Pairing and Vortex Lattices for Interacting Fermions in Optical Lattices with a Large Magnetic Field

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(Received 11 January 2010; revised manuscript received 14 March 2010; published 7 April 2010)

We study the structure of a pairing order parameter for spin-$1/2$ fermions with attractive interactions in a square lattice under a uniform magnetic field. Because the magnetic translation symmetry gives a unique degeneracy in the single-particle spectrum, the pair wave function has both zero and finite-momentum components coexisting, and their relative phases are determined by a self-consistent mean-field theory. We present a microscopic calculation that can determine the vortex lattice structure in the superfluid phase for different flux densities. Phase transition from a Hofstadter insulator to a superfluid phase is also discussed.

DOI: 10.1103/PhysRevLett.104.145301 PACS numbers: 67.85.−d, 03.75.Ss

Optical lattices and synthetic magnetic fields are two of major tools to create strongly interacting many-body systems in cold atoms [1,2]. In conventional solid state materials, the accessible magnetic flux per unit cell n_B is very small, $n_B \ll 1$, even for the strongest magnetic field attainable in the laboratory ($\lesssim 45T$). Hence, as in most conventional metals, the electron density n is several orders larger than n_B so that the magnetic field can be treated semiclassically, or, as in the two-dimensional electron gases, $n \sim n_B \ll 1$, the density is so low that only the bottom of an electron band is populated, and the effective mass approximation is sufficient to account for the lattice effect. In cold-atom systems, because the magnetic field is synthetically generated by rotation [3] or by engineering atom-light interactions [4,5], and the lattice spacing is of the order of half a micron, one can access the regime $n \sim n_B \sim 1$, where both the lattice and the magnetic field should be treated on an equal footing and in a quantum-mechanical manner. Consequently, such a system exhibits the famous Hofstadter butterfly single-particle spectrum [6].

For neutral atoms in lattices, the interaction is dominated by on-site interactions as in the Hubbard model. Hereafter, we shall refer to the model describing interacting cold atoms in optical lattices with large magnetic field as the Hofstadter-Hubbard (HH) model. Recently, many works have focused on the bosonic HH model [7], which reveal a number of interesting phenomena, including vortex lattice states and possible fractional quantum Hall states. However, so far little attention has been paid to the fermionic HH model.

The subject of this Letter is the properties of the paired superfluid phase in the fermionic HH model with attractive interactions. For $n_B \sim 1$, the pairing problem differs from type-II superconductors in a fundamental way. In type-II superconductors the separation between the vortices is much larger than the size of Cooper pairs; hence, one can locally apply the BCS scenario to define a local order parameter $\Delta(\mathbf{r})$, and understand the vortex lattice by coupling this "coarse grained" order parameter to the magnetic field. In the HH model considered here, magnetic field modifies the single-particle dispersion in an important way. Despite a strong magnetic field, there is always a well-defined Fermi surface and Bloch states in the magnetic Brillouin zone (MBZ) in the Hofstadter model. Therefore, with attractive interaction, BCS pairing always occurs as an instability of the Fermi liquid. This enables us to reach the regime where the pair size is comparable to the distance between vortices; hence, any discussion of pairing must include the effect of the magnetic field at the microscopic level. We shall show that such a microscopic theory requires the definition of an order parameter with multiple components, and will discuss how this order parameter naturally describes the configuration of vortices. The main points of our analysis are highlighted as follows.

(1) We first review that for $n_B = p/q$, where p and q are coprime integers, each single-particle state in the Hofstadter spectrum is q-fold degenerate due to magnetic translation symmetry [8,9]. This degeneracy enforces that a comprehensive formulation of BCS theory in this case must contain Cooper pairs with both zero and a set of finite momenta, and treat them on an equal footing.

(2) We show that the magnetic translation symmetry also imposes relations between pairing order parameters of different momentum. These relations are verified numerically by self-consistently solving the BCS mean-field Hamiltonian.

(3) The relative phases between different pairing order parameters determined from self-consistent solutions can also be understood from a more intuitive and simpler Ginzburg-Landau argument.

(4) We determine the structure of vortices in the superfluid ground state using the information from (2). The unit cell of the superfluid phase is enlarged to $q \times q$, whose symmetry is lower than that of the original Hamiltonian. Hence, the superfluid ground state has discrete degeneracy, related to the symmetry of the vortex lattice.

(5) For certain fermion densities, a critical interaction strength is predicted for a quantum phase transition from a Hofstadter insulator to a superfluid phase.

The Model.—We consider a two-component Fermi gas in a two-dimensional optical lattice potential so that an s-band tight binding model accurately describes the dynamics. Both components are coupled to the same gauge field $\vec{A} = (0, px/q)$ in the Landau gauge. Note that there is no Zeeman shift associated with a synthetic magnetic field. The single-particle Hamiltonian is given by

$$H_0 = -t \sum_{\langle ii' \rangle \sigma} (e^{i2\pi A_{ii'}} c^\dagger_{i\sigma} c_{i'\sigma} + \text{H.c.}), \tag{1}$$

where $i = (i_x, i_y)$ labels the lattice sites, and $\langle ii' \rangle$ represents all the nearest neighboring bonds. In the Landau gauge, $A_{ii'} = pi_x/q$ if $i - i'$ is along the y direction and $A_{ii'} = 0$ if $i - i'$ is along the x direction. Let $T_{\hat{x}}$ and $T_{\hat{y}}$ be magnetic translations of one lattice spacing along x and y direction (see Refs. [8,9] for definition). The Hamiltonian H_0 commutes with both $T_{\hat{x}}$ and $T_{\hat{y}}$, but these two operators do not commute $T_{\hat{x}} T_{\hat{y}} = \exp\{i2\pi p/q\} T_{\hat{y}} T_{\hat{x}}$. One can choose the set of commuting operators as H_0, $T_{\hat{y}}$, and $(T_{\hat{x}})^q \equiv T_{q\hat{x}}$, in effect, enlarging the unit cell in real space to contain q sites in the x direction. Thus, MBZ becomes $k_x \subset [-\pi/q, \pi/q)$ and $k_y \subset [-\pi, \pi)$. Denoting the common eigenstate by ψ_{n,k_x,k_y} and the eigenenergy by $\epsilon_{nk_xk_y}$, the magnetic Bloch theorem yields $T_{q\hat{x}} \psi_{n,k_x,k_y} = \exp\{iqk_x\} \psi_{n,k_x,k_y}$, $T_{\hat{y}} \psi_{n,k_x,k_y} = \exp\{ik_y\} \psi_{n,k_x,k_y}$, where n is the band index. Thus, for $T_{l\hat{x}}$, $l = 1, \ldots, q-1$, $T_{l\hat{x}} \psi_{n,k_x,k_y}$ is a degenerate eigenstate of ψ_{n,k_x,k_y}, and since $T_{\hat{y}}(T_{l\hat{x}} \psi_{n,k_x,k_y}) = \exp\{i(k_y + 2\pi lp/q)\}(T_{l\hat{x}} \psi_{n,k_x,k_y})$, we have the following properties [8,9]

$$\begin{aligned} \psi_{n,k_x,k_y}(x + l\lambda, y) &\propto \psi_{n,k_x,k_y+2\pi lp/q}(x, y), \\ \epsilon_{n,k_x,k_y} &= \epsilon_{n,k_x,k_y+2\pi lp/q}. \end{aligned} \tag{2}$$

For $p/q = 1/3$, the spectrum and Fermi surface shown in Figs. 1(a) and 1(b) clearly display a threefold degeneracy.

In addition to H_0, we consider the on-site interaction between different spin components $H_{\text{int}} = U \sum_i c^\dagger_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} c_{i\uparrow}$. The Hamiltonian for the HH model discussed below is then given by $H_{\text{HH}} = H_0 + H_{\text{int}}$.

Generalized BCS Theory.—We start by diagonalizing the noninteracting Hamiltonian H_0. First we relabel the sites to reflect the enlargement of the unit cell. For a site $i = (i_x, i_y)$, we let $i_x = j_x q + \beta$, where j_x is an integer labeling the magnetic unit cell and $\beta = 0, \ldots, q-1$ denotes the q-inequivalent sites within each cell. So, magnetic unit cells are uniquely labeled by j_x and $j_y = i_y$. With this notation we identify $c_{j\beta\sigma} = c_{i\sigma}$. Fourier transformation in the variable j yields $c_{\mathbf{k}\beta\sigma}$, where $\mathbf{k}$ is limited inside the MBZ. We now define a new set of operators $d_{\mathbf{k}n\sigma}$ through $c_{\mathbf{k}\beta\sigma} = \sum_n g^n_\beta(\mathbf{k}) d_{\mathbf{k}n\sigma}$, under which H_0 becomes diagonalized as $H_0 = \sum_{n\mathbf{k}\sigma} \epsilon_{n\mathbf{k}} d^\dagger_{\mathbf{k}n\sigma} d_{\mathbf{k}n\sigma}$. Note that diagonalization of H_0 is equivalent to solving Harper's equation [6], and $g^n_\beta(\mathbf{k})$ is the βth component of the nth eigenvector of Harper's equation at wave vector $\mathbf{k}$ [6]. $d^\dagger_{\mathbf{k}n}$ is the operator that creates a particle in the nth magnetic subband at wave vector $\mathbf{k}$. As an example, $\epsilon_{n\mathbf{k}}$ is plotted in Fig. 1(a) for $p/q = 1/3$. In terms of $d_{\mathbf{k}n\sigma}$, H_{int} becomes

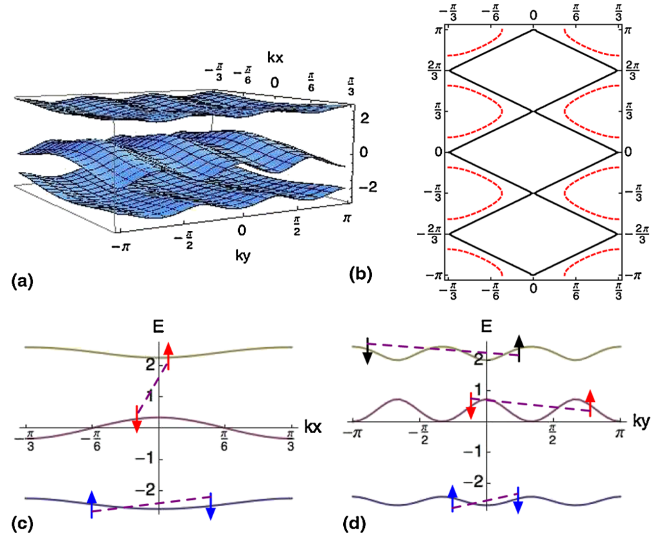

FIG. 1 (color online). (a) Three magnetic bands for $p/q = 1/3$. (b) The Fermi surface of a half-filled (black solid line) and slightly away from half-filled (red dashed line) system. (c) and (d) the band dispersion along k_x direction with $k_y = 0$ (c) and along k_y direction with $k_x = 0$ (d). Various possible pairings included in our BCS theory are also illustrated in (c) and (d), which include intra- and interband pairing (c), and pairing with nonzero center-of-mass momentum (d).

$$\begin{aligned} H_{\text{int}} = U \sum_\beta \sum_{\mathbf{Q}} \sum_{\mathbf{k},\mathbf{k}'} \sum_{n_1,\ldots,n_4} & g^{n_1 *}_\beta\left(\frac{\mathbf{Q}}{2} + \mathbf{k}\right) g^{n_2 *}_\beta\left(\frac{\mathbf{Q}}{2} - \mathbf{k}\right) \\ \times & g^{n_3}_\beta\left(\frac{\mathbf{Q}}{2} - \mathbf{k}'\right) g^{n_4}_\beta\left(\frac{\mathbf{Q}}{2} + \mathbf{k}'\right) d^\dagger_{(\mathbf{Q}/2)+\mathbf{k},n_1,\uparrow} d^\dagger_{(\mathbf{Q}/2)-\mathbf{k},n_2,\downarrow} \\ \times & d_{(\mathbf{Q}/2)-\mathbf{k}',n_3,\downarrow} d_{(\mathbf{Q}/2)+\mathbf{k}',n_4,\uparrow}, \end{aligned} \tag{3}$$

where the momentum sum is restricted to the MBZ. We should focus on the "on-shell" Cooper processes. Importantly, due to the q-fold degeneracy, we not only consider $\mathbf{Q} = 0$ terms in Eq. (3), but also need to consider all the terms with $\mathbf{Q} = (0, 2\pi lp/q)$, where $l = 0, \ldots, q-1$, since $-\mathbf{k}$ and $\mathbf{k} + \mathbf{Q}$ also have the same kinetic energy. Consequently, nonzero center-of-mass momentum pairing needs to be included as well. Besides, intraband Cooper pairs have a nonvanishing coupling to the interband Cooper pairs. For instance, in Eq. (3), if $n_1 = n_2$ but $n_3 \neq n_4$, the interaction coefficient is nonzero. Hence, intraband pairing must induce interband pairing. All of these pairing scenarios under consideration are schematically illustrated in Figs. 1(c) and 1(d), and a comprehensive BCS theory in this problem must treat all of these possibilities on an equal footing. Therefore, we introduce totally q^2 order parameters $\vec{\Delta}^l = (\Delta_0^l, \ldots, \Delta_{q-1}^l)$ given by

$$\Delta_\beta^l = -U \sum_{n,n',\mathbf{k}} g_\beta^n\left(\mathbf{k} + \frac{\mathbf{Q}}{2}\right) g_\beta^{n'}\left(-\mathbf{k} + \frac{\mathbf{Q}}{2}\right) \times \langle d_{-\mathbf{k}+(\mathbf{Q}/2),n',\downarrow} d_{\mathbf{k}+(\mathbf{Q}/2),n,\uparrow}\rangle, \tag{4}$$

where l, $\beta = 0, \ldots, q-1$. The site index β denotes q inequivalent sites along the x direction of each magnetic unit cell, and the index l represents the center-of-mass momentum of the pair $\mathbf{Q} = (0, 2\pi l p/q)$, which represents the order parameter modulation along the y direction. For instance, for $p/q = 1/3$, there are three different center-of-mass momenta, which are $\mathbf{Q}_{l=0} = (0, 0)$, $\mathbf{Q}_{l=1} = (0, 2\pi/3)$, and $\mathbf{Q}_{l=2} = (0, 4\pi/3)$. With Δ_β^l, the mean-field Hamiltonian becomes

$$H_{\rm MF} = \sum_{n\mathbf{k}\sigma} \epsilon_{n\mathbf{k}\sigma} d_{n\mathbf{k}\sigma}^\dagger d_{n\mathbf{k}\sigma} - \sum_{l,\beta}\left\{\sum_{n,n',\mathbf{k}}\left(\Delta_\beta^l g_\beta^{n*}\left(\mathbf{k}+\frac{\mathbf{Q}}{2}\right) \times g_\beta^{n'*}\left(-\mathbf{k}+\frac{\mathbf{Q}}{2}\right) d_{\mathbf{k}+(\mathbf{Q}/2),n,\uparrow}^\dagger d_{-\mathbf{k}+(\mathbf{Q}/2),n',\downarrow}^\dagger + \text{H.c.}\right) + \frac{|\Delta_\beta^l|^2}{U}\right\}. \tag{5}$$

The real space order parameter for site $i = (i_x, i_y)$ is given by $\Delta_i = \sum_{l=0}^{q-1} \Delta_{i_x(\text{mod}q)}^l e^{i2\pi l p i_y/q}$; therefore the unit cell in the superfluid phase is enlarged to $q \times q$ in real space (see Fig. 2).

Solution to BCS Theory.—We start with q^2 random complex numbers as initial Δ_β^l and iteratively solve the BCS mean-field Hamiltonian [Eq. (5)] until a self-consistent solution is reached. We find for a convergent solution, the q^2 order parameters are not completely independent. In fact, these q^2 order parameters break up into q sets of q order parameters with the same magnitude. Taking $p/q = 1/3$ or $1/4$ as examples, their relations are summarized in Table I.

These structures can be understood from the symmetry properties discussed above. The system is invariant under translation by one lattice site along the x direction and a simultaneous translation of k_y by $2\pi p/q$. Under this operation, $\Delta_\beta^l \to \Delta_{\beta'}^{l'}$, where $\beta' = \beta + 1(\text{mod}q)$ and $l' = l + 2(\text{mod}q)$; thus, these two orders parameters must be equal up to a relative phase. To verify these relations, we show in Fig. 3(a) that our numerical solutions satisfy $I_{ll'} = |\vec{\Delta}^{l'}\Gamma\Delta^{l\dagger}|/(|\vec{\Delta}^l||\vec{\Delta}^{l'}|) = 1$ for $l' = l + 2(\text{mod}q)$, where Γ is a $q \times q$ matrix with $\Gamma_{ij} = \delta_{i+1(\text{mod}q),j}$. This symmetry imposed relation works for any p/q, which implies that if Δ^0 is nonzero, all $\Delta^{2n(\text{mod}q)}$ are nonzero; i.e., the zero and finite momentum component must coexist.

The self-consistent solution also determines the relative phases. For $p/q = 1/3$, we find six degenerate solutions with $(\theta_1, \theta_2) = (\pm 2\pi/3, \pm 2\pi/3)$, $(0, \pm 2\pi/3)$, $(\pm 2\pi/3, 0)$; for $p/q = 1/4$, we find $\theta_{1,2} = \pm\pi/2$ and either $a, b \neq 0$, $c = d = 0$ or $c, d \neq 0$, $a = b = 0$; therefore, there are totally four degenerate solutions. One can see from Fig. 2 that this degeneracy can also be inferred naturally from the geometry of the vortex configuration.

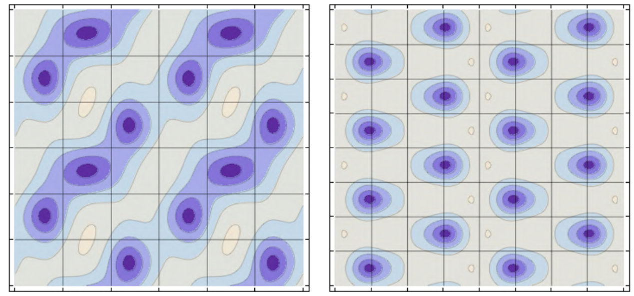

FIG. 2 (color online). Structure of the vortex lattice found from self-consistently solving the BCS mean-field Hamiltonian, where $n_B = 1/3$ for (a) and $n_B = 1/4$ for (b). This is a contour plot of pairing order parameter $\Delta(\mathbf{r})$. The gray area means low superfluid density and locates the center of vortex cores. The intersection of vertical and horizontal straight lines indicates lattice sites. In both plots we use $U = -5.5t$ and $n_\uparrow = n_\downarrow = 1/3$ for (a) and $n_\uparrow = n_\downarrow = 1/2$ for (b).

The most favorable relative phases can also be understood by a simple Ginzburg-Landau (GL) argument. This GL theory should work well particularly nearby the phase transition point discussed below, where the order parameter is small. For those order parameters that definitely coexist, we first write down the most general coupling form between them by momentum conservation, and then determine the most favorable relative phases by minimizing energy. For instance, for $p/q = 1/3$, Δ^0, Δ^1, and Δ^2 all coexist, and then one can write $E_{\rm GL} \propto \Delta^{0*}\Delta^{0*}\Delta^1\Delta^2 + \Delta^{1*}\Delta^{1*}\Delta^0\Delta^2 + \Delta^{2*}\Delta^{2*}\Delta^0\Delta^1$ + c.c.; thus the energy depends on the phases as $\cos(2\theta_1 - \theta_2) + \cos(2\theta_2 - \theta_1) + \cos(\theta_1 + \theta_2)$, and one can easily show that the angles listed above are its minima. For $p/q = 1/4$, Δ^0 and Δ^2 definitely coexist, thus one shall write down $E \propto \Delta^{0*}\Delta^{0*}\Delta^2\Delta^2$ + c.c., which gives the energy-phase relative as $\cos 2\theta_1$, whose minima occur at $\theta_1 = \pm\pi/2$.

TABLE I. Pairing order parameters for $p/q = 1/3$ (upper) and $p/q = 1/4$ (lower). a, b, c, and d denotes some complex numbers depending on details, like the fermion density and U/t.

Δ_β^l	$\beta = 0$	$\beta = 1$	$\beta = 2$
$l = 0$	a	b	c
$l = 1$	$be^{i\theta_1}$	$ce^{i\theta_1}$	$ae^{i\theta_1}$
$l = 2$	$ce^{i\theta_2}$	$ae^{i\theta_2}$	$be^{i\theta_2}$

Δ_β^l	$\beta = 0$	$\beta = 1$	$\beta = 2$	$\beta = 3$
$l = 0$	a	b	a	b
$l = 1$	c	d	c	d
$l = 2$	$be^{i\theta_1}$	$ae^{i\theta_1}$	$be^{i\theta_1}$	$ae^{i\theta_1}$
$l = 3$	$de^{i\theta_2}$	$ce^{i\theta_2}$	$de^{i\theta_2}$	$ce^{i\theta_2}$

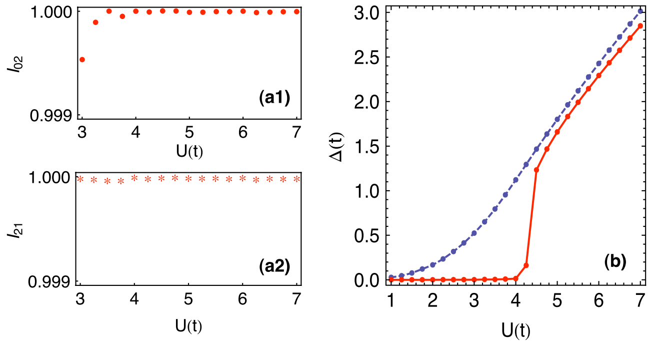

FIG. 3 (color online). (a) For $p/q = 1/3$, I_{02}, and I_{21} (see text for definition) equal to unity within numerical accuracy, which verifies the symmetry relations. (b) Average superfluid order parameter $\Delta = \bar{\Delta}_i$ as a function of U for $p/q = 1/3$, $n = 1/3$ (red solid line) shows a phase transition and $n = 1/2$ (blue dashed line) does not.

Vortex configuration: To study the configuration of vortices in the superfluid ground state, we first note that in presence of magnetic field the Wannier wave function at each site should be chosen as $\varphi(\mathbf{r} - \mathbf{R}_\mathbf{j}) = e^{i2\pi p(x/\lambda)(y/\lambda - j_y)/q}\varphi^0(\mathbf{r} - \mathbf{R}_\mathbf{j})$, where φ^0 is a Wannier wave function in the absence of magnetic field, λ is the lattice spacing. The real space profile of the order parameter $\Delta(\mathbf{r}) = \sum_j \Delta_j \varphi(\mathbf{r} - \mathbf{R}_\mathbf{j})$ is contour plotted in Fig. 2 for two different flux densities. We also verify that the phase of $\Delta(\mathbf{r})$ winds 2π around each vortex core. There are six (four) space group symmetry-related configurations for Figs. 2(a) and 2(b), which corresponds to six (four) degenerate mean-field solutions. Hence, we have presented a systematical way to determine the configuration of vortices in a BCS superfluid from a microscopic theory, which can be verified experimentally with the standard imaging technique in cold-atom experiments.

Insulator (semimetal) to superfluid transition.—For $n_B = p/q$, and for the fermion density of each spin component $n = \nu/q$, where ν is an integer from $1, \ldots, q-1$, the system is usually a Hofstadter insulator in the absence of interactions. Except for the case that q is an even integer and $n = 1/2$, the system is a semimetal since there are Dirac nodes at the Fermi energy. In both cases, since the Fermi energy is either in the band gap (Hofstadter insulator), or the density-of-state linearly vanishes (Hofstadter semimetal) at the Fermi energy, there is no Cooper instability for infinitesimally small attractive interactions. Thus, it requires a critical interaction strength to turn the system into a paired superfluid through a second-order phase transition, as shown in Fig. 3(b). In this calculation, we also fix the fermion density by judging chemical potential. This transition is driven by the competition between pairing-energy gain and the single-particle energy cost to excite particles across the band gap, which was first discussed in Ref. [10] for a lattice system without magnetic field. Without the magnetic field, to realize the transition one needs to tune the interaction close to a Feshbach resonance to achieve strong pairing strength comparable to the band gap; while in this case, since the magnetic band gap is controlled by original band width t, the transition can be achieved by varying U/t, as routinely done in cold-atom experiments. This transition is accomplished by a change in compressibility and can be measured directly from *in situ* density profile, which has been successfully used in studying the boson Hubbard model.

H. Z. is grateful to Fei Ye for the discussion of magnetic translation group, and we thank Jason Ho for helpful correspondence. H. Z. is supported by the Basic Research Young Scholars Program of Tsinghua University, NSFC Grant No. 10944002 and 10847002. R. O. U. is supported by TÜBİTAK. M. O. O. is supported by TUBITAK-KARIYER Grant No. 104T165.

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[1] For a review, see I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[2] The physics of rotating bosons has been recently reviewed in A. L. Fetter, Rev. Mod. Phys. **81**, 647 (2009).
[3] V. Schweikhard *et al.*, Phys. Rev. Lett. **92**, 040404 (2004); S. Tung, V. Schweikhard, and E. A. Cornell, Phys. Rev. Lett. **97**, 240402 (2006).
[4] G. Juzeliūnas and P. Öhberg, Phys. Rev. Lett. **93**, 033602 (2004); G. Juzeliūnas, J. Ruseckas, P. Öhberg, and M. Fleischhauer, Phys. Rev. A **73**, 025602 (2006); J. Ruseckas, G. Juzeliūnas, P. Öhberg, and M. Fleischhauer, Phys. Rev. Lett. **95**, 010404 (2005); E. J. Mueller, Phys. Rev. A **70**, 041603(R) (2004).
[5] Y.-J. Lin *et al.*, Nature (London) **462**, 628 (2009).
[6] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[7] M. Niemeyer, J. K. Freericks, and H. Monien, Phys. Rev. B **60**, 2357 (1999); D. Jaksch and P. Zoller, New J. Phys. **5**, 56 (2003); M. Ö. Oktel, M. Niţă, and B. Tanatar, Phys. Rev. B **75**, 045133 (2007); C. J. Wu, H. D. Chen, J. P. Hu, and S. C. Zhang, Phys. Rev. A **69**, 043609 (2004); R. O. Umucalılar and M. Ö. Oktel, *ibid.* **76**, 055601 (2007); D. S. Goldbaum and E. J. Mueller, *ibid.* **79**, 021602 (2009); **77**, 033629 (2008); K. Kasamatsu, *ibid.* **79**, 021604(R) (2009); A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **94**, 086803 (2005); G. Möller and N. R. Cooper, Phys. Rev. Lett. **103**, 105303 (2009); T. Đurić and D. K. K. Lee, Phys. Rev. B **81**, 014520 (2010).
[8] J. Zak, Phys. Rev. **134**, A1602 (1964); **134**, A1607 (1964); I. Dana, Y. Avron, and J. Zak, J. Phys. C **18**, L679 (1985).
[9] For a review, see D. Xiao, M. C. Chang, and Q. Niu, arXiv:0907.2021 [Rev. Mod. Phys. (to be published)], Sec. VIII.
[10] H. Zhai and T. L. Ho, Phys. Rev. Lett. **99**, 100402 (2007).