

Reliable decentralised control of delayed MIMO plants

A.N. Gündeş^{a*} and H. Özbay^b

^aDepartment of Electrical and Computer Engineering, University of California, Davis, CA 95616, USA;

^bDepartment of Electrical and Electronics Engineering, Bilkent University, Ankara 06800, Turkey

(Received 7 January 2009; final version received 28 July 2009)

Reliable decentralised proportional–integral–derivative controller synthesis methods are presented for closed-loop stabilisation of linear time-invariant plants with two multi-input, multi-output (MIMO) channels subject to time delays. The finite-dimensional part of plants in the classes considered here are either stable or they have at most two poles in the unstable region. Closed-loop stability is maintained with only one of the two controllers when the other controller is turned off and taken out of service.

Keywords: delay systems; decentralised control; reliable stabilisation; PID controller

1. Introduction

In this work, a stabilising controller synthesis method is developed for linear time-invariant (LTI), multi-input multi-output (MIMO) systems that are subject to time delays. The controller structure is a two-channel block-decentralised controller configuration, where each of the two channels may have multiple inputs and outputs. The challenging objectives of decentralised closed-loop stabilisation, reliable stability in the case of complete failure of either one of the two channels and integral action are all achieved with simple low-order controllers.

In addition to closed-loop stability, an important performance objective is asymptotic tracking of step-input references with zero steady-state error, which is achieved by designing controllers with integral action. The simplest integral action controllers are in the proportional–integral–derivative (PID) form (Goodwin, Graebe, and Salgado 2001), which are first order if the derivative term is zero (PI) or second order if the derivative term is non-zero. Although PID controllers are widely used in many control applications and preferred due to easy implementation and tuning, their simplicity also presents a major restriction that they can control only certain classes of unstable plants since the controller order cannot exceed two. For the delay-free case, and even without the decentralisation constraint, a complete characterisation of unstable plants that can be stabilised using PID controllers is not available. It was shown in Gündeş and Özgüler (2007) that strong stabilisability of the plant is a necessary but not sufficient condition. Several unstable delay-free plant classes that admit

PID controllers are identified in Gündeş and Özgüler (2007), where the zeros in the unstable region are essentially restricted to be either all larger or all smaller than the positive real poles of the plant; a dual classification allows the zeros to be anywhere in the complex plane while restricting the poles that are in the unstable region. Stability and feedback stabilisation of delay systems have been extensively investigated and many delay-independent and delay-dependent stability results are available (Niculescu 2001; Gu, Kharitonov, and Chen 2003). Most of the tuning and internal model control techniques used in process control systems apply to delay systems (Skogestad 2003), which inherit the results on robust control of infinite-dimensional systems (Foiás, Özbay, and Tannenbaum 1996). The more specialised problem of existence of stabilising PID controllers for delay systems is not easy to solve (see e.g. Silva, Datta, and Bhattacharyya (2005)). For stable plants and for unstable plants with up to two poles in the unstable region, a non-decentralised PID controller synthesis method was developed in Gündeş, Özbay, and Özgüler (2007) for delays that affect the inputs and outputs (I/O delays). Although these earlier synthesis approaches used in Gündeş et al. (2007) form the basic motivation for some of the results presented in this article, the method developed here allows arbitrary delay terms to affect different entries of the plant's transfer-matrix for the stable case, and also deals with a more challenging problem due to the decentralised controller configuration and reliability considerations.

A control system's reliability against complete failure of certain channels is a practical engineering

*Corresponding author. Email: angundes@ucdavis.edu

consideration and an important design requirement. Reliable stabilisation guarantees closed-loop stability even when some control channels are affected by failures and feedback is not available from those sensors. It is assumed that a controller that fails is set equal to zero; i.e. the failure is recognised and the failed controller is taken out of service (with its states reset to zero). If the controller design incorporates integral action as in the case of PID controllers, then asymptotic tracking of constant reference inputs with zero steady-state error is achieved in those channels that remain operational, but closed-loop stability is still maintained. In the decentralised setting, PID controller designs were considered for two-by-two delay-free plants in Aström, Johansson, and Wang (2002) and Tavakoli, Griffin, and Fleming (2006), where the channels have single-input single-output (SISO). The reliable control problem of maintaining closed-loop stability when one controller fails was studied in Gündeş and Özgüler (2002) for delay-free plants which had unstable poles only at the origin; the controllers achieved integral action but their order was generally high and not restricted as in PID. A more recent work presented reliable decentralised PID controllers in Gündeş, Mete, and Palazoğlu (2009) for several more general unstable delay-free plant classes that allow PID stabilisation. The work summarised thus far did not incorporate delay terms in reliable decentralised stabilisation and the results obtained were for finite-dimensional systems.

The goal in this article is to establish existence of decentralised reliably stabilising PID controllers and to present controller designs for MIMO systems subject to time delays. Since the main objective is to characterise controllers that reliably stabilise the system, we do not consider performance issues but allow freedom in the design parameters, which can be used towards satisfaction of performance criteria. We propose systematic decentralised PID synthesis procedures for the following classes of delayed MIMO systems:

- (1) For plants whose finite dimensional part is stable, completely different delay terms may affect each of the MIMO transfer-matrix entries; i.e. $e^{-h_{ij}s}$ multiplies the ij -th entry of the finite-dimensional part of the plant's transfer-matrix. We propose decentralised PID designs that are reliable against the failure of any one of the two MIMO controllers. The main result in this section (Proposition 1) is motivated by similar methods as in Gündeş et al. (2007), which presented a non-decentralised synthesis without reliability considerations and only applied to I/O delays. In contrast, the

result in this work is applicable to the most general delay considerations possible for this plant class and has a completely different decentralised feedback configuration.

- (2) For plants whose finite-dimensional part is unstable, arbitrary delay terms enter the numerator matrix in the coprime factorisation of the plant's transfer-matrix. In the case of unstable plants, due to the order constraints of (second order) PID controllers, we allow up to two poles in the unstable region to be present in any of the transfer-matrix entries, whereas the transmission-zeros may be anywhere, and there may be any number of poles in the stable region. The main results in this section (Propositions 2 and 3) show that decentralised PID controllers exist for these classes of MIMO plants with delays, and develop systematic synthesis procedures that explicitly characterise reliable designs with wide range of parameter choices, where constant reference inputs are tracked asymptotically only in the channel that remains operational but closed-loop stability is always maintained.

We apply the systematic methods of Propositions 1–3 to systems containing delays to illustrate the reliable decentralised PID controller synthesis. In Example 1, we achieve a fully reliable design where stability is maintained when either one of the channels fails. In Example 2, due to the instabilities that cannot be compensated in the case of failures, it is shown that a decentralised design is achieved but is not reliable against failure of either channel. In Example 3, a partially reliable design is achieved where the main channel remains active and closed-loop stability is maintained if the secondary channel fails. In each example, simulation results are shown for the chosen controller parameters. The freedom in these parameters is specified in the synthesis methods. These parameters can be varied to achieve other performance specifications and to achieve desired responses. Our objective is to establish closed-loop stabilisability with decentralised structure and PID order constraints and hence, we do not explore fully the issues of how the choice of free parameters affect the system's performance.

We use the following standard notation:

Notation: Let \mathbb{C} , \mathbb{R} and \mathbb{R}_+ , denote complex, real and positive real numbers. The extended closed right-half complex plane is $\mathcal{U} = \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 0\} \cup \{\infty\}$; \mathbf{R}_p denotes real proper rational functions (of s); $\mathbf{S} \subset \mathbf{R}_p$ is the stable subset with no poles in \mathcal{U} ; $\mathcal{M}(\mathbf{S})$ is the set of matrices with entries in \mathbf{S} ; I_r is the $r \times r$ identity matrix. The space \mathcal{H}_∞ is the set of all bounded analytic

functions in \mathbb{C}_+ . For $h \in \mathcal{H}_\infty$, the norm is defined as $\|h\|_\infty = \text{ess sup}_{s \in \mathbb{C}_+} |h(s)|$, where ess sup denotes the essential supremum. A matrix-valued function H is in $\mathcal{M}(\mathcal{H}_\infty)$ if all its entries are in \mathcal{H}_∞ ; in this case $\|H\|_\infty = \text{ess sup}_{s \in \mathbb{C}_+} \bar{\sigma}(H(s))$, where $\bar{\sigma}$ denotes the maximum singular value. From the induced L^2 gain point of view, a system whose transfer-matrix is H is stable iff $H \in \mathcal{M}(\mathcal{H}_\infty)$. A square transfer-matrix $H \in \mathcal{M}(\mathcal{H}_\infty)$ is unimodular iff $H^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. We drop (s) in transfer-matrices such as $G(s)$. Since all norms of interest here are \mathcal{H}_∞ norms, we drop the norm subscript, i.e. $\|\cdot\|_\infty \equiv \|\cdot\|$. We use coprime factorisations over \mathbf{S} ; i.e. for $G \in \mathbf{R}_p^{r \times r}$, $G = Y^{-1}X$ denotes a left-coprime factorisation (LCF), where $X, Y \in \mathbf{S}^{r \times r}$, $\det Y(\infty) \neq 0$.

2. Problem description

Consider the two-channel decentralised feedback system $\text{Sys}(\widehat{G}, C_D)$ with two MIMO channels in Figure 1, where $C_D = \text{diag}[C_1, C_2] \in \mathbf{R}_p^{r \times r}$ is the decentralised controller and \widehat{G} is the delayed plant transfer-function partitioned as

$$\widehat{G} = \begin{bmatrix} \widehat{G}_{11} & \widehat{G}_{12} \\ \widehat{G}_{21} & \widehat{G}_{22} \end{bmatrix}. \quad (1)$$

It is assumed that the feedback system is well posed and that the delay-free part of the plant and the controller have no unstable hidden-modes. The finite-dimensional part of the plant is $G \in \mathbf{R}_p^{r \times r}$, where each channel has as many inputs as outputs, i.e. $G_{ij} \in \mathbf{R}_p^{r_j \times r_j}$, $G_{ij} \in \mathbf{R}_p^{r_i \times r_j}$, $i, j \in \{1, 2\}$, and $\text{rank } G = r$. Let $G = Y^{-1}X$ be an LCF of G . Then we assume that \widehat{G} can be written as

$$\widehat{G} = Y^{-1}\widehat{X}, \quad \text{where } \widehat{X}_{ij} = e^{-h_{ij}s} X_{ij}, \quad i, j = 1, \dots, r. \quad (2)$$

Therefore, the ij -th entry \widehat{X}_{ij} of \widehat{X} may contain all different delay terms and that the delays are known. If the finite-dimensional part G of the delayed plant \widehat{G} is stable, then (2) implies that the entries of \widehat{G} may contain all different arbitrary known delay terms. If the finite-dimensional part G of the delayed plant \widehat{G} is not stable, then we assume that the delayed plant

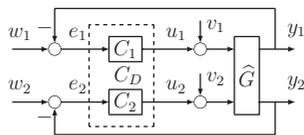


Figure 1. The two-channel decentralised system $\text{Sys}(\widehat{G}, C)$ with delays.

transfer-function \widehat{G} has restrictions on the number of poles in the unstable region.

For the system $\text{Sys}(\widehat{G}, C_D)$, let $w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $v := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ denote the input and output vectors. The closed-loop transfer-matrix H_{cl} from (w, v) to (u, y) is

$$H_{\text{cl}} = \begin{bmatrix} C_D(I + \widehat{G}C_D)^{-1} & -C_D(I + \widehat{G}C_D)^{-1}\widehat{G} \\ \widehat{G}C_D(I + \widehat{G}C_D)^{-1} & (I + \widehat{G}C_D)^{-1}\widehat{G} \end{bmatrix}. \quad (3)$$

Definition 1: (a) The feedback system $\text{Sys}(\widehat{G}, C_D)$ is stable if the closed-loop map H_{cl} is in $\mathcal{M}(\mathcal{H}_\infty)$. (b) The controller C_D stabilises \widehat{G} if C_D is proper and $\text{Sys}(\widehat{G}, C_D)$ is stable. (c) The controller C_D that stabilises \widehat{G} is partially reliable if the system $\text{Sys}(\widehat{G}, 0, C_2)$ is also stable, i.e. the transfer-function from (w_2, v) to (y, u_2) is in $\mathcal{M}(\mathcal{H}_\infty)$. (d) The controller C_D that stabilises \widehat{G} is fully reliable if the system $\text{Sys}(\widehat{G}, 0, C_2)$ is also stable (i.e. the transfer-function from (w_2, v) to (y, u_2) is in $\mathcal{M}(\mathcal{H}_\infty)$), and the system $\text{Sys}(\widehat{G}, C_1, 0)$ is also stable (i.e. the transfer-function from (w_1, v) to (y, u_1) is in $\mathcal{M}(\mathcal{H}_\infty)$).

For existence of partially reliable controllers, the finite-dimensional part G of the plant \widehat{G} must satisfy additional requirements (Gündes et al. 2009). In addition to the decentralised structure of the controller C_D , we restrict our attention to *proper* PID controllers of the following form (Goodwin et al. 2001): For $j=1, 2$,

$$C_j = K_{pj} + \frac{1}{s} K_{Ij} + \frac{s}{\tau_j s + 1} K_{Dj}, \quad (4)$$

where K_{pj} , K_{Ij} , $K_{Dj} \in \mathbb{R}^{r_j \times r_j}$ are the proportional, the integral, and the derivative constants, respectively, and $\tau_j \in \mathbb{R}_+$, where C_j has integral-action when $K_{Ij} \neq 0$. We include subsets of PID controllers obtained by setting one or two of these three constants to zero; e.g. (4) is a PI controller when $K_{Dj} = 0$.

3. Reliable controller synthesis

Partially or fully reliable decentralised PID controllers can be designed for stable MIMO plants with delays. In Section 3.1 we explore decentralised design for stable MIMO plants, where arbitrary delay terms may affect different entries of the plant's transfer-matrix. In Section 3.2, we consider decentralised PID controller synthesis for MIMO plants with one or two poles in the region of instability \mathcal{U} , including the origin. Some restrictions on the number of \mathcal{U} -poles are necessary since for plants with an arbitrary number of \mathcal{U} -poles, existence of PID controllers is not guaranteed even when the plant is delay-free. Many plants that have more than two poles in the unstable region cannot be

stabilised using PID controllers (e.g. $\frac{1}{(s-p)^3}$ or $\frac{1}{(s-p)(s^2+p^2)}$ for $p \geq 0$).

3.1 Stable plants with time delays

If the finite-dimensional part G of the delayed plant \widehat{G} is stable, then it is possible to design decentralised PID controllers that are partially or fully reliable. The delay terms enter the entries of the plant's transfer-matrix arbitrarily. Note that $\widehat{G}(0) = G(0)$. In Proposition 1, we first design the controller C_2 to stabilise \widehat{G}_{22} and then we design C_1 to stabilise the system \widehat{W} defined by

$$\widehat{W} := \widehat{G}_{11} - \widehat{G}_{12}C_2(I + \widehat{G}_{22}C_2)^{-1}\widehat{G}_{21}, \quad (5)$$

which contains C_2 . When G is stable, \widehat{W} is also stable. This method provides a partially reliable decentralised design. If C_1 is designed to stabilise \widehat{W} and \widehat{G}_{11} simultaneously, then the decentralised controller becomes fully reliable.

The synthesis in Proposition 1 is based on similar methods as in the non-decentralised design ideas in (Günder et al. 2007), where the delays were restricted to have diagonal I/O structures. Here, Proposition 1 applies to a more general case with arbitrary delay terms; furthermore, it provides a systematic synthesis approach of decentralised reliable controller design for plants containing arbitrary delay terms.

Proposition 1: Let \widehat{G} be as in (1), where $G \in \mathbf{S}^{r \times r}$ is stable, and let $\text{rank}\widehat{G}(0) = \text{rank}G(0) = r$. For C_j to be a PD controller, let $M_j = 0$. For C_j to be a PID controller (with non-zero integral constant), let $M_j = I$.

- (a) *Partially reliable design:* Let $\text{rank}\widehat{G}_{22}(0) = \text{rank}G_{22}(0) = r_2$. Choose any $\widehat{K}_{P2}, \widehat{K}_{D2} \in \mathbb{R}^{r_2 \times r_2}, \tau_2 > 0$. Define

$$\widehat{C}_2 := \widehat{K}_{P2} + \frac{s}{\tau_2 s + 1} \widehat{K}_{D2} + \frac{1}{s} G_{22}(0)^{-1} M_2. \quad (6)$$

Then for any $\beta_2 \in \mathbb{R}_+$ satisfying (7), the PID controller C_2 in (7) stabilises \widehat{G}_{22} :

$$C_2 = \beta_2 \widehat{C}_2, \quad 0 < \beta_2 < \left\| \frac{1}{s} [s \widehat{G}_{22}(s) \widehat{C}_2 - M_2] \right\|^{-1}. \quad (7)$$

Let \widehat{W} be defined by (5). Choose any $\widehat{K}_{P1}, \widehat{K}_{D1} \in \mathbb{R}^{r_1 \times r_1}, \tau_1 > 0$. Define

$$\widehat{C}_1 := \widehat{K}_{P1} + \frac{s}{\tau_1 s + 1} \widehat{K}_{D1} + \frac{1}{s} \widehat{W}(0)^{-1} M_1. \quad (8)$$

Then for any $\beta_1 \in \mathbb{R}_+$ satisfying (9), the PID controller C_1 in (9) stabilises \widehat{W} :

$$C_1 = \beta_1 \widehat{C}_1, \quad 0 < \beta_1 < \left\| \frac{1}{s} [s \widehat{W}(s) \widehat{C}_1 - M_1] \right\|^{-1}. \quad (9)$$

With C_2 and C_1 as in (7) and (9), respectively, $C_D = \text{diag}[C_1, C_2]$ is a partially reliable decentralised PID controller for the delayed plant \widehat{G} . For $\widehat{K}_{Dj} = 0$, the controllers (7) and (9) become P controllers (if $M_j = 0$) or PI controllers (if $M_j = I$); for $\widehat{K}_{Pj} = 0$, (7) and (9) become D controllers (if $M_j = 0$) or ID controllers (if $M_j = I$).

- (b) *Fully reliable design:* Let $\text{rank}\widehat{G}_{jj}(0) = \text{rank}G_{jj}(0) = r_j, j \in \{1, 2\}$. Let $\widehat{W}(0)G_{11}(0)^{-1}$ have all positive real eigenvalues. Let C_2 be as in (7). Let C_1 be as in (9) with β_1 satisfying

$$0 < \beta_1 < \min \left\{ \left\| \frac{1}{s} [s \widehat{W}(s) \widehat{C}_1 - M_1] \right\|^{-1}, \left\| \frac{1}{s} [s \widehat{G}_{11}(s) \widehat{C}_1 - G_{11}(0) \widehat{W}(0)^{-1} M_1] \right\|^{-1} \right\}. \quad (10)$$

Then $C_D = \text{diag}[C_1, C_2]$ is a fully reliable decentralised PID controller for the delayed plant \widehat{G} .

Remark: The control procedure in Proposition 1 motivates the ‘optimal’ design of some of the free parameters, such as \widehat{K}_{P2} and \widehat{K}_{P1} . However, how the choice of the free design parameters would eventually affect the system's performance cannot be generalised. The focus here is on reliable stability and a full performance analysis is not considered. Consider the optimal PI controller $\widehat{C}_2(s) = \widehat{K}_{P2} + \frac{1}{s} G_{22}(0)^{-1}$. The proportional gain \widehat{K}_{P2} will be optimised so that the allowable interval for β_2 is the largest, i.e. so that the bound for β_2 in (7) is maximised:

$$\left\| \frac{1}{s} \left[s \widehat{G}_{22}(s) \left(\widehat{K}_{P2} + \frac{1}{s} G_{22}(0)^{-1} \right) - I \right] \right\|^{-1}. \quad (11)$$

Re-arranging terms in (11), defining $\widehat{K}_{P2} = G_{22}(0)^{-1} \widetilde{K}_{P2}$, and $F_{22}(s) := \widehat{G}_{22}(s) G_{22}(0)^{-1}$, we are interested in finding the optimal \widetilde{K}_{P2} such that (12) is minimised:

$$\left\| \frac{1}{s} [F_{22}(s) - I] + F_{22}(s) \widetilde{K}_{P2} \right\|. \quad (12)$$

This problem was studied and a formula for the optimal solution was obtained for a class of SISO functions $F_{22}(s)$ in Özbay and Günder (2007). Similarly, a PI controller C_1 can be derived by optimising \widehat{K}_{P1} to maximise the bound for β_1 in (9). With $\widehat{W}(s) \widehat{W}(0)^{-1}$ replacing F_{22} , the optimisation problem is again in the form (12).

In Example 1, we apply the synthesis procedure of Proposition 1 to design a partially and fully reliable decentralised control system that manipulates the flow rate of two drugs (dopamine and sodium nitropruside) to regulate two outputs (main arterial pressure

and cardiac output) for critical care patients. A simplified model is used representing the input–output behaviour for a particular patient (Bequette 2003). The free parameters \widehat{K}_{Pj} and \widehat{K}_{Dj} are chosen completely arbitrarily and adjusted based on the simulations to obtain faster step responses with acceptable damping. A generalisation of how these arbitrary selections would affect the system’s response is not possible but can be studied on a case-by-case basis.

Example 1: Let $\widehat{G} = \begin{bmatrix} \frac{-6}{0.67s+1} e^{-0.75s} & \frac{3}{2s+1} e^{-s} \\ \frac{12}{0.67s+1} e^{-0.75s} & \frac{5}{5s+1} e^{-s} \end{bmatrix} \in \mathcal{H}_\infty^{2 \times 2}$.

Following Proposition 1, partially and fully reliable decentralised PID controllers can be designed with non-zero K_{Ij} since $\text{rank}G(0) = 2$, $G_{jj}(0) \neq 0$, $\widehat{W}(0) \times G_{11}(0)^{-1} = 2.2 > 0$ when we have a non-zero integral action in C_2 . First design C_2 : Choose $\widehat{K}_{P2} = 1$, $\widehat{K}_{D2} = 0.2$, $\tau_2 = 0.1$. With $\beta_2 = 0.6$ satisfying (7), the PID controller in (7) is $C_2 = 0.6 + 0.12/s + 0.12s/(0.1s + 1)$. Now design C_1 : Choose $\widehat{K}_{P1} = -0.15$, $\widehat{K}_{D1} = -0.1$, $\tau_1 = 0.1$. With $\beta_1 = 0.1$ satisfying (9), the PID controller in (9) is $C_1 = -0.015 - 1/(132s) - 0.01s/(0.1s + 1)$. Then $C_D = \text{diag}[C_1, C_2]$ is a partially reliable decentralised controller; it is also fully reliable since $\beta_1 = 0.1$ also satisfies (10) with this \widehat{K}_{P1} . Figure 2(a) shows the closed-loop step responses for the outputs y_1 (dashed), y_2 (solid), with unit-step references applied at both w_1, w_2 . The controller $C_D = \text{diag}[C_1, C_2]$ is active with both channels operational, and both achieve asymptotic tracking with zero steady-state error. Figure 2(b) shows the step responses when $C_1 = 0$, with only the second channel operational. Since the controller is $C_D = \text{diag}[0, C_2]$, the output y_1 does not track the step reference due to lack of integral action in the first channel. Figure 2(c) shows the step responses when $C_2 = 0$, with only the first channel operational. Since the controller is $C_D = \text{diag}[C_1, 0]$, the output y_2 does not track the step reference due to lack of integral action in the second channel.

3.2 Unstable plants with time delays

Although strong stabilisability is a necessary but insufficient condition for PID stabilisability, a complete characterisation of unstable plants that can be stabilised using these simple controllers is not available even for the delay-free and non-decentralised cases. Assuming full-feedback structure, several classes of unstable delay-free plants were identified as PID stabilisable in Gündes and Özgüler (2007), where restrictions were imposed on either the plant’s right-half-plane transmission-zeros or the plant’s right-half-plane poles. For the case of plants with I/O delays under full-feedback, it was shown in Gündes

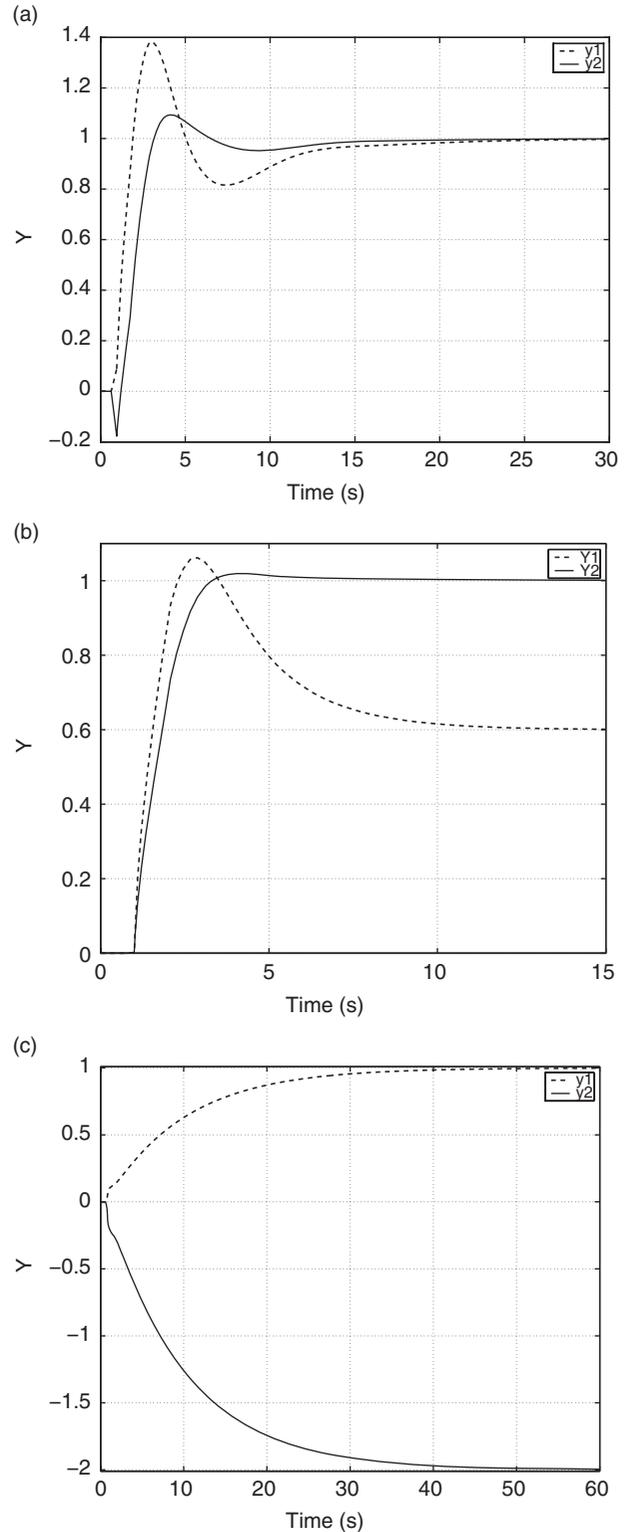


Figure 2. Example 1 step-responses with (a) $C_D = \text{diag}[C_1, C_2]$, (b) $C_D = \text{diag}[0, C_2]$, (c) $C_D = \text{diag}[C_1, 0]$.

et al. (2007) that plants whose finite-dimensional part has at most two positive real poles (while there is no restriction on the poles with negative real part) can be stabilised using non-decentralised PID controllers.

If the finite-dimensional part G of the delayed plant \widehat{G} is unstable, then let the finite-dimensional part $G(s) \in \mathbf{R}_p^{r \times r}$ of the plant have full (normal) rank. Let G have no transmission-zeros at $s=0$. Without loss of generality, it can be assumed that G has an LCF as in (13):

$$G = Y^{-1}X = \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix}^{-1} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}. \quad (13)$$

We assume that different delay terms may affect any arbitrary entry of the numerator factor X of G in (13) and the delayed plant \widehat{G} can be written as:

$$\widehat{G} = Y^{-1}\widehat{X} = \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix}^{-1} \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix}. \quad (14)$$

Note that I/O delays would be a special case of the more general delay description in (14). We consider two cases of delayed plants \widehat{G} , where the finite-dimensional part G is unstable with restrictions on the number of \mathcal{U} -poles. In all cases, G may have any number of poles in the stable region. In the proposed methods, we first design C_2 to stabilise G_{22} and then C_1 to stabilise the system \widehat{W} defined in (5), which contains C_2 . The channels can be re-ordered to exchange the roles of G_{11} and G_{22} . In Case 1, \widehat{W} is unstable; in Case 2, \widehat{W} is stable. In Case 1, G has one \mathcal{U} -pole $p_{11} \in \mathbb{R}_+$ that appears in G_{22} , and has another \mathcal{U} -pole $p_{21} \in \mathbb{R}_+$ that appears in G_{11} (and possibly various other entries) but not in G_{22} (unless $p_{11} = p_{21}$). In this case, a partially reliable decentralised design that relies on closed-loop stability with only C_2 active and $C_1 = 0$ is not possible because of the instability that is not reflected in G_{22} . In Case 2, G has at most two \mathcal{U} -poles that appear in G_{22} and these poles may appear in various other entries of G . Since all instabilities of G are reflected in G_{22} , a partially reliable decentralised design with $C_1 = 0$ can be achieved in this case.

3.2.1 Case 1

In this case for the finite-dimensional part G of \widehat{G} in (13), we assume

$$Y_{11} = \frac{(s-p_{11})}{a_1s+1}I_{r_1}, \quad Y_{22} = \frac{(s-p_{21})}{a_2s+1}I_{r_2}, \quad Y_{12} = 0, \quad (15)$$

where $p_{11}, p_{21} \geq 0$ are the non-negative real poles of G , $a_j \in \mathbb{R}_+$, $j=1,2$. Let $\text{rank}X_{ij}(p_{j1}) = \text{rank}(s-p_{j1}) \times G(s)|_{s=p_{j1}} = r_j$ for $j=1,2$. Therefore, all entries of G_{jk} , $j, k \in \{1,2\}$ have a pole at p_{j1} . All entries of G have the same pole if $p_{11} = p_{21}$. For PID controller design with non-zero integral constant, also assume that G_{22} has no transmission zeros at $s=0$, i.e. $\text{rank}X_{22}(0) = \text{rank}(s-p_{21})G(s)|_{s=0} = r_2$; this assumption is not

necessary for PD controller design. Since each Y_{jj} is diagonal, the delayed plant \widehat{G} can be written as

$$\widehat{G} = \begin{bmatrix} Y_{11}^{-1} & 0 \\ 0 & Y_{22}^{-1} \end{bmatrix} \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix}. \quad (16)$$

Under certain assumptions on the poles $p_{11}, p_{21} \in \mathbb{R}_+$, there exist decentralised PID controllers for the delayed plant \widehat{G} . A systematic decentralised PID controller synthesis method is developed for this case in Proposition 2.

Proposition 2: Let \widehat{G} be as in (16). For $j=1,2$, let $G_{jj} = Y_{jj}^{-1}X_{jj} \in \mathbf{R}_p^{r_j \times r_j}$, $\text{rank}X_{jj}(p_{j1}) = \text{rank}(s-p_{j1}) \times G_{jj}(s)|_{s=p_{j1}} = r_j$, where $p_{j1} \geq 0$. Let $G_{jk} = Y_{jj}^{-1}X_{jk}$, $k=1,2$. For C_j to be a PD controller, let $M_j=0$. For C_j to be a PID controller (with non-zero integral constant), let $M_j=I$ and let G have no transmission-zeros at $s=0$, i.e. $\text{rank}X(0) = \text{rank}(YG(s))|_{s=0} = r$, and let $\text{rank}X_{22}(0) = \text{rank}(s-p_{21})G_{22}(s)|_{s=0} = r_2$. For the designs of C_1 and C_2 choose any $\widehat{K}_{Dj} \in \mathbb{R}^{r_j \times r_j}$, $\tau_j > 0$, $j = \{1,2\}$.

Step 1: Design C_2 : Define

$$\widehat{C}_2 := X_{22}(0)^{-1} + \frac{s}{\tau_2s+1} \widehat{K}_{D2}, \quad (17)$$

$$\Phi_{\Lambda 2} := \frac{1}{s} \left[(s-p_{21})\widehat{G}_{22}(s)\widehat{C}_2 - I \right]. \quad (18)$$

If $0 \leq p_{21} < \|\Phi_{\Lambda 2}\|^{-1}$, then for any $\alpha_2 \in \mathbb{R}_+$ satisfying (19), the PD controller C_{pd2} in (19) stabilises \widehat{G}_{22} :

$$C_{pd2} = (\alpha_2 + p_{21})\widehat{C}_2, \quad 0 < \alpha_2 < \|\Phi_{\Lambda 2}\|^{-1} - p_{21}. \quad (19)$$

If $\widehat{K}_{D2} = 0$, (19) is a P controller. With C_{pd2} as in (19), let $H_{pd2} := \widehat{G}_{22}(I + C_{pd2}\widehat{G}_{22})^{-1}$, where $H_{pd2}(0)^{-1} = \alpha_2 X_{22}(0)^{-1}$. Then for any $\gamma_2 \in \mathbb{R}_+$ satisfying (20), the PID controller C_2 in (20) stabilises \widehat{G}_{22} :

$$C_2 = C_{pd2} + \frac{\gamma_2 \alpha_2}{s} X_{22}(0)^{-1} M_2, \quad (20)$$

$$0 < \gamma_2 < \left\| \frac{1}{s} [H_{pd2}(s)H_{pd2}(0)^{-1} - I] \right\|^{-1}.$$

Step 2: Design C_1 : Let $\widehat{W} :=: Y_{11}^{-1}\widehat{W}_{11}$ be defined by (5). Define

$$\widehat{C}_1 := \widehat{W}_{11}(0)^{-1} + \frac{s}{\tau_1s+1} \widehat{K}_{D1}, \quad (21)$$

$$\Phi_{\Lambda 1} := \frac{1}{s} \left[(s-p_{11})\widehat{W}(s)\widehat{C}_1 - I \right]. \quad (22)$$

If $0 \leq p_{11} < \|\Phi_{\Lambda 1}\|^{-1}$, then for any $\alpha_1 \in \mathbb{R}_+$ satisfying (23), let C_{pd1} be given by (23):

$$C_{pd1} = (\alpha_1 + p_{11})\widehat{C}_1, \quad 0 < \alpha_1 < \|\Phi_{\Lambda 1}\|^{-1} - p_{11}. \quad (23)$$

If $\widehat{K}_{D1} = 0$, (23) is a P controller. With C_{pd1} as in (23), let $H_{pd1} := \widehat{W}(I + C_{pd1}\widehat{W})^{-1}$, where $H_{pd1}(0)^{-1} = \alpha_1 \widehat{W}_{11}(0)^{-1}$. For any $\gamma_1 \in \mathbb{R}_+$ satisfying (24), let C_1 be as in (24):

$$C_1 = C_{pd1} + \frac{\gamma_1 \alpha_1}{s} \widehat{W}_{11}(0)^{-1} M_1, \quad (24)$$

$$0 < \gamma_1 < \left\| \frac{1}{s} [H_{pd1}(s)H_{pd1}(0)^{-1} - I] \right\|^{-1}.$$

With C_2 , C_1 as in (20), (24), $C_D = \text{diag}[C_1, C_2]$ is a decentralised PID controller for the delayed plant \widehat{G} . For $\widehat{K}_{Dj} = 0$, (20), (24) are P controllers (if $M_j = 0$) or PI controllers (if $M_j = I$); for $\widehat{K}_{Pj} = 0$, (20), (24) are D controllers (if $M_j = 0$) or ID controllers (if $M_j = I$).

In Example 2, we apply the synthesis procedure in Proposition 2 to design decentralised PID controllers for an MIMO distillation column with arbitrary delays in the channels. A full-feedback proportional control design was considered for this system in Gündes et al. (2007) for the special case of $h_1 = h_4$, $h_2 = h_3$ affecting input channels. Here, we choose $\widehat{K}_{Dj} = 0$ and design PI controllers for both channels. The selection of the parameters \widehat{K}_{Dj} obviously affect the response. Since these effects are different for each particular case, we do not fully analyse performance issues but only establish closed-loop stability.

Example 2: Let $\widehat{G} = \begin{bmatrix} \frac{3.04}{s} e^{-h_1 s} & \frac{-278.2e^{-h_2 s}}{s(s+6)(s+30)} \\ \frac{0.052}{s} e^{-h_3 s} & \frac{206.6e^{-h_4 s}}{s(s+6)(s+30)} \end{bmatrix}$, which can be written in the form of (14):

$$\widehat{G} = \begin{bmatrix} \frac{s}{a_1 s + 1} & 0 \\ 0 & \frac{s}{a_1 s + 1} \end{bmatrix}^{-1} \times \begin{bmatrix} \frac{3.04e^{-h_1 s}}{a_1 s + 1} & \frac{-278.2e^{-h_2 s}}{(s+6)(s+30)(a_1 s + 1)} \\ \frac{0.052e^{-h_3 s}}{a_1 s + 1} & \frac{206.6e^{-h_4 s}}{(s+6)(s+30)(a_1 s + 1)} \end{bmatrix},$$

$a_1 > 0$, $Y_{11} = Y_{22}$ and $p_{11} = p_{21} = 0$. Let $h_1 = h_3 = 0.5$, $h_2 = h_4 = 0.6$. Choose $\widehat{K}_{D2} = 0$. With $\widehat{C}_2 = X_{22}(0)^{-1} = 180/206.6$, take $\alpha_2 = 0.5$ satisfying (19). Then take $\gamma_2 = 0.1$ satisfying (20). The PI controller $C_2 = \alpha_2 X_{22}(0)^{-1}(1 + \gamma_2/s) = 0.4356 + 0.04356/s$ stabilises \widehat{G}_{22} . Now choose $\widehat{K}_{D1} = 0$. With $\widehat{C}_1 = \widehat{W}_{11}(0)^{-1} = 1/3.11$, take $\alpha_1 = 1.3$ satisfying (23). Then take $\gamma_1 = 0.15$ satisfying (24). The PI controller $C_1 = \alpha_1 \widehat{W}_{11}(0)^{-1}(1 + \gamma_1/s) = 0.418 + 0.0627/s$ stabilises \widehat{W} . With the decentralised PI controller $C_D = \text{diag}[C_1, C_2]$ stabilising \widehat{G} , Figure 3 shows the closed-loop step responses for the outputs y_1 (dashed), y_2 (solid), with unit-step references applied at both w_1, w_2 .

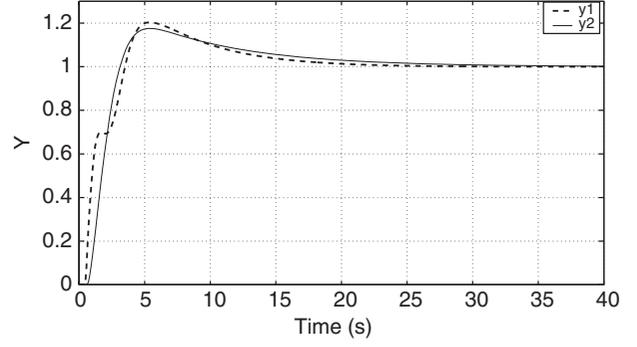


Figure 3. Example 2 step-responses with $C_D = \text{diag}[C_1, C_2]$.

3.2.2 Case 2

In this case for the finite-dimensional part G of \widehat{G} , we assume Y_{12} to be either diagonal or zero, let $Y_{11} = I_{r_1}$. Let

$$d := \prod_{i=1}^{\ell} (a_i s + 1), \quad n := \prod_{i=1}^{\ell} (s - p_{2i}), \quad (25)$$

$Y_{22} = \frac{g}{f} I_{r_2}$, $X_{22} = Y_{22} G_{22}$ and $\ell \in \{1, 2\}$, $a_i \in \mathbb{R}_+$, $i \in \{1, \ell\}$. Let $\text{rank } X_{22}(p_{2i}) = \text{rank } nG(s)|_{s=p_{2i}} = r_j$ for $i \in \{1, \ell\}$. Therefore, G has one or two \mathcal{U} -poles at $p_{2i} \in \mathcal{U}$, and all \mathcal{U} -poles of G appear in G_{22} ; they may also appear in any of the other entries of G . If $\ell = 1$, then $p_{21} \geq 0$; if $\ell = 2$, then the two poles are either real ($p_{21}, p_{22} \geq 0$) or they are a complex-conjugate pair ($p_{21} = \bar{p}_{22} \in \mathcal{U}$). For PID controller design with non-zero integral constant, also assume that G_{22} has no transmission zeros at $s=0$, i.e. $\text{rank } X_{22}(0) = \text{rank } nG_{22}(s)|_{s=0} = r_2$; this assumption is not necessary for PD controller design. Since Y_{22} is diagonal, and Y_{12} is diagonal when it is not zero, the delayed plant \widehat{G} can be written as

$$\widehat{G} = \begin{bmatrix} I & Y_{12} \\ 0 & Y_{22} \end{bmatrix}^{-1} \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{21} & \widehat{X}_{22} \end{bmatrix}. \quad (26)$$

Under certain assumptions on the \mathcal{U} -poles, there exist decentralised PID controllers for the delayed plant \widehat{G} . Furthermore, closed-loop stability can be maintained with $C_1 = 0$. A systematic reliable decentralised PID controller synthesis is developed in Proposition 3, where, for the controller C_2 that stabilises \widehat{G}_{22} , we consider real and complex-conjugate pairs of poles as two separate cases:

Case (a): The two \mathcal{U} -poles are real, i.e. $p_{2i} \in \mathbb{R}$, $p_{2i} \geq 0$, $i = 1, 2$.

Case (b): The two \mathcal{U} -poles are a complex-conjugate pair, i.e. $p_{21} = \bar{p}_{22}$, $n = s^2 - (p_{21} + p_{22})s + p_{21}p_{22} = s^2 - 2fs + g^2$, $f \geq 0$, $g > 0$, $f < g$. In this case, $X_{22}(0) = g^2 G_{22}(0)$.

Proposition 3: Let \widehat{G} be as in (26). With n, d as in (25), $G_{22} = Y_{22}^{-1}X_{22} \in \mathbf{R}_p^{r_2 \times r_2}$, $\text{rank } X_{22}(p_{2i}) = \text{rank } nG(s)|_{s=p_{2i}} = r_2$, $i \in \{1, \ell\}$, $\ell \in \{1, 2\}$. For C_j to be a PD controller, let $M_j = 0$. For C_j to be a PID controller (with $K_I \neq 0$), let $M_j = I$ and let $\text{rank } X(0) = \text{rank}(YG(s))|_{s=0} = r$, $\text{rank } X_{22}(0) = \text{rank } nG_{22}(s)|_{s=0} = r_2$.

Step 1: Design C_2 : If $\ell = 1$, design the PID controller C_2 that stabilises \widehat{G}_{22} as in (20) of Proposition 2. If $\ell = 2$, choose any $\tau_2 > 0$. Define

$$\Psi_{\Lambda 1} := \frac{1}{s} \left[\frac{n}{(\tau_2 s + 1)} \widehat{G}_{22}(s) X_{22}(0)^{-1} - I \right]. \quad (27)$$

Consider two cases: (a) Let $p_{2i} \geq 0$, $i \in \{1, 2\}$. Let $\widehat{F}_2 := (s - p_{22}) \widehat{G}_{22}(s) X_{22}(0)^{-1}$. If $0 \leq p_{21} < \|\Psi_{\Lambda 1}\|^{-1}$, then define $\Psi_{\Lambda 2}$ as in (28) for any $\alpha_1 \in \mathbb{R}_+$ satisfying (29):

$$\Psi_{\Lambda 2} := \frac{1}{s} \left[\alpha_1 \left(I + \frac{(\alpha_1 + p_{21}) \widehat{F}_2}{\tau_2 s + 1} \right)^{-1} \widehat{F}_2 - I \right], \quad (28)$$

$$0 < \alpha_1 < \|\Psi_{\Lambda 1}\|^{-1} - p_{21}. \quad (29)$$

If $0 \leq p_{22} < \|\Psi_{\Lambda 2}\|^{-1}$, then let $K_{P2} = (\alpha_1 \alpha_2 - p_{21} p_{22}) X_{22}(0)^{-1}$, $K_{D2} = (\alpha_1 + p_{21})(1 + \tau_2 p_{22}) X_{22}(0)^{-1}$ for any $\alpha_2 \in \mathbb{R}_+$ satisfying (30):

$$0 < \alpha_2 < \|\Psi_{\Lambda 2}\|^{-1} - p_{22}. \quad (30)$$

Then a PD controller that stabilises \widehat{G}_{22} is given by

$$C_{pd2} = \left[\begin{array}{c} (\alpha_1 \alpha_2 - p_{21} p_{22}) \\ + \frac{(\alpha_1 + p_{21})(1 + p_{22} \tau_2) s}{\tau_2 s + 1} \end{array} \right] X_{22}(0)^{-1}. \quad (31)$$

With C_{pd2} as in (31), let $H_{pd2} := \widehat{G}_{22}(I + C_{pd2} \widehat{G}_{22})^{-1}$, where $H_{pd2}(0)^{-1} = \alpha_1 \alpha_2 X_{22}(0)^{-1}$. Then the PID controller C_2 in (32) stabilises \widehat{G}_{22} for any $\gamma_2 \in \mathbb{R}_+$ satisfying (32):

$$C_2 = C_{pd2} + \frac{\gamma_2 \alpha_1 \alpha_2}{s} X_{22}(0)^{-1} M_2, \quad (32)$$

$$0 < \gamma_2 < \left\| \frac{1}{s} [H_{pd2}(s) H_{pd2}(0)^{-1} - I] \right\|^{-1}.$$

(b) Let $p_{21} = \bar{p}_{22} \in \mathbb{C}$, $n = s^2 - (p_{21} + p_{22})s + p_{21} p_{22} = s^2 - 2fs + g^2$, $f \geq 0$, $g > 0$, $f < g$. If $f + 2g < \|\Psi_{\Lambda 1}\|^{-1}$, then let

$$K_{P2} = [\delta_1 \delta_2 + \delta_1(g - f) + \delta_2 g - fg] X_{22}(0)^{-1},$$

$$K_{D2} = (\delta_1 + \delta_2 + f + 2g) X_{22}(0)^{-1} - \tau_2 K_{P2},$$

for any $\delta_1, \delta_2 \in \{\mathbb{R}_+ \cup 0\}$ satisfying (33):

$$0 \leq \delta_1 + \delta_2 < \|\Psi_{\Lambda 1}\|^{-1} - (f + 2g). \quad (33)$$

Then a PD controller that stabilises \widehat{G}_{22} is given by

$$C_{pd2} = \frac{\left\{ \begin{array}{c} [(\delta_1 + \delta_2 + f + 2g)s + \delta_1 \delta_2] \\ + \delta_1(g - f) + \delta_2 g - fg \end{array} \right\}}{g^2(\tau_2 s + 1)} G_{22}(0)^{-1}. \quad (34)$$

With C_{pd2} , as in (34), let $H_{pd2} := \widehat{G}_{22}(I + C_{pd2} \widehat{G}_{22})^{-1}$, where $H_{pd2}(0)^{-1} = (\delta_1 + g)(\delta_2 + g - f) X_{22}(0)^{-1}$ and $X_{22}(0) = g^2 G_{22}(0)$. Then the PID controller C_2 in (35) stabilises \widehat{G}_{22} for any $\gamma_2 \in \mathbb{R}_+$ satisfying (32):

$$C_2 = C_{pd2} + \frac{\gamma_2(\delta_1 + g)(\delta_2 + g - f)}{s} X_{22}(0)^{-1} M_2,$$

$$0 < \gamma_2 < \left\| \frac{1}{s} [H_{pd2}(s) H_{pd2}(0)^{-1} - I] \right\|^{-1}. \quad (35)$$

Step 2: Design C_1 : If $\ell = 1$, let C_2 be as in (20). If $\ell = 2$, let C_2 be as in (32) or (35) when $p_{2i} \in \mathbb{R}_+$ or $p_{2i} \in \mathcal{U} \setminus \mathbb{R}_+$, respectively. Let \widehat{W} be defined by (5). Choose any $\widehat{K}_{P1}, \widehat{K}_{D1} \in \mathbb{R}^{r_1 \times r_1}$, $\tau_1 > 0$. Define

$$\widehat{C}_1 := \widehat{K}_{P1} + \frac{s}{\tau_1 s + 1} \widehat{K}_{D1} + \frac{1}{s} \widehat{W}(0)^{-1} M_1. \quad (36)$$

For $\beta_1 \in \mathbb{R}_+$ satisfying (37), let C_1 be as in (37):

$$C_1 = \beta_1 \widehat{C}_1, \quad 0 < \beta_1 < \left\| \frac{1}{s} [s \widehat{W}(s) \widehat{C}_1 - M_1] \right\|^{-1}. \quad (37)$$

With C_1 as in (37), $C_D = \text{diag}[C_1, C_2]$ is a partially reliable decentralised PID controller for the delayed plant \widehat{G} .

In Example 3, we apply the synthesis procedure in Proposition 3 to design decentralised PID controllers for the delayed version of a chemical reactor model adopted from El-Farra, Mhaskar, and Christofides (2004), where the concentration of the inlet reactant and the rate of heat input are manipulated to regulate the outlet reactant concentration and the reactor temperature. The linearisation around one of the operating points gives an unstable plant transfer-matrix G , which is the finite-dimensional part of \widehat{G} .

Example 3: Let $\widehat{G} = \frac{1}{d(s)} \begin{bmatrix} \frac{10}{6} s - \frac{1}{8} & -\frac{1}{512} e^{-h_2 s} \\ 4e^{-h_1 s} & (4s + \frac{1}{8}) e^{-h_2 s} \end{bmatrix}$, with $d(s) = 100(s - \frac{1}{16})(s + \frac{3}{160})$, $h_1 = 0.25$ s and $h_2 = 0.5$ s. Note that G has poles at $p_{21} = 1/16 \in \mathcal{U}$ and $p = -3/160 \notin \mathcal{U}$; i.e. $\ell = 1$. Since the only \mathcal{U} -pole $p_{21} \in \mathcal{U}$ is reflected in G_{22} , the transfer-matrix \widehat{G} can be written in the form of (26):

$$\widehat{G} = \begin{bmatrix} 1 & \frac{1}{96} \\ 0 & \frac{(s - \frac{1}{16})}{(a_1 s + 1)} \end{bmatrix}^{-1} \begin{bmatrix} \frac{10}{6} e^{-0.25s} & \frac{1}{24} e^{-0.5s} \\ \frac{4}{(a_1 s + 1)} e^{-0.25s} & \frac{4s + \frac{1}{8}}{(a_1 s + 1)} e^{-0.5s} \end{bmatrix}$$

$$\times \frac{1}{100(s + \frac{3}{160})}$$

$a_1 > 0$, $Y_{11} = 1$, $Y_{22} = \frac{s-p_{21}}{a_1 s+1}$. First we design $C_{pd2} = (\alpha_2 + p_{21})X_{22}^{-1}(0)(1 + \tilde{K}_{d2}\frac{s}{\tau_2 s+1})$, where $X_{22}^{-1}(0) = 15$ and the parameters $\tilde{K}_{d2} = X_{22}^{-1}(0)\tilde{K}_{d2}$, τ_2 are optimised so that the allowable range of the gain $(\alpha_2 + p_{21})$, determined by (19), is maximised. We find that optimal choices are $\tilde{K}_{d2} = 22$ and $\tau_2 = 32$ give rise to $0 < (\alpha_2 + p_{21}) < 2$. We select $(\alpha_2 + p_{21}) = 1$; hence $\alpha_2 = 1 - \frac{1}{16}$, and $C_{pd2}(s) = 15(1 + \frac{22s}{32s+1})$. With this choice of C_{pd2} , the allowable range for the integral action gain is computed from (20) as $0 < \gamma_2 < \frac{1}{23}$. We choose $C_2(s) = 15(1 + \frac{1}{30s} + \frac{22s}{32s+1})$. In the second step we design a PI controller $C_1(s)$ in the form $C_1(s) = \beta_1 \tilde{W}(0)^{-1}(\tilde{K}_{p1} + \frac{1}{s})$, where $\tilde{W}(0)^{-1} = \frac{15}{8}$ and \tilde{K}_{p1} is optimised to maximise the allowable range for β_1 . We find that optimal choice $\tilde{K}_{p1} = \frac{100}{3}$ gives $0 < \beta_1 < 1.1631$. We choose $C_1(s) = 1.5(\frac{100}{3} + \frac{1}{s})$. Clearly, the controller parameters can further be optimised in the ranges specified above.

Figure 4(a) shows the step responses for the outputs y_1 , y_2 , with unit-steps applied at both w_1 , w_2 and both channels of $C_D = \text{diag}[C_1, C_2]$ are active. Figure 4(b) shows the step responses when C_1 fails, i.e. $C_D = \text{diag}[0, C_2]$ with only the second

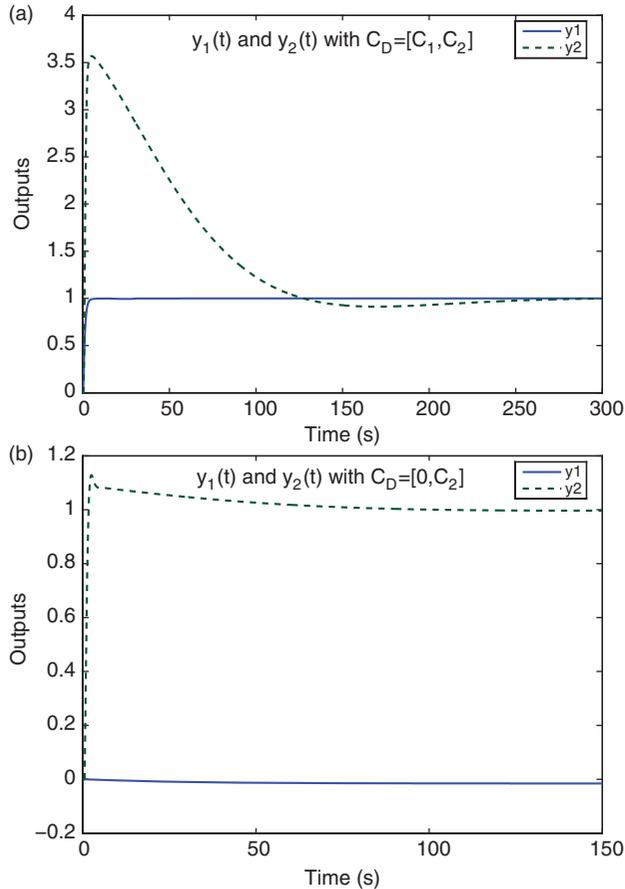


Figure 4. Example 3 step-responses with (a) $C_D = \text{diag}[C_1, C_2]$, (b) $C_D = \text{diag}[0, C_2]$.

channel operational. The partially reliable design guarantees closed-loop stability when $C_1 = 0$ and asymptotic tracking with zero steady-state error is achieved since integral action is present in the second channel.

4. Conclusions

We derived reliable PID controllers for LTI plants with two decentralised MIMO channels subject to delays. For stable plants, the decentralised controllers are designed to be partially or fully reliable to provide closed-loop stability even when either one of the controllers is set to zero. For plants with only one or two unstable poles (with no restriction on the number of stable poles) we presented systematic methods to define the PID controller parameters explicitly. Reliable stabilisation is also achieved for unstable plants if the main channel that always remains operational contains all plant poles that are in the unstable region. Our systematic synthesis method explicitly defines the PID parameters for reliable closed-loop stabilisation but we do not explore how specific choices of these free parameters affect the performance since this is an issue to study for particular applications rather than the general case. Since the parameters are chosen based on sufficient conditions for stability, this introduces a certain amount of conservativeness. Considering the difficulty of the problem due to restrictions imposed by the decentralised structure, order limitations of PID controllers and the presence of arbitrary delay terms in the plant's transfer-matrix entries, conservative results for performance considerations are to be expected while there is freedom in the choice of parameters for stability.

Plants whose finite-dimensional part has more than two poles in the unstable region do not necessarily admit PID controllers even if they are strongly stabilisable. This is true even for plants with no delays. Further assumptions are needed on such plants, which would impose restrictions on the plant's transmission-zeros. The reliable decentralised PID synthesis methods presented here may be extended to delayed plants with more than two MIMO channels. Performance implications for choices within the stabilising parameters can also be explored for specific applications of the synthesis methods presented here.

References

- Aström, K.J., Johansson, K.H., and Wang, Q.-G. (2002), 'Design of decoupled PI controllers for two-by-two systems', *IEE Proceedings on Control Theory Applications*, 149, 74–81.
- Bequette, B.W. (2003), *Process Control Modeling, Design and Simulation*, Englewood Cliff, NJ: Prentice Hall.

El-Farra, N.H., Mhaskar, P., and Christofides, P.D. (2004), ‘Hybrid predicative control of nonlinear systems: method and applications to chemical processes’, *International Journal of Robust Nonlinear Control*, 14, 199–225.

Foias, C., Özbay, H., and Tannenbaum, A. (1996), *Robust Control of Infinite Dimensional Systems*, LNCIS (Vol. 209), London: Springer-Verlag.

Goodwin, G.C., Graebe, S.F., and Salgado, M.E. (2001), *Control System Design*, Englewood Cliff, NJ: Prentice Hall.

Gu, K., Kharitonov, V.L., and Chen, J. (2003), *Stability of Time-Delay Systems*, Boston: Birkhäuser.

Gündeş, A.N., and Özgüler, A.B. (2002), ‘Two-channel decentralised integral-action controller design’, *IEEE Transactions on Automatic Control*, 47, 2084–2088.

Gündeş, A.N., and Özgüler, A.B. (2007), ‘PID stabilisation of MIMO plants’, *IEEE Transactions on Automatic Control*, 52, 1502–1508.

Gündeş, A.N., Özbay, H., and Özgüler, A.B. (2007), ‘PID controller synthesis for a class of unstable MIMO plants with I/O delays’, *Automatica*, 43, 135–142.

Gündeş, A.N., Mete, A.N., and Palazoğlu, A. (2009), ‘Reliable decentralised PID controller synthesis for MIMO processes’, *Automatica*, 45, 353–363.

Niculescu, S.-I. (2001), *Delay Effects on Stability: A Robust Control Approach*, LNCIS (Vol. 269), Heidelberg: Springer-Verlag.

Özbay, H., and Gündeş, A.N. (2007), ‘Resilient PI and PD controller designs for a class of unstable plants with I/O delays’, *Applied and Computational Mathematics*, 6, 18–26, Online version: http://www.elm.az/acm/v_6_n_1_2007/18-26.pdf

Silva, G.J., Datta, A., and Bhattacharyya, S.P. (2005), *PID Controllers for Time-delay Systems*, Boston: Birkhäuser.

Skogestad, S. (2003), ‘Simple analytic rules for model reduction and PID controller tuning’, *Journal of Process Control*, 13, 291–309.

Tavakoli, S., Griffin, I., and Fleming, P.J. (2006), ‘Tuning of decentralised PI (PID) controllers for TITO processes’, *Control Engineering Practice*, 14, 1069–1080.

Appendix

Proof of Proposition 1: (a) The decentralised PID controller $C_D = ND^{-1}$, where $D = \text{diag}[D_1, D_2]$, with $D_j = I - \frac{\beta_j}{s+\beta_j} M_j$, $N_j = C_j D_j$, $\beta_j \in \mathbb{R}_+$, $j = 1, 2$, stabilises $\widehat{G} \in \mathcal{M}(\mathcal{H}_\infty)$ if and only if $U_D := D + \widehat{G}N$ is unimodular. Similarly, C_2 stabilises \widehat{G}_{22} if and only if U_2 is unimodular, where

$$\begin{aligned} U_2 &:= D_2 + \widehat{G}_{22}N_2 = D_2 + \widehat{G}_{22}C_2D_2 \\ &= I + \beta_2 \left[\widehat{G}_{22} \left(\widehat{K}_{P2} + \frac{s}{\tau_2 s + 1} \widehat{K}_{D2} \right) D_2 \right. \\ &\quad \left. + \frac{s}{s + \beta_2} \frac{(\widehat{G}_{22}G_{22}(0)^{-1} - I)}{s} M_2 \right]. \end{aligned}$$

Note that U_2 is unimodular if (7) is satisfied. When $M_2 = 0$, $U_2 = I + \beta_2 \widehat{G}_{22}(\widehat{K}_{P2} + \frac{s}{\tau_2 s + 1} \widehat{K}_{D2})$ is unimodular if (7) holds. Hence, C_2 in (7) stabilises \widehat{G}_{22} and $C_2(I + \widehat{G}_{22}C_2)^{-1} \in$

$\mathcal{M}(\mathcal{H}_\infty)$ implies $\widehat{W} \in \mathcal{M}(\mathcal{H}_\infty)$; $C_2(I + \widehat{G}_{22}C_2)^{-1}(0) = G_{22}(0)^{-1}$ implies $\widehat{W}(0) = G_{11}(0) - G_{12}(0)G_{22}(0)^{-1}G_{21}(0)$. By (9), $U_w := D_1 + \widehat{W}C_1D_1$ is unimodular; hence, C_1 stabilises \widehat{W} . Therefore, $\text{Sys}(\widehat{G}, C_D)$ is stable and C_D is partially reliable since $\text{diag}[0, C_2]$ also stabilises \widehat{G} . (b) By assumption, $\Theta := G_{11}(0)\widehat{W}(0)^{-1}$ has positive real eigenvalues implies $\|sI(sI + \beta_1\Theta)^{-1}\| = 1$ for $\beta_1 > 0$. Define $\tilde{D}_1 = I - \beta_1\Theta(sI + \beta_1\Theta)^{-1}M_1$, $\tilde{N}_1 = C_1\tilde{D}_1$. Then U_1 is unimodular, where

$$\begin{aligned} U_1 &:= \tilde{D}_1 + \widehat{G}_{11}\tilde{N}_1 = I + \beta_1 \left[\widehat{G}_{11} \left(\widehat{K}_{P1} + \frac{s\widehat{K}_{D1}}{\tau_1 s + 1} \right) \tilde{D}_1 \right. \\ &\quad \left. + \frac{(\widehat{G}_{11}\widehat{W}(0)^{-1} - \Theta)}{s} sI(sI + \beta_1\Theta)^{-1} M_1 \right]. \end{aligned}$$

Hence, C_1 stabilises \widehat{G}_{11} and C_D is fully reliable since $\text{diag}[C_1, 0]$ also stabilises \widehat{G} . \square

Proof of Proposition 2: (i) By (14), the decentralised PID controller $C_D = ND^{-1}$ (as in the proof of Proposition 1) stabilises \widehat{G} if and only if $U_D := YD + \widehat{X}N$ is unimodular. The PD controller C_{pd2} stabilises \widehat{G}_{22} if and only if U_{pd} is unimodular, where

$$\begin{aligned} U_{pd} &:= Y_{22} + \widehat{X}_{22}C_{pd2} = \frac{(s - p_{21})}{a_2 s + 1} [I + \widehat{G}_{22}C_{pd2}] \\ &= \frac{(s - p_{21})}{a_2 s + 1} [I + (\alpha_2 + p_{21})\widehat{G}_{22}C_{pd2}] \\ &= \left[I + \frac{(\alpha_2 + p_{21})s}{s + \alpha_2} \Phi_{\Lambda 2} \right] \frac{(s + \alpha_2)}{(a_2 s + 1)}. \end{aligned}$$

A sufficient condition for U_{pd} to be unimodular is that (19) holds. Hence, C_{pd2} in (19) stabilises \widehat{G}_{22} and $H_{pd2} := U_{pd}^{-1}\widehat{X}_{22} = \widehat{G}_{22}(I + C_{pd2}\widehat{G}_{22})^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, where $H_{pd2}(0)^{-1} = G_{22}^{-1}(0) + K_{p2} = X_{22}(0)^{-1}Y_{22}(0) + (\alpha_2 + p_{21})X_{22}(0)^{-1} = \alpha_2 X_{22}(0)^{-1}$. Using similar steps as in the proof of Proposition 1, the I-controller $K_{I2}/s = \gamma_2 H_{pd2}(0)^{-1}/s$ stabilises H_{pd2} for any $\gamma_2 \in \mathbb{R}$ satisfying (20). Therefore, $C_2 = C_{pd2} + K_{I2}/s$ in (20) stabilises \widehat{G}_{22} and $C_2(I + \widehat{G}_{22}C_2)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. Now since $U_2 := \frac{s}{s+\gamma_2} Y_{22} + \widehat{X}_{22} \frac{s}{s+\gamma_2} C_2$ is unimodular, U_D is unimodular if and only if $Y_{11}D_1 + \widehat{W}_{11}N_1$ is unimodular, equivalently, C_1 stabilises \widehat{W} . Using a C_1 design for \widehat{W} similar to C_2 for \widehat{G}_{22} , it follows that C_D stabilises \widehat{G} . \square

Proof of Proposition 3: (a) Let $p_{2i} \in \mathbb{R}_+$. If $\ell = 2$, let

$$\begin{aligned} V_1 &:= \frac{(s - p_{11})}{a_1 s + 1} I + (\alpha_1 + p_{11}) \frac{(a_2 s + 1)}{\tau_2 s + 1} \widehat{X}_{22} X_{22}(0)^{-1} \\ &= \frac{(s - p_{11})}{a_1 s + 1} \left[I + \frac{(\alpha_1 + p_{11})}{\tau_2 s + 1} \widehat{F}_2 \right] \\ &= \left[I + \frac{(\alpha_1 + p_{11})s}{s + \alpha_1} \Psi_{\Lambda 1} \right] \frac{(s + \alpha_1)}{a_1 s + 1}. \end{aligned}$$

If (29) holds then V_1 is unimodular. By (31), $C_{pd2} = (\alpha_1 + p_{11}) \frac{(s - p_{21})}{\tau_2 s + 1} X_{22}(0)^{-1} + \alpha_1 (\alpha_2 + p_{21}) X_{22}(0)^{-1}$. Define V_{pd} as

$$\begin{aligned} V_{pd} &:= Y_{22} + \widehat{X}_{22}C_{pd} = \frac{(s - p_{21})}{a_2 s + 1} \left[\frac{(s - p_{11})}{a_1 s + 1} I \right. \\ &\quad \left. + (\alpha_1 + p_{11}) \frac{(a_2 s + 1)}{\tau_2 s + 1} \widehat{X}_{22}(s) X_{22}(0)^{-1} \right] \end{aligned}$$

$$+ \alpha_1(\alpha_2 + p_{21})\widehat{X}_{22}(s)X_{22}(0)^{-1} \\ = V_1 \left[\frac{(s-p_{21})}{a_2s+1} I + \alpha_1(\alpha_2 + p_{21})V_1^{-1}\widehat{X}_{22}(s)X_{22}(0)^{-1} \right] =: V_1 V_2.$$

Since V_1 is unimodular, V_{pd} is unimodular if and only if V_2 is unimodular, where

$$V_2 = \frac{(s-p_{21})}{a_2s+1} \left[I + \frac{(a_2s+1)}{s-p_{21}} \alpha_1(\alpha_2 + p_{21})V_1^{-1}\widehat{X}_{22}(s)X_{22}(0)^{-1} \right] \\ = \frac{(s-p_{21})}{a_2s+1} \left[I + \alpha_1(\alpha_2 + p_{21}) \left(I + \frac{(a_1s+1)}{\tau_2s+1} \widehat{F}_2 \right)^{-1} \right. \\ \left. \times \widehat{G}_{22}(s)X_{22}(0)^{-1} \right] = \left[I + \frac{(\alpha_2 + p_{21})}{s + \alpha_2} (\alpha_1 V_1^{-1} \widehat{X}_{22}(s)X_{22}(0)^{-1})^{-1} \right. \\ \left. \times (a_2s+1) - I \right] \frac{(s+\alpha_2)}{a_2s+1} = \left[I + \frac{(\alpha_2 + p_{21})s}{s + \alpha_2} \Psi_{\Lambda 2} \right] \frac{(s+\alpha_2)}{a_2s+1}.$$

If (30) holds then V_2 is unimodular. Hence, C_{pd2} in (31) stabilises \widehat{G}_{22} and $H_{pd2} := V_{pd}^{-1}\widehat{X}_{22} = \widehat{G}_{22}(I + C_{pd2}\widehat{G}_{22})^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, where $H_{pd2}(0)^{-1} = G_{22}^{-1}(0) + K_{P2} = X_{22}(0)^{-1} \times Y_{22}(0) + (\alpha_1\alpha_2 - p_{11}p_{21})X_{22}(0)^{-1} = \alpha_1\alpha_2 X_{22}(0)^{-1}$. Using similar steps as in the proof of Proposition 1, the I-controller $K_{I2}/s = \gamma_2 H_{pd2}(0)^{-1}/s$ stabilises H_{pd2} for any $\gamma_2 \in \mathbb{R}$ satisfying (20). Therefore, $C_2 = C_{pd2} + K_{I2}/s$ in (32) stabilises \widehat{G}_{22} .

(b) Let $p_{2i} \in \mathcal{U} \setminus \mathbb{R}_+$. Define $y := (s + \delta_1 + g)(s + \delta_2 + g - f)$, where $g - f > 0$ by assumption. Let $x := y - n = (\delta_1 + \delta_2 + f + 2g)s + \delta_1\delta_2 + \delta_1(g - f) + \delta_2g - fg$. Then $\|\frac{xy}{y}\| \leq (\delta_1 + \delta_2 + f + 2g)$, where $\frac{p_{11} + p_{21}}{2} + 2\sqrt{p_{11}p_{21}} = f + 2g$. If (33) holds,

then $\|\frac{xy}{y}\Psi_{\Lambda 1}\| \leq (\delta_1 + \delta_2 + f + 2g)\|\Psi_{\Lambda 1}\| < 1$ implies V_{pd} is unimodular, where

$$V_{pd} := Y_{22} + \widehat{X}_{22}C_{pd2} = \frac{n}{d} [I + \widehat{G}_{22}C_{pd2}] = \frac{y}{d} \left[I + \frac{x}{y} \Psi_{\Lambda 1} \right].$$

Hence, C_{pd2} in (34) stabilises \widehat{G}_{22} and $H_{pd2} := V_{pd}^{-1}\widehat{X}_{22} = \widehat{G}_{22}(I + C_{pd2}\widehat{G}_{22})^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$, where $H_{pd2}(0)^{-1} = G_{22}^{-1}(0) + K_{P2} = (g^2I + K_{P2})X_{22}(0)^{-1}$. For any $\gamma_2 \in \mathbb{R}$ satisfying (20), the I-controller $K_{I2}/s = H_{pd2}(0)^{-1}\gamma_2/s$ stabilises H_{pd2} . Therefore, $C_2 = C_{pd2} + K_{I2}/s$ in (35) stabilises \widehat{G}_{22} . Now we prove C_1 guarantees stability of the overall system, with C_2 as in (32) when $p_{2i} \in \mathbb{R}_+$ and as in (35) when $p_{2i} \notin \mathbb{R}_+$; hence $C_2(I + \widehat{G}_{22}C_2)^{-1} \in \mathcal{M}(\mathcal{H}_\infty)$. Write $C_2 = (N_2U_2^{-1} + D_2U_2^{-1})^{-1}$, with U_2 unimodular, where

$$U_2 := \frac{s}{s + \gamma_2} Y_{22} + \widehat{X}_{22} \frac{s}{s + \gamma_2} C_2.$$

Then $\widehat{W} = \widehat{X}_{11} - (Y_{12}D_2 + \widehat{X}_{12}N_2)U_2^{-1}\widehat{X}_{21} \in \mathcal{M}(\mathcal{H}_\infty)$. Therefore, $U_D := \frac{s}{s+\beta} Y + \widehat{X} \frac{s}{s+\beta} C_D$ is unimodular for any $\beta \in \mathbb{R}_+$ if and only if $\frac{s}{s+\beta} I_{r1} + \frac{s}{s+\beta} \widehat{W} C_1$ is unimodular for $b > 0$, equivalently, C_1 stabilises \widehat{W} . Designing C_1 for \widehat{W} as in Proposition 1, C_D stabilises \widehat{G} . Furthermore, C_D is partially reliable because $U_D = \begin{bmatrix} I_{r1} & Y_{12}D_2 + \widehat{X}_{12}N_2 \\ 0 & U_2 \end{bmatrix}$ is unimodular when $C_1 = 0$. \square