A gPC-based approach to uncertain transonic aerodynamics

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Abstract

The present paper focus on the stochastic response of a two-dimensional transonic airfoil to parametric uncertainties. Both the freestream Mach number and the angle of attack are considered as random parameters and the generalized Polynomial Chaos (gPC) theory is coupled with standard deterministic numerical simulations through a spectral collocation projection methodology. The results allow for a better understanding of the flow sensitivity to such uncertainties and underline the coupling process between the stochastic parameters. Two kinds of non-linearities are critical with respect to the skin-friction uncertainties: on one hand, the leeward shock movement characteristic of the supercritical profile and on the other hand, the boundary-layer separation on the aft part of the airfoil downstream the shock. The sensitivity analysis, thanks to the Sobol’ decomposition, shows that a strong non-linear coupling exists between the uncertain parameters. Comparisons with the one-dimensional cases demonstrate that the multi-dimensional parametric study is required to get the correct shape and magnitude of the standard deviation distributions of the flow quantities such as pressure and skin-friction.

1. Introduction

Uncertainty quantification (UQ) of the influence of uncertain parameters onto physical systems is a major issue in order to properly predict the system response to random inputs. Several benefits can be obtained from such studies. For example, it allows to introduce realistic safety margins depending on the solution sensitivity to the random inputs. One of the main interests is also to study the coupling process between several uncertain parameters which cannot be investigated through linearized approaches like Adjoint-state-based methods. Moreover, UQ leads to a classification of the most influential parameters on the system response and allows for the identification of extreme behaviors under specific coupling. Moreover, UQ leads to a classification of the most influential parameters on the system response.

Stochastic aerodynamics, i.e. the study of aerodynamic properties of an immersed solid body in the presence of uncertainties, is a recent field compared with classical deterministic aerodynamics. While deterministic aerodynamics is mostly concerned with an almost exact prediction of the flow, stochastic aerodynamics aims at predicting the most probable flow features, the mean flow features (averaging being performed over the different values taken by the uncertain parameters) but also extreme events. Most related works were carried out in the incompressible flow framework for wind engineering oriented studies, e.g. see Solari and Piccardo [21] and Pagnigni and Solari [18] and references given therein. In these works, the main purpose is to model turbulent wind gusts that are responsible for structural loads and to predict the induced structure deformations and/or vibrations. An important point is that in these works the aerodynamic forces exerted on the solid body are obtained via easy-to-solve surrogate models, but not computed using a Navier–Stokes solver. This approach is relevant when the emphasis is put on integrated parameters such as drag and lift for bluff bodies, for which accurate response surfaces can be built.

The present paper addresses another issue of stochastic aerodynamics which is of great interest for aerospace engineering related studies, i.e. the prediction of a transonic flow around a 2D clean wing in the presence of external flow related uncertainties. The emphasis is put on the features of the Reynolds-averaged mean flow, which are the useful and meaningful data used for aerodynamic analysis and shape optimization. The problem of wind gust buffeting is not considered here. For such an analysis, global or oversimplified surrogate models are no longer relevant, and high fidelity Navier–Stokes simulations must be carried out, since engineers are interested in getting a detailed prediction of the flow structure for shape optimization purposes.

Walters and Huyse [24] have underlined the necessity of uncertainty quantification in computational fluid dynamics (CFD) and have reviewed the few methods available. Among them, the Polynomial Chaos (PC) theory introduced by Wiener [26] has been applied to fluid mechanics problems. It has appeared well suited to
get insight into the influence of random variables on relevant aero-
dynamic issues.

The PC approach is very well suited for the representation of
gaussian processes. Its extension to non-gaussian random uncer-
tainties by Xiu and Karniadakis [29] is called generalized Polyno-
mial Chaos (gPC). A development of the whole basic principles
can be found in the book by Ghanem and Spanos [5] and an outlook
of uncertainty quantification in CFD through the use of (gPC) rep-
resentation can be found in Knio and Le Maître [8].

This methodology has been successfully applied in the last dec-
ade to a wide range of fluid mechanics problems, allowing
researchers to assess the sensitivity of a system to random/uncer-
tain external conditions as well as the influence of numerical
parameters.

Le Maître et al. [10] have investigated the transport and mixing
process in a microchannel whereas oscillations of random ampli-
tude have been used by Wan and Karniadakis [25] to study the
chaotic heat transfer enhancement in a grooved channel. Lucor and
Karniadakis [14] have investigated the influence of a uncertain
freestream velocity on the vortex shedding process behind a circu-
lar cylinder while Le Maître et al. [11] have investigated the Ray-
leigh–Bénard instability with random wall temperature. More
recently, Ko et al. [9] have performed simulations of a 2D mixing
layer with random boundary conditions to access the shear layer
growth sensitivity to stochastic inflow forcing. Moreover, a differ-
ent approach has been used by Lucor et al. [16] who have studied
the sensitivity of a LES solution to subgrid-scale-model parametric
uncertainty. They have quantified the influence of an uncertain
Smagorinsky constant value on the energy spectra in the case of
isotropic homogeneous decaying turbulence. In this work, the
authors attempt to measure the sensitivity of the system response
not to external conditions/inputs uncertainty but to the model
uncertainty itself. This is not the point of the present paper. We
emphasize that our goal is not to access the quality of our deter-
minsic solver for a particular or several configurations. Instead,
we wish to quantify the robustness of the simulated response to
external parametric uncertainty.

Uncertainty quantification is essentially a study of errors, both
their description and their consequences. It can be viewed as the
determination of error bars to be assigned to the numerical solu-
tion algorithms. This problem is particularly difficult for non-linear
hyperbolically dominated flows, Yu et al. [30]. Very few papers
deal with stochastic compressible flows. Mathelin et al. have ap-
died the Galerkin PC representation to quasi-one-dimensional
supersonic flow, Mathelin et al. [17]. For this problem, they have
also derived a collocation technique that reduces the computa-
tional burden associated with high-order non-linearities. Some re-
search work in the supersonic regime have also been performed by
Lin et al. [12] dealing with 2D Euler equations for a stochastic
wedge flow (random inflow velocity and random oscillations of
the wedge around its apex). Loeven et al. [13] make use of a deter-
minsic compressible RANS code which is coupled to a probabilis-
tic collocation solver to propagate freestream aerodynamic (Mach
number) uncertainty through a subsonic steady flow around a
NACA0012 airfoil. The Mach number takes a uniform distribution
form with a 5% coefficient of variation and the angle of attack is
deterministic, $\alpha = 5$ deg. The stochastic solution converges fast
and exhibits no spatial discontinuity. For compressible flows with
shocks, such as transonic flows, global gPC approximation can suf-
fice from lack of robustness due to stochastic oscillating systems
involving long-term integration and/or discontinuities in the ran-
dom space. Some work has been pursued to tackle this issue by
designing adaptive stochastic method that can handle discontinu-
ties. For stochastic collocation techniques, we can mention among
others the work of Foo et al. [4], Witteveen and Bijl [27]. Poëtre
et al. [19] propose a stochastic intrusive approach to tackle shocks
in compressible gas dynamics. Their gPC-based technique relies on
the decomposition of the entropic variable of the flow and does not
depend on a special discretization of the random space.

For the optimization of stochastic compressible flows, it is cru-
cial to ensure that the optimal model response is robust with re-
spect to the inherent uncertainties associated with the design
variables, constraints and the objective function. Traditional opti-
mization techniques together with UQ are computationally expen-
sive and time consuming when it comes to identify what drives the
response variability. A stochastic optimization framework combining
stochastic surrogate model representation and optimization
algorithm is proposed by Lucor et al. [15]. A gPC stochastic repre-
sentation is used as the surrogate model. This approach allows
both sensitivity and optimization analysis. The stochastic optimi-
mization method is applied to a multi-layer reacting flow device.
The geometric configuration is assumed to be uncertain and the
structure design is optimized to maximize the energy transfer be-
tween the reacting flow and the device moving parts.

The aim of the present study is to quantify the response of the
flow around a classical bi-dimensional airfoil to uncertain flow
conditions in the transonic regime with the use of a stochastic col-
location spectral projection based on the gPC theory. The new dif-
ficult technical problem addressed in this article is therefore to
assess the capability of a pseudo-spectral method like gPC to accu-
rately capture the non-linear stochastic behavior of flows with
strong discontinuities like shocks, the shock being very sensitive
to uncertainties. This sensitivity result in dramatic changes in both
shock location and shock intensity, making the gPC convergence
process much more complex that for non-bifurcating smooth
flows. The first part of the article is devoted to the simulations
overview. The main part of the study focus on the physical analysis
of the simulations for a 2-parameter stochastic case where both
the infinite Mach number $M_{\infty}$ and the angle of attack $\alpha$ are as-
sumed to be stochastic parameters with uniform distribution.

Then, conclusions are drawn.

2. Numerical procedure

2.1. Simulations overview

The bi-dimensional airfoil retained to perform the current study
is the supercritical OAT15A profile with a chord $c = 0.23$ m. The
freestream conditions are the same as those previously used for
wind tunnels experiments [Jacquen et al. [6]] as well as numerous
numerical simulations with $P_i = 1$ bar and $T_i = 300$ K. The Mach
number $M_{\infty}$ and the angle of attack $\alpha$ are, respectively equal to
0.73 and 2.5 degrees. The Reynolds number $Re$, based on $c$ is equal
to $3 \times 10^6$. In the following, the deterministic simulation with
($M_{\infty}, \alpha$) = (0.73, 2.5) will be referred to as the reference simulation.
A realistic range of variation for the uncertain parameters will be
chosen as to make sure that buffeting does not occur within the
parametric region.

Since the emphasis is put on the Reynolds-Averaged flow fea-
tures, Reynolds-Averaged Navier–Stokes (RANS) are retained as
the relevant mathematical model in this work. Let us also empha-
size that stochastic convergence is expected for RANS solution, but
would not for instantaneous turbulent fields due to the chaotic
nature of the latter. The compressible RANS equations are solved
using the ElsA aerodynamic solver developed at ONERA for the past
ten years, Cambier and Veuillot [1]. A Jameson spatial scheme is
used along with the one equation Spalart–Allmaras model. The 2D
mesh is composed of two blocks the size of which are, respec-
tively $385 \times 161$ cells (C block surrounding the airfoil) and
$129 \times 369$ cells in the wake, leading to $110,000$ cells and has been
extended to 80 chords in all directions. The mesh and the accuracy
of the CFD tool has been carefully checked in past studies, Deck [2].
The capability of the turbulence model, the numerical scheme and computational grid to accurately predict the RANS steady solution at each quadrature point of the gPC projection (see below) has been assessed. Moreover, we have checked that the stochastic predictions of our study are not too sensitive to our RANS parameters as long as we use a sufficiently resolved spatial discretization.

2.2. Polynomial Chaos representation and collocation projection

The following section presents a brief description of the mathematical framework of stochastic spectral method employing expansions of the random inputs and solution based on Askey-type orthogonal polynomial functionals of random vectors. The (gPC) theory is a generalization of the Hermite Chaos originally proposed by Wiener [26] and the reader should refer to Ghanem and Spanos [5] for more details. The stochastic collocation spectral method essentially transform the stochastic problem to a high-dimensional deterministic problem through the use of appropriate projections. The dimensionality of the new system is a function of the noise level of the random input and the order of accuracy required from the representation.

We consider a probability space $\Omega$, where $\Omega$ denotes the event space, $\mathcal{A}$ is its $\sigma$-algebra and $\mathcal{F}$ its probability measure. Let $p(\omega)$ be a random field, i.e. mappings $p : \Omega \to V$ from the probability space into a function space $V$. If $V = \mathbb{R}$, $p(\omega)$ are random variables, and if $V$ is a function space over a time and/or space interval, random fields are stochastic processes. $V$ is a Hilbert space with dual $V^*$, norm $\| \cdot \|$ and inner product $(\cdot, \cdot) : V \times V \to \mathbb{R}$. As $V$ is densely embedded in $V^*$, we abuse notation and denote by $(\cdot, \cdot)$ also the $V \times V$ duality pairing.

In the present study, we will consider second-order random fields, i.e. $p : \Omega \to V$ is a second-order random field over $V$ if

$$\mathbb{E}[p^2] = \mathbb{E} \left[ p(\omega) \right] < \infty,$$

where $\mathbb{E}$ denotes the expectation of a random variable $Y \in L^1(\Omega, \mathbb{R})$ and is defined by

$$\mathbb{E}[Y] = \int_{\omega \in \Omega} Y(\omega) d\mathbb{P}(\omega) = \int Y(\xi) W(\xi) d\xi,$$

with $W(\xi)$ is the measure of the random variable $\xi$ denoting the density of the law $\mathcal{F}(\omega)$ with respect to the Lebesgue measure $d\xi$ and with integration taken over a suitable domain, determined by the range of $\xi$.

The (gPC) representation is a useful means of representing second-order random fields $p(\omega)$ parametrically through the set of random variables $\{\xi_j(\omega)\}_{j=1}^N$ through the events $\omega \in \Omega$. We have:

$$p(\omega) = \sum_{j=0}^N p_j \phi_j(\xi_j(\omega)).$$

Here $\{\phi_j(\xi_j(\omega))\}$ are mutually orthogonal polynomials satisfying the orthogonality relation:

$$\phi_i \phi_j = \delta_{ij} \mathbb{E}[\xi_j^2].$$

Practically, the expansion in Eq. (3) is then truncated to a finite-dimensional space based on a “finite-dimensional noise assumption”: i.e. only a finite number $N$ of random variables $\{\xi_j(\omega)\}_{j=1}^N$ are used. Further, the highest polynomial order $P$ is selected based on accuracy requirements. Consequently, if we denote by $p(\mathbf{x})$, the spatial pressure field, the finite-term gPC expansion reads as follows:

$$p(\mathbf{x}, \xi) = \sum_{j=0}^{M} p_j(\mathbf{x}) \phi_j(\xi_j(\omega)), \quad \text{with} \quad M = \frac{(N + P)!}{N! P!}.$$

The probabilistic collocation method was first introduced by Tatang et al. [23]. It consists in constructing polynomial approximations of the solution from a nodal set of collocation points. After collocation projections, the resulting set of deterministic equations is always uncoupled and each solution is obtained with a deterministic numerical solver. The continuous solution is then interpolated on the data points using multi-dimensional tensor product Lagrange basis. Evaluation of the solution moments requires integrating those Lagrange basis, which can be quite cumbersome, unless we choose the nodal set of points to be a cubature set points. By choosing the cubature weight function to coincide with the joint density of the random input, the computation of the moments becomes straightforward. Nevertheless, the interpolation error is hard to control with this approach, Xiu and Hesthaven [28].

In this study, we use the gPC-collocation method which is a pseudo-spectral method with gPC polynomial basis. It is somewhat similar to the previous approach as it relies on evaluating the solution at finite number of quadrature points but is not based on Lagrange interpolation. Instead, we construct the expansion (5) based on the solver’s evaluations during a post-processing stage. In the case where both $M$ and $x$ are random variables ($N = 2$), the coefficients $p_j(\mathbf{x})$ can be directly expressed as:

$$\forall j \in \{0, \ldots, M - 1\}, \quad p_j(\mathbf{x}) = \frac{\langle p(\mathbf{x}, \omega) \phi_j(M_{\omega}(\omega), \mathbf{x}(\omega)) \rangle}{\langle \phi_j^2(\mathbf{x}(\omega)) \rangle}$$

This method is non-intrusive in the sense that we project the stochastic solution directly onto each member of the orthogonal basis chosen to span the random space. It has the advantage not to require modifications to the existing deterministic solver. The global error of the final representation can be seen as a superposition of an aliasing error (coming from the interpolation), a finite-term projection error (due to the truncated representation) and a numerical error due to the inherent numerical approximation of the deterministic solver.

Different ways of dealing with high-dimensional integrations can be considered depending on the prevalence of accuracy vs. efficiency, Keese [7]. Here, we use a full numerical Gauss quadrature due to its efficiency for moderate $N$ values. The number of quadrature points $N_Q$, representing the number of simulations and relying on the regularity of the function to integrate, is fixed by the user.

In the present study, statistical moments up to the second-order have been investigated thanks to the following expression:

$$\mu_{p(\mathbf{x})} = \langle p(\mathbf{x}, \omega) \rangle = p_0(\mathbf{x}),$$

$$\sigma_{p(\mathbf{x})}^2 = \langle (p(\mathbf{x}, \omega) - p_0(\mathbf{x}))^2 \rangle = \sum_{j=1}^{M} \langle p_j^2(\mathbf{x}) \phi_j^2 \rangle.$$

Probability density functions (pdf) of the solutions can easily be evaluated as well from (5).

2.3. Simulations under uncertainties

In this study, we consider the case of a 2D foil in a randomly perturbed flow in transonic regime. We suppose that the stochastic perturbation affects the magnitude and direction of the incoming inflow velocity. Such random fluctuations can be characteristic of some gust of wind. This translates to a random component in both the flow speed (or Mach number) and the angle of incidence of the flow (or angle of attack of the profile). In aerelasticity, turbulent wind gusts can be modeled by time-dependent forcings represented by random processes, Soong and Grigoriu [22]. In this case, the emphasis is put on the structural response to the stochastic excitation. In these studies, the stochastic model can be quite elaborate but the flow around the airfoil is not directly computed. This approximation may prove disastrous in the case of a transonic
regime that is highly sensitive to perturbations implying shock-waves and separations.

Our approach is different, in the sense that we consider simpler models for the random parameters but we use a Navier–Stokes solver to propagate these uncertainties to the flow around the foil. We treat the uncertain input parameters to our simulations as i.i.d random variables. The non-linearity of the system then transforms these uncorrelated random variables to spatial random processes. The range of variation of the uncertain parameters is chosen as to avoid the buffeting region for which the RANS solver would not be accurate. This requirement suggests considering bounded supports. Once the bounds of the intervals are chosen we are left with the choice of relevant distributions. While different distributions for different parameters are not incompatible with the gPC formulation, we prefer to consider uniform distributions for both parameters. This choice means that we do not favor any particular parametric value within the domain of interest. Moreover, the choice of a uniform distribution is justified as it is the maximum entropy distribution for any continuous random variable on an interval of compact support. In other words, an assumption of any other prior distribution satisfying the constraints will have a smaller entropy, thus containing more information and less uncertainty than the uniform distribution, Jaynes [3]. The uncertain parameters retained for the current study are \( M_1 \) and \( \alpha \). The Mach number \( M_1 \) has a 0.73 mean value and a ±5% variability and the angle of attack \( \alpha \) has a 2.5 degrees mean value and a ±20% variability. Due to the choice of uniform distributions for the inputs and without any \textit{a priori} knowledge of the outputs pdf solution, an appropriate basis from a mathematical point of view is the Legendre polynomial basis Xiu and Karniadakis [29]. This latter one will be used in the following as our expansion basis. Additional studies were also completed for mono-dimensional cases (i.e. the uncertainties on \( M_1 \) and \( \alpha \) have been studied separately) and will be used for comparison in the sensitivity analysis in the last section.

3. Stochastic response to uncertainties

3.1. Numerical convergence

A full convergence study of the stochastic problem has been performed through the analysis of the aerodynamics coefficients as well as their first and second-order statistical moments. The two relevant parameters are \( N_q \) and \( P \). Due to the dependence of the standard deviation accuracy onto the number of terms \( \frac{M}{C^0} \) and so indirectly on the order \( P \), it is of primary importance to check the influence of \( P \) on the magnitude of \( \sigma \).

In this work, \( N_q \) and \( P \) have been determined sequentially in a two-step process. First, the minimum acceptable value for \( N_q \) is identified so that converged mean values of pressure and skin-friction coefficients on the foil surface are obtained. Figs. 2 and 3 respectively plot the values for these two coefficients vs. the chord abscissa for an uncertain Mach number with uniform distribution within the range 0.73 ± 8%. Converged solutions are easily obtained (with a very few number of Gauss points) along most of the profile except in the region of the shock movement for the pressure and
also in the separated area zone for the skin-friction coefficient. In these two critical regions an \( N_q \) value of 40 is required. Using this value of \( N_q = 40 \), the standard deviations obtained along the chord for the pressure and the skin-friction are plotted for different values of the gPC polynomial order \( P \) in Figs. 4 and 5. Similarly to the previous step, fast convergence is reached in linear regions, i.e. regions without shock and/or separation. In the critical non-linear regions an 18th polynomial order may be required.

Some results from the literature suggest that global pseudo-spectral gPC approximation based on collocation is not appropriate in the case of discontinuous or sharp solutions. In this case, the continuous interpolated solution (5) may exhibit some oscillations inducing irregular and unphysical patterns in the spatial distribution of the solution moments or pdf, e.g. stair-like profile for the mean solution (cf. Figs. 2 and 3) at low \( N_q \). The location of these irregularities coincide with the collocation points, see for instance Witteveen and Bijl [27]. The phenomenon is particularly noticeable for local physical quantity (such as \( C_f \)), more sensitive to discretization errors.

In the following paragraph, we refer to the simplified diagram of Fig. 6 for visual assistance. Given a fixed spatial discretization grid of typical resolution size \( \Delta x \) along the chord, the accuracy of the gPC approximation depends on \( P \) and \( N_q \). Let us call \( p_q = p(x_{sq}, Z_q) \) the value of the discontinuous solution at the location of the shock \( x_{sq} \) obtained for the collocation point \( Z_q \). When the number \( N_q \) of collocation points is not sufficient and the response of the system is very sensitive to the uncertainty, it may happen that the typical distance between two neighboring shocks \( \Delta x_{sq} \) is much larger than \( \Delta x \). In this case, \( \Delta x_{sq} \gg \Delta x \) and the problem described hereinbefore appears, Poëtte et al. [19]. However, several of our studies (not all presented here) have shown that the profiles recover regularity when we increase \( N_q \) as \( \Delta x_{sq} \rightarrow \Delta x \). This is the case here (cf. Figs. 2 and 3 at high \( N_q \)). For some higher moments (cf. Fig. 5) and some pdf contours (cf. Fig. 8), some small oscillations may remain along the distribution, but the right profile magnitude is captured for sufficiently high \( P \). In the case where \( \Delta x_{sq} \ll \Delta x \), one faces aliasing error as the shocks are not assigned to the correct cell.

It is in general difficult to predict the appropriate \( N_q \) as the average \( \Delta x_{sq} \) is not known a priori. This latter depends on the distribution of the chosen quadrature rule as well as the sensitivity of the response to the parametric uncertainty. This sensitivity relates to the span length of the geometric envelop in which all probable discontinuous events may take place. Non-linearity of the system, monotonicity of the solution with respect to the parametric variation and airfoil geometry will affect differently this range.

In conclusion, it is somewhat possible to alleviate the problem but there exists a strong coupling between the discretization in physical space and the collocation grid in random space. As a re-

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**Fig. 3.** Mean value distribution of the skin-friction along the chord.

**Fig. 4.** Standard deviation of the pressure coefficient along the chord.

**Fig. 5.** Standard deviation of the skin-friction along the chord.

**Fig. 6.** Schematic illustrating the dependence between physical (\( \Delta x \)) and stochastic (\( \Delta x_{sq} \)) discretizations. When \( \Delta x_{sq} \gg \Delta x \), i.e. the parametric (here \( M_\infty \)) distribution points distribution (dotted cyan curves) is such that the corresponding discontinuous realizations are far apart (blue curves), the distribution of the gPC solution moments may exhibit some irregular and unphysical patterns (black curve). These irregularities are smoothed out (red curve) if \( \Delta x_{sq} \approx \Delta x \). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
3.2. Mean fields, standard deviations and PDF distributions

Figs. 7 and 8 present the mean and standard deviation distribution of wall data along the airfoil for the stochastic bi-dimensional case as well as their associated PDFs distributions and PDFs profiles for five locations on the leeward side. Reference cases are also included.

Let’s focus on the wall pressure coefficient $K_p$ first. For $0.35 < x/c < 0.65$, we notice that stochastic solutions greatly depart from the reference solution. We also notice that the uncertain mean solution differs from the deterministic one. The main discrepancy consists in a less pronounced compression region surrounding the mean shock position. This result is consistent with the fact that the shock location is not fixed for different low Mach number realisations. Indeed, the shock moves upstream as the Mach number value decreases from the averaged value of 0.73 to its lower bound. No clear influence of the parametric uncertainty can be observed for $x/c > 0.65$ as well as for $x/c < 0.35$ which is the most upstream shock position for the uncertainty range investigated here. These observations have to be related to the behavior of the standard deviations. For $0 < x/c < 0.35$, $\sigma_{K_p}$ results from a linear response of the flow to the uncertainty as the shock-wave never penetrates this area whatever $M_\infty$ values in our range. Downstream this location, higher magnitudes of the standard deviation are observed. It appears that the strong spatial non-linearities introduced by the shocks translate to the random domain. The dependency of the shock location to $M_\infty$ accounts for the high sensitivity of the flow in this wide region. Further downstream, $\sigma_{K_p}$ becomes very weak except for $x/c > 0.9$ where trailing-edge effects can be observed.

The skin-friction behavior is quite different and exhibits discrepancies with the reference solution for both upstream and downstream locations compared to the deterministic shock position. For $x < 0.65$, these differences can be explained in a similar way as the ones for the pressure coefficient. The uncertainty range has a diffusion-like effect on the skin-friction gradient in the shock region thus explaining the lower $C_f$ values ahead of $x/c = 0.55$. For $x/c > 0.65$, the stochastic solution presents higher $C_f$ magnitude than the deterministic case meaning that the uncertain parameters deeply influence the boundary-layer state in the second half of the airfoil. The analysis of Mach fields (not shown here) for several realizations within the uncertainty range shows that the shock position has the tendency to shift downstream when $M_\infty$ is raised up from its lower bound to its mean value of 0.73. For higher $M_\infty$ values, its position remains stationary and only the boundary-layer state behind the shock is altered, leading to separation. Such evolution can be clearly evidenced when looking at the Mach contours of the mono-dimensional cases. It shows why the skin-friction...
The coefficient is much more sensitive to the uncertainty than the pressure coefficient in the second half of the profile. This trend is similar for the standard deviation distributions where high magnitudes of $r_{Cf}$ can be observed both upstream and downstream of the reference shock position.

Additional knowledge can be brought with the use of the PDFs. It can be very useful to detect some rare events that cannot be revealed from the standard deviation distributions. For the wall pressure, the PDFs distribution is almost centered around the most probable value which also corresponds to the mean stochastic $Kp$ value for $x/c < 0.35$. This is no more the case in the region where the flow response is non-linear. The PDFs exhibit a wider range of $Kp$ values with two dominant peaks. As a consequence, in this region, the mean stochastic $Kp$ differs from the most probable $Kp$ magnitude. For $x/c > 0.65$, the shock-wave is always located upstream independently of $M_\infty$ in the uncertainty bounds considered and a narrow peak is observed on the PDFs. The $Kp$ values are almost insensitive to the uncertainty as previously observed on both the mean stochastic $Kp$ values and $\sigma_{Kp}$. It is obvious that the skin-friction behaves quite differently because of the quite large variations observed in all the possible $Cf$ values for $x/c > 0.35$. Once again, in this region of the airfoil, it is clear that the most probable $Cf$ magnitude highly differs from the mean stochastic value.

This is in accordance with the results shown in Fig. 9 which depicts the spatial distribution of the Mach number coefficient of variation $c_v$. The coefficient of variation is a non-dimensional number and is a measure of dispersion of a probability distribution. It is defined as the ratio between the standard deviation $\sigma$ and the mean stochastic value $\mu$. Two distinct areas with high $c_v$ magnitudes can be isolated. The first one, for $0.35 < x/c < 0.65$, corresponds to the region where non-linear variations of the pressure coefficient were underlined (Fig. 7) due to the variable shock position. In this region, dispersion as high as 30% can be observed. The second area of high variability is located in the boundary-layer behind the reference shock position. On the lower figure, a coefficient superior to one can be observed very locally whereas all the boundary-layer area exhibits $c_v$ values superior to 0.5. These high dispersion values agree well with the observations previously drawn dealing with the boundary-layer separation downstream the shock when $M_\infty > 0.73$.

3.3. Coupling process

A sensitivity analysis can be performed by using the Sobol' decomposition (Saltelli and Sobol' [20]). It allows to determine the relative influence of each stochastic parameter on the system within the uncertainty range investigated. Thanks to the methodology used in the present study, the polynomial chaos based Sobol' indices can be directly calculated from the expansion coefficients.

Using the Sobol' decomposition of the total variance, we can write:

$$\sigma_{Cf}^2_{total} = \sigma_{M}^2 + \sigma_{a}^2 + \sigma_{M-a}^2$$  \hspace{1cm} (9)
where \( \sigma_{\text{Total}} \) is the total standard deviation whereas \( \sigma_M \) and \( \sigma_\alpha \) are the partial standard deviations respectively due to the Mach number and the angle of attack uncertainties. The \( \sigma_{M-\alpha} \) term is the standard deviation resulting from the coupling process between the 2 stochastic parameters.

Fig. 10 presents the distribution of the partial standard deviations as well as the reference standard deviations from the mono-dimensional cases which have been noted, respectively \( \sigma_M(\text{Ref.}) \) and \( \sigma_\alpha(\text{Ref.}) \). The observations which can be drawn from both figures (a) and (b) are similar. \( \sigma_{\text{Total}} \) and \( \sigma_M \) exhibit the same shape and almost the same amplitudes meaning that \( \sigma_M \) dominates the whole standard deviation, meaning that the compressibility effect are the most influential. This result could have been guessed looking at the mono-dimensional cases, far less expensive in computing resources. The physical rationale for that is that variations in the Mach number induces a change of both the shock location and the shock strength on the suction-side, these two effects having a deep impact on the boundary-layer state downstream the shock. But there is also a significant coupling between compressibility effects and incidence effects resulting in a coupling term \( \sigma_{M-\alpha} \) with superior magnitude to \( \sigma_\alpha \) in the interaction region \( 0.3 < x/c < 0.65 \). Moreover, when both the mono- and bi-dimensional cases are compared, discrepancies appear both in magnitude and shape of the standard deviation. A 20% increase of the pick value is observed in the 2-parameter case compared to the mono-dimensional one for both the pressure and the skin-friction coefficients. One can also observe the absence of the bump in the standard deviation distributions \( \sigma_M(\text{Ref.}) \) and \( \sigma_\alpha(\text{Ref.}) \) due to the coupling process in the bi-dimensional case. This coupling process is much more evident on \( \sigma_{M-\alpha} \) related to the skin-friction coefficient where the coupling term reaches a 40% value of the whole standard deviation in the interaction region. The conclusion is that non-linear stochastic interactions between the shock displacement and the suction-side boundary-layer downstream the shock are more important than isolated angle of attack effects.

4. Conclusions

The present work was aimed at investigating the sensitivity of a 2D transonic flow around an airfoil to uncertain parameters with the use of the Polynomial Chaos methodology. The stochastic inputs chosen in the present study are the infinite Mach number and the angle of attack due to the sensitivity of the flow to these variables in the transonic regime. The physical response of the flow to such uncertainties is studied based on bi-dimensional chaos simulations. The stochastic collocation methodology
has succeeded in providing converged solutions up to the second-order. However, a spatial discontinuity can be made due to the different non-linearities involved in the flow. Thus, the pressure discontinuities (shocks) require the use of high-order polynomial expansions due to the stochastic parameters on the steep dependency of the shock position. On the contrary, the existence of non-linearities through the appearance of separation behind the mean shock position does not require high-order terms. Discrepancies appear between the most probable pressure and skin-friction distributions and the deterministic case in the interaction region where the shock moves dependently of the stochastic parameters which demonstrate the influence of the uncertainties on the response of the flow. The analysis of the PDFs distributions is also helpful to evidence the highly non-linear regions of the flow and investigate the rare events which can occur in these areas. Another important observation is that prediction of the range of variation around the stochastic mean value and the standard deviation is not accurate in such a non-linear case: extreme events occur which are much stronger. Then, a detailed analysis of the coupling between the random parameters thanks to the Sobol’ decomposition has been performed and revealed to be a powerful tool to analyse the sensitivity of the flow. The partial standard deviations differ from their mono-parameter counterparts both in shapes and magnitudes revealing that the study of the whole multi-parameter case is required in order to get accurate results. Another important conclusion drawn from Fig. 1 is that the most probable value (i.e. the one with the highest probability) of KP and CF at a given location may be very different from the mean stochastic value (found by integrating the pdf), because the pdf exhibit several significant peaks, each peak being associated to a pattern of the solution. Therefore, recovery of the full pdf profile appears to be mandatory for safety studies.

There is currently work in progress dealing with advanced methodologies aimed at reducing the cost of the quadrature evaluation for large multi-parameter stochastic space through the use of cubatures.

This work has been undertaken through a cooperation program between Université Pierre et Marie Curie (UPMC), the French national aerospace research center ONERA and Airbus Industries.

References


