\) is generated by \(c'_3\), and \(\bar{A}[m]/A_6\) is generated by \(c_1, c_2, c_3\); by (3.20) and Lemma 4.2,

\[\bar{A}_2c'_3 \mapsto A_5/A_4, \quad A_1c_1 \oplus A_2c_2 \oplus \mathbb{Z}c_3 \mapsto \bar{A}[m]/A_6, \quad (4.5)\]
For the last summand $\mathbb{Z}c_3$, we use the fact that
\[(t_1 - 1)c_3 = -(t_1 - 1)a_2 = 0 \mod A_6.\]

Thus, $\ell(\mathbb{A}[m]/A_6) \leq 2m + 1$, and statements (5) and (7) are proved.

The module $A_3/A_1$ is generated by $\tilde{a}_1, \tilde{a}_2, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3$, and relations (3.10), (3.12), (3.14) imply
\[\tilde{a}_2 = -\tilde{c}_1, \quad \tilde{a}_1 = -\tilde{c}_2, \quad (t_1 - 1)(\tilde{c}_3 + \tilde{a}_2) = (t_2 - 1)\tilde{a}_1.\]

We can retain three generators $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ only, rewriting the last relation in the form
\[(t_1 - 1)(\tilde{c}_3 - \tilde{c}_1) + (t_2 - 1)\tilde{c}_2 = 0. \quad (4.6)\]

Note also that $\varphi_m(t_3)A_3 = 0$, see (4.3).

In $A_3/A_2$, we have $(t_1 - 1)\tilde{c}_3 = (t_1 - 1)\tilde{a}_2 = 0$, hence also $(t_1 - 1)\tilde{c}_1 = 0$. Then (4.6) implies $(t_2 - 1)\tilde{c}_2 = 0$, and
\[\tilde{A}_3\tilde{c}_1 \oplus \tilde{A}_3\tilde{c}_2 \oplus \tilde{A}_3\tilde{c}_3 \to A_3/A_2, \quad (4.7)\]
see Lemma 4.2. This gives us statement (3).

The module $A_2/A_1$ is generated by $\tilde{c}'_1$ and $\tilde{c}'_3$. By (4.3) and (4.1), we have
\[\varphi_m(t_i)(A_2/A_1) = 0 \quad \text{for all } i = 1, 2, 3. \quad (4.8)\]

Relations (3.11) and (4.6) imply $(t_1t_3 - 1)(\tilde{c}'_3 - \tilde{c}'_1) = 0$; using (4.1), this can be rewritten as $(t_3 - t_2)\tilde{c}'_3 = (t_1 - t_2)\tilde{c}'_1$. Let
\[u := (t_3 - t_2)\tilde{c}'_3 = (t_1 - t_2)\tilde{c}'_1\]
and consider the cyclic submodule $A'_2 \subset A_2/A_1$ generated by $u$. By Lemma 4.2,
\[\tilde{A}_2\tilde{c}'_1 \oplus \tilde{A}_2\tilde{c}'_3 \to (A_2/A_1)/A'_2. \quad (4.9)\]

On the other hand, $A'_2 \subset A\tilde{c}'_1 \cap A\tilde{c}'_3$; hence, $t_3^{-1} = t_2 = t_1^{-1}$ on this module and, by Lemma 4.2 again,
\[\tilde{A}_2u \to A'_2 \quad \text{if } m \text{ is odd}. \quad (4.10)\]

This fact proves statement (2) in the case of $m$ odd.

If $m = 2k$ is even, (4.10) still holds, but we need a stronger statement. Note that $\varphi_m(t)$ is divisible by $\varphi_k(t^2)$. Furthermore, one has a polynomial identity
\[t^{m-2} \sum_{r=0}^{m-1} t^{1-r}\varphi_r(t^2) = t\varphi_{k-1}(t^2)\varphi_m(t) + \varphi_k(t^2), \quad (4.11)\]
which is easily established by multiplying both sides by \( t^2 - 1 \). On the submodule \( A'_2 \)
we have \( t_2 = t_1^{-1} \), see (4.1); hence, \( s := t_2 t_1^{-1} = t_2^2 \). Then, representing \( u \) in the form
\( u = t_1 (1 - s) \xi_1' \), we have
\[
\xi_1'(1 - s) c_1' = t_2^r (1 - s^r) c_1' = (t_1^r - t_2^r) c_1', \quad r \in \mathbb{Z}. \tag{4.12}
\]
Summing up over \( r = 0, \ldots, m - 1 \) and using (4.8) and (4.11) at \( t = t_2 \), we conclude that
\[
\varphi_k(t_2^m) u = 0, \quad i.e.,
\]
\[
A_2 u / \varphi_k(t_2^m) \to A'_2 \quad \text{if } m = 2k \ \text{is even}, \tag{4.13}
\]

obtaining a stronger inequality \( \ell(A'_2) \leq \deg \varphi_k(t_2^m) = m - 2 \).

The final inequality in the statement of the lemma is the sum of items (1)–(7).

\section*{4.2. Proof of Theorem 1.3}

We assume that \( m \geq 3 \). By (2.4), it suffices to show that \( T(\tilde{L}[m]) = 0 \), where \( \tilde{L}[m] := L[m] + R_0[m] \) is the divisor introduced in Section 3.4. Since \( \Phi[m] \) is simply connected,
we can use Theorem 2.2, reducing the problem in question to proving the inequality
\( \ell(H_1(\Phi[m] \setminus \tilde{L}[m])) \leq \text{rk } K(\tilde{L}[m]) \).

According to [1,14], \( \text{rk } S_m = 3(m - 1)(m - 2) + 1 + \delta_m \). On the other hand, \( H_2(\tilde{L}[m]) \)
the free abelian group generated by the classes of the \( 3m^2 \) lines and the additional
class \( [R_0[m]] \). Hence, \( \text{rk } K(\tilde{L}[m]) = 9m - 6 - \delta_m \), and the statement follows from (3.21)
and Lemma 4.4.

\begin{remark}
It follows from the proof that all inequalities in the statement of Lemma 4.4
are, in fact, equalities, \( i.e., \) no relation has been lost, even though some relations were
multiplied by non-units. Furthermore, all epimorphisms (4.5), (4.7), (4.9), (4.10), (4.13)
are isomorphisms.
\end{remark}

\begin{remark}
We only use the inequality \( \text{rk } S_m \leq 3(m - 1)(m - 2) + 1 + \delta_m \), \( i.e., \) the fact
that there is at least a certain number of relations between the components. In general,
it would suffice to prove the inequality \( \ell(A[\alpha]) \leq \text{rk } K(V[\alpha]) + |G| - 1 \), see Lemma 3.18.
\end{remark}

\begin{remark}
The rank \( \text{rk } S_m \) can easily be computed directly, by tensoring the module
by \( \mathbb{C} \) and counting the irreducible summands, which are all of dimension 1 (multi-
eigenspaces of the three commuting finite order operators \( t_1, t_2, t_3 \)).
\end{remark}

\begin{remark}
By (3.26), when computing the torsion, one can replace \( A[\alpha] \) with the
smaller module \( A^0[\alpha] \). \textit{A posteriori,} \( A^0[m] \) is the \( A[m]- \text{module spanned by the three generators} \ c_1, c_2, c_3 \text{ subject to a single relation} \)
\[
(t_1 - 1)(t_3 - 1)c_1 = (t_2 - 1)(t_3 - 1)c_2 + (t_1 - 1)(t_3 - 1)c_3,
\]
see [5]. In this form, some of the results of this paper generalize to Fermat varieties of higher dimension, see [6]. Note, though, that this one-relator presentation of $A^0[\alpha]$ does not extend to more general Delsarte surfaces; see [5] for further details.

4.3. A toy example

In conclusion, we consider a very simple example, answering the generalized question, see Problem 2.6, in the special case of a covering ramified over at most three lines.

**Theorem 4.18.** If the covering $\Phi[\alpha] \to \Phi$ is unramified over at least one of the lines $R_j$, $j = 0, 1, 2, 3$, then $T(V[\alpha]) = 0$.

**Proof.** We can assume that the covering is unramified over $R_3$, i.e., the epimorphism $\alpha: G \to G$ sends $g_3$ to zero. Then, obviously, $\text{Ker} \alpha = \mathbb{Z}g_3 \oplus (G_{12} \cap \text{Ker} \alpha)$ and, by Proposition 3.22, we have $H_1(\Phi[\alpha]) = 0$, i.e., Theorem 2.2 is applicable.

By (3.16), we have $t_3 = 1$ on $A[\alpha]$, and relations (3.10), (3.12), (3.14) become

$$(t_2 - 1)a_3 = (t_1 - 1)a_3 = 0, \quad (t_1 - 1)(c_3 + a_2) = (t_2 - 1)a_1.$$

Introducing the generator $a_2' := c_3 + a_2$ instead of $a_2$, we see that the submodule $A^0[\alpha] \subset A[\alpha]$ introduced in Section 3.5 is a direct summand (as a $A$-module), and all relations in $A^0[\alpha]$ are $t_3 = 1$ and (3.9), (3.11), (3.13). The three latter translate into independent relations $(t_2 - 1)c_1 = (t_1 - 1)c_2 = (t_1t_2 - 1)c_3 = 0$, and $A^0[\alpha]$ is a direct sum of three group rings:

$$A^0[\alpha] = \mathbb{Z}[\mathbb{G}/\alpha(g_2)]c_1 \oplus \mathbb{Z}[\mathbb{G}/\alpha(g_1)]c_2 \oplus \mathbb{Z}[\mathbb{G}/\alpha(g_1 + g_2)]c_3.$$

By (3.26), one has $T(V[\alpha]) \cong \text{Tors } A^0[\alpha] = 0$. □

**Corollary 4.19 (of (2.5) and Theorem 4.18).** If a covering $pr: \Phi[\alpha] \to \Phi$ as in Theorem 4.18 has degree $m$ prime to 6, then $\text{Pic } \Phi[\alpha] = S(V[\alpha])$. □

4.4. Proof of Theorem 1.4 and Corollary 1.5

Corollary 1.5 is an immediate consequence of Theorem 1.4 and the fact that Pic $\Sigma_{p,q}$ is rationally generated by the classes of the lines, see [3]. In view of Theorem 2.2, the statement of Theorem 1.4 is purely homological, and we can deform $\Sigma_{p,q}$ to the Fermat surface $\Phi[m]$; then, the $m^2$ lines in question deform to the components of $L_1[m]$, and $S_{p,q} = S(L_1[m])$. Similar to (2.4), the latter group equals $S(L_1[m])$, where $L_1[m] := L_1[m] + R_0[m]$. Patching $L_2[m]$ and $L_3[m]$, cf. Section 3.4, we conclude that

$$T(L_1[m]) = \text{Tors } A'[m], \quad A'[m] := A[m]/(Ac_2 + Ac_3).$$
Filtering this module as in Lemma 4.4 and analyzing the proof of the lemma, we see that statements (1), (4), and (6) hold without change, whereas the other statements can be rewritten as follows:

(2) \( \ell(A_2/A_1) = 0 \) due to (4.6),
(3) \( \ell(A_3/A_2) \leq m - 1 \), see (4.7),
(5) \( \ell(A_5/A_4) = 0 \), see (4.5),
(7) \( \ell(A_7/A_6) \leq m \), see (4.5).

Summing this up, we obtain \( \ell(\bar{A}'[m]) \leq m^3 + 2m - 2 \). On the other hand, one has \( \text{rk} \mathbf{S}(\bar{L}_1[m]) = (m - 1)^2 + 1 \), see [3]; hence, \( \text{rk} \mathbf{K}(\bar{L}_1[m]) = 2m - 1 \) and, as in Section 4.2, we conclude that \( \bar{A}'[m] \) is a free abelian group. \( \square \)

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References

