

Any l-state analytical solutions of the Klein-Gordon equation for the Woods-Saxon potential

V. H. Badalov^{1,*}, H. I. Ahmadov², and S. V. Badalov³

¹*Institute for Physical Problems Baku State University,*

Z. Khalilov st. 23, AZ-1148, Baku, Azerbaijan

²*Department of Mathematical Physics,*

Faculty of Applied Mathematics and Cybernetics,

Baku State University, Z. Khalilov st. 23, AZ-1148, Baku, Azerbaijan

³*Department of Physics, Bilkent University, 06800, Ankara, Turkey*

Abstract

The radial part of the Klein-Gordon equation for the Woods-Saxon potential is solved. In our calculations, we have applied the Nikiforov-Uvarov method by using the Pekeris approximation to the centrifugal potential for any l states. The exact bound state energy eigenvalues and the corresponding eigenfunctions are obtained for a particle bound on the various values of the quantum numbers n and l .

PACS numbers: 03.65.-w; 03.65.Fd; 03.65.Ge; 03.65.Pm

Keywords: Klein-Gordon equation, Nikiforov-Uvarov method, Pekeris approximation, Exact solutions

*Electronic address: E-mail:badalovvatan@yahoo.com

I. INTRODUCTION

An analytical solution of the radial part of the Klein-Gordon equation is of high importance spinless in relativistic quantum mechanics, because the wave function contains all necessary information for full description of a quantum system. There are only few potentials for which the radial part of the Klein-Gordon equation can be solved explicitly for all n and l . So far, many methods were developed, such as supersymmetry (SUSY) [1,2] and the Pekeris approximation [3-8], to solve radial part of the Klein-Gordon equation exactly or quasi-exactly for $l \neq 0$ within these potentials.

The one-dimensional Klein-Gordon (KG) equation is investigated for the PT -symmetric generalized Woods-Saxon (WS) potential [9] and Hulthen [10] and is solved by using the Nikiforov-Uvarov (NU) method which is based on solving the second-order linear differential equations by reduction to a generalized equation of hypergeometric type.

The radial part of the Klein-Gordon equation for the Woods-Saxon potential [11] can not be solved exactly for $l \neq 0$. It is well known that the Woods-Saxon potential is one of the important short-range potentials in physics. And this potential were applied to numerous problems, in nuclear and particle physics, atomic physics, condensed matter and chemical physics. Therefore, it would be interesting and important to solve the radial of the Klein - Gordon equation for Woods-Saxon potential for $l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interacting systems.

In this work, we solve the radial part of the Klein-Gordon equation for the standard Woods-Saxon potential using NU method [12], and obtain the energy eigenvalues and corresponding eigenfunctions for any l states.

II. NIKIFOROV-UVAROV METHOD

The Nikiforov-Uvarov (NU) method is based on the solutions of general second-order linear equations with special orthogonal functions. It has been extensively used to solve the non-relativistic Schrödinger equation and other Schrödinger-like equations. The one-dimensional Schrödinger equation or similar second-order differential equations can be written using NU method in the following form:

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0, \quad (2.1)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials, at most second-degree, and $\tilde{\tau}(z)$ is a first-degree polynomial.

Using Eq(2.1) the transformation

$$\psi(z) = \Phi(z)y(z) \quad (2.2)$$

one reduces it to the hypergeometric-type equation

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0. \quad (2.3)$$

The function $\Phi(z)$ is defined as the logarithmic derivative [10]

$$\frac{\Phi'(z)}{\Phi(z)} = \frac{\pi(z)}{\sigma(z)}, \quad (2.4)$$

where $\pi(z)$ is at most the first-degree polynomial.

The another part of $\psi(z)$, namely $y(z)$, is the hypergeometric-type function, that for fixed n is given by the Rodriguez relation:

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \quad (2.5)$$

where B_n is the normalization constant and the weight function $\rho(z)$ must satisfy the condition [12]

$$\frac{d}{dz} (\sigma(z)\rho(z)) = \tau(z)\rho(z), \quad (2.6)$$

with $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$.

For accomplishment of the conditions imposed on function $\rho(z)$ the classical orthogonal polynomials, it is necessary, that polynomial $\tau(z)$ becomes equal to zero in some point of an interval (a, b) and derivative of this polynomial for this interval at $\sigma(z) > 0$ will be negative, i.e. $\tau'(z) < 0$.

The function $\pi(z)$ and the parameter λ required for this method are defined as follows:

$$\pi(z) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \quad (2.7)$$

$$\lambda = k + \pi'(z). \quad (2.8)$$

On the other hand, in order to find the value of k , the expression under the square root must be the square of a polynomial. This is possible only if its discriminant is zero. Thus, the new eigenvalue equation for the Eq.(2.1) is [12]:

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \quad (n = 0, 1, 2, \dots), \quad (2.9)$$

After the comparison of Eq.(2.8) with Eq.(2.9), we obtain the energy eigenvalues.

III. SOLUTIONS OF THE KLEIN-GORDON EQUATION WITH THE WOODS-SAXON POTENTIAL

The standard Woods-Saxon potential [11] is defined by

$$V(r) = -\frac{V_0}{1 + \exp\left(\frac{r-R_0}{a}\right)}, \quad a \ll R_0, \quad 0 \leq r < \infty. \quad (3.1)$$

This potential was used for description of interaction of a neutron with a heavy nucleus. The parameter R_0 is interpreted as radius of a nucleus, the parameter a characterizes thickness of the superficial layer inside, which the potential falls from value $V = 0$ outside of a nucleus up to value $V = -V_0$ inside a nucleus. At $a = 0$, one gets the simple potential well with jump of potential on the surface of a nucleus.

In the spherical coordinates, the stationary Klein-Gordon equation with Woods-Saxon potential is [13]

$$-\hbar^2 c^2 \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \left[\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \right\} \psi = [(E - V(r))^2 - m_0^2 c^4] \psi. \quad (3.2)$$

where m_0 is the rest mass of a scalar particle.

The terms in the square brackets with the overall minus sign are the dimensionless angular momentum squared operator, \hat{L}^2 . Defining $\psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$, we obtain the radial part of the Klein-Gordon equation [13]

$$\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[\frac{(E - V)^2 - m_0^2 c^4}{\hbar^2 c^2} - \frac{l(l+1)}{r^2} \right] R(r) = 0. \quad (3.3)$$

where l is the angular momentum quantum number.

After introducing the new function

$$u(r) = rR(r),$$

Eq.(3.3) takes the form

$$\frac{d^2 u(r)}{dr^2} + \left[\frac{(E - V)^2 - m_0^2 c^4}{\hbar^2 c^2} - \frac{l(l+1)}{r^2} \right] u(r) = 0. \quad (3.4)$$

The Eq.(3.4) has the same form as the equation for a particle in one dimension, except for two important differences. First, there is a repulsive effective potential proportional to

the eigenvalue of $\hbar^2 l(l+1)$. Second, the radial function must satisfy the boundary conditions $u(0) = 0$ and $u(\infty) = 0$.

If in Eq.(3.4) introduce the notations

$$x = \frac{r - R_0}{R_0}, \quad \alpha = \frac{R_0}{a},$$

then the Woods-Saxon potential is given by the expression

$$V_{WS} = -\frac{V_0}{1 + \exp(\alpha x)}.$$

It is known that the radial part Klein-Gordon equation cannot be solved exactly for this potential at the value $l \neq 0$ using the standard methods as SUSY and NU. From Eq.(3.6) it is seen, that the centrifugal potential $V_l(r)$ is inverse square potentials, which cannot be solved analytically. Therefore, in order to solve this problem we can take the most widely used and convenient for our purposes Pekeris approximation. This approximation is based on the expansion of the centrifugal barrier in a series of exponentials depending on the internuclear distance, taking into account terms up to second order, so that the effective l dependent potential preserves the original form. It should be pointed out, however, that this approximation is valid only for low vibrational energy states. By changing the coordinates $x = \frac{r-R_0}{R_0}$ or $r = R_0(1+x)$, the centrifugal potential is expanded in the Taylor series around the point $x = 0$ ($r = R_0$)

$$V_l(r) = \frac{\hbar^2 l(l+1)}{2m_0 r^2} = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2} \frac{1}{(1+x)^2} = \delta (1 - 2x + 3x^2 - 4x^3 + \dots), \quad (3.5)$$

where $\delta = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2}$.

According to the Pekeris approximation, we shall replace potential $V_l(r)$ with expression [7,8]:

$$V_l^*(r) = \delta \left(C_0 + \frac{C_1}{1 + \exp \alpha x} + \frac{C_2}{(1 + \exp \alpha x)^2} \right), \quad (3.6)$$

where C_0 , C_1 and C_2 are some constants.

In order to define the constants C_0 , C_1 and C_2 , we also expand this potential in the Taylor series around the point $x = 0$ ($r = R_0$):

$$V_l^*(x) = \delta \left[\left(C_0 + \frac{C_1}{2} + \frac{C_2}{4} \right) - \frac{\alpha}{4} (C_1 + C_2) x + \frac{\alpha^2}{16} C_2 x^2 + \frac{\alpha^3}{48} (C_1 + C_2) x^3 - \frac{\alpha^4}{96} C_2 x^4 + \dots \right]. \quad (3.7)$$

Comparing equal powers of x Eqs.(3.5) and (3.7), we obtain the constants C_0 , C_1 and C_2 [7,8]:

$$C_0 = 1 - \frac{4}{\alpha} + \frac{12}{\alpha^2}, \quad C_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2}, \quad C_2 = \frac{48}{\alpha^2}.$$

Instead of solving the radial part Klein-Gordon equation for the centrifugal potential $V_l(r)$ given by Eq.(3.5), we now solve the radial part Klein-Gordon equation for the new centrifugal potential $V_l^*(r)$ given by Eq.(3.6) obtained using the Pekeris approximation. Having inserted this new centrifugal potential into Eq.(3.4), we obtain

$$\frac{d^2u}{dr^2} + \left[\left(\frac{E^2 - m_0^2c^4}{\hbar^2c^2} - \frac{l(l+1)C_0}{\alpha^2a^2} \right) + \frac{\left(\frac{2EV_0}{\hbar^2c^2} - \frac{l(l+1)C_1}{\alpha^2a^2} \right)}{1 + e^{\frac{r-R_0}{a}}} + \frac{\left(\frac{V_0^2}{\hbar^2c^2} - \frac{l(l+1)C_2}{\alpha^2a^2} \right)}{\left(1 + e^{\frac{r-R_0}{a}} \right)^2} \right] u = 0. \quad (3.8)$$

We use the following dimensionless notations

$$\epsilon^2 = - \left(\frac{(E^2 - m_0^2c^4)a^2}{\hbar^2c^2} - \frac{l(l+1)C_0}{\alpha^2} \right); \quad \beta^2 = \frac{2EV_0a^2}{\hbar^2c^2} - \frac{l(l+1)C_1}{\alpha^2}; \quad \gamma^2 = - \left(\frac{V_0^2a^2}{\hbar^2c^2} - \frac{l(l+1)C_2}{\alpha^2} \right), \quad (3.9)$$

with real $\epsilon > 0$ ($E^2 \leq m_0^2c^2$) for bound states; β is real, γ is real and positive.

If we rewrite Eq.(3.8) by using a new variable of the form

$$z = \left(1 + \exp \left(\frac{r - R_0}{a} \right) \right)^{-1},$$

we obtain

$$u''(z) + \frac{1-2z}{z(1-z)}u'(z) + \frac{-\epsilon^2 + \beta^2z - \gamma^2z^2}{(z(1-z))^2}u(z) = 0, \quad (0 \leq z \leq 1), \quad (3.10)$$

with $\tilde{\tau}(z) = 1 - 2z$; $\sigma(z) = z(1 - z)$; $\tilde{\sigma}(z) = -\epsilon^2 + \beta^2z - \gamma^2z^2$.

In the NU-method the new function $\pi(z)$ is

$$\pi(z) = \pm \sqrt{\epsilon^2 + (k - \beta^2)z - (k - \gamma^2)z^2}. \quad (3.11)$$

The constant parameter k can be found employing the condition that the expression under the square root has a double zero, i.e., its discriminant is equal to zero. So, there are two possible functions for each k

$$\pi(z) = \pm \begin{cases} \left(\epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) z - \epsilon, & \text{for } k = \beta^2 - 2\epsilon^2 + 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}, \\ \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) z - \epsilon, & \text{for } k = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \end{cases} \quad (3.12)$$

According to the NU-method, from the four possible forms of the polynomial $\pi(z)$ we select the one for which the function $\tau(z)$ has the negative derivative and root lies in the interval $(0,1)$. Therefore, the appropriate functions $\pi(z)$ and $\tau(z)$ have the following forms

$$\pi(z) = \epsilon - \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) z, \quad (3.13)$$

$$\tau(z) = 1 + 2\epsilon - 2 \left(1 + \epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) z, \quad (3.14)$$

and

$$k = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \quad (3.15)$$

Then, the constant $\lambda = k + \pi'(z)$ is written as

$$\lambda = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2} - \epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \quad (3.16)$$

An alternative definition of λ_n (Eq.(2.9)) is

$$\lambda = \lambda_n = 2 \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) n + n(n+1). \quad (3.17)$$

Having compared Eqs.(3.16) and (3.17)

$$\beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2} - \epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2} = 2 \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) n + n(n+1), \quad (3.18)$$

we obtain

$$\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} + n + \frac{1}{2} - \frac{\sqrt{1 + 4\gamma^2}}{2} = 0, \quad (3.19)$$

or

$$\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} - n' = 0. \quad (3.20)$$

Here

$$n' = -n + \frac{\sqrt{1 + 4\gamma^2} - 1}{2}, \quad (3.21)$$

n being the radial quantum number ($n = 0, 1, 2, \dots$).

After substituting α, γ, C_2 into Eq.(3.21), we obtain

$$n' = -n + \frac{\sqrt{1 + \frac{192a^4l(l+1)}{R_0^4} - \frac{4V_0^2a^2}{\hbar^2c^2}} - 1}{2}. \quad (3.22)$$

From Eq.(3.20), we find

$$\epsilon = \frac{1}{2} \left(n' + \frac{\beta^2 - \gamma^2}{n'} \right). \quad (3.23)$$

Because for the bound states $\epsilon > 0$, we get

$$n' > 0. \quad (3.24)$$

If $n' > 0$, there exist bound states, otherwise, there are no bound states at all. By using Eq.(3.21) this relation can be recast into the form

$$0 \leq n < \frac{\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 1}{2} \quad (3.25)$$

i.e. it gives the finite coupling value.

The condition $\gamma > 0$ gives the define coupling value for the potential depth V_0

$$0 < V_0 < \frac{4\hbar c a \sqrt{3l(l+1)}}{R_0^2}. \quad (3.26)$$

From Eqs. (3.23) and (3.9), we obtain

$$E_{nl} = -\frac{V_0}{2} \left(1 - \frac{l(l+1)(C_1+C_2)}{\alpha^2 \left(n'^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right)} \right) \pm cn' \sqrt{\frac{m_0^2 c^2 + \frac{\hbar^2 l(l+1)}{R_0^2} C_0}{n'^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2}} - \frac{\hbar^2}{4a^2} \left(1 - \frac{l(l+1)(C_1+C_2)}{\alpha^2 \left(n'^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right)} \right)^2} \quad (3.27)$$

Thus, substituting the expressions of α , C_0 , C_1 , C_2 and n' into Eq.(3.27), one can find the energy eigenvalues E_{nl}

$$E_{nl} = -\frac{V_0}{2} \left(1 - \frac{32l(l+1)a^3}{R_0^3 \left[\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right]} \right) \pm c \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n - 1 \right) \left\{ \frac{m_0^2 c^2 + \frac{\hbar^2 l(l+1)}{R_0^2} \left(1 - \frac{4a}{R_0} + \frac{12a^2}{R_0^2} \right)}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2}} - \frac{\hbar^2}{16a^2} \left(1 - \frac{32l(l+1)a^3}{R_0^3 \left[\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right]} \right)^2 \right\}^{\frac{1}{2}} \quad (3.28)$$

If all two conditions (3.25) and (3.26) are satisfied simultaneously, the bound states exist. From Eq.(3.25) is seen that if $l = 0$, then one gets $n < 0$. Hence, the Klein-Gordon equation

for the standard Woods-Saxon potential with zero angular momentum has no bound states. For larger values of V_0 ($V_0 > \frac{4\hbar a c \sqrt{3l(l+1)}}{R_0^2}$) the condition (3.25) is not satisfied. Therefore, no bound states exist for these values of V_0 .

According to Eq.(3.28) the energy eigenvalues depend on the depth of the potential V_0 , the width of the potential R_0 , and the surface thickness a . Any energy eigenvalue must be less than V_0 . If constraints imposed on n and V_0 are satisfied, the bound states appear. From Eq.(3.26) is seen that the potential depth increases when the parameter a increases, but the parameter R_0 is decreasing for given l quantum number and vice versa. Therefore, one can say that the bound states exist within this potential. Thus, the energy spectrum Eq.(3.28) are limited, i.e. we have only the finite number of energy eigenvalues.

The binding energy of a bound Klein-Gordon particle is defined as [13]:

$$E_b = E_{nl} - m_0 c^2. \quad (3.29)$$

By using the empirical values $r_0 = 1.285 fm$ and $a = 0.65 fm$ taken from Ref.[14] the potential depth $V_0 = (40.5 + 0.13A) MeV$ and the radius of the nucleus $R_0 = r_0 A^{1/3} fm$ are calculated for the atomic mass number of target nucleus $A = 40; 56; 66; 92; 140; 208$ and pions with mass $m_0 c^2 = 139.570 MeV$. In the Table 1. energies of the bound states obtained numerically for the spherical standard Woods-Saxon potential for some values of l and n are given.

In addition, we have seen that there are some restrictions on the potential parameters in order to obtain bound state solutions. We also point out that the exact results obtained for the standard Woods-Saxon potential may have some interesting applications for studying different quantum mechanical and nuclear scattering problems.

However, in the non-relativistic limit with the mapping $E_{nl}^{(R)} - m_0 c^2 \rightarrow E_{nl}^{(NR)}$, according to Appendix A for bound state energy eigenvalues, we obtain [8]:

$$E_{nl} = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2} \left(1 + \frac{12a^2}{R_0^2} \right) - \frac{\hbar^2}{2m_0 a^2} \left[\frac{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right)^2}{16} + \frac{4 \left(\frac{m_0 V_0 a^2}{\hbar^2} - \frac{4l(l+1)a^3}{R_0^3} \right)^2}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right)^2} + \frac{m_0 V_0 a^2}{\hbar^2} \right] \quad (3.30)$$

Now, we are going to determine the radial eigenfunctions of this potential. Having substituted $\pi(z)$ and $\sigma(z)$ into Eq.(2.4) and then solving first-order differential equation, one can find the finite function $\Phi(z)$ in the interval $(0, 1)$

$$\Phi(z) = z^\epsilon (1-z) \sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \quad (3.31)$$

It is easy to find the second part of the wave function from the definition of weight function:

$$\rho(z) = z^{2\epsilon} (1-z)^2 \sqrt{\epsilon^2 - \beta^2 + \gamma^2}, \quad (3.32)$$

and substituting into Rodrigues relation (2.4), we get

$$y_n(z) = B_n z^{-2\epsilon} (1-z)^{-2\sqrt{\epsilon^2 - \beta^2 + \gamma^2}} \frac{d^n}{dz^n} \left[z^{n+2\epsilon} (1-z)^{n+2\sqrt{\epsilon^2 - \beta^2 + \gamma^2}} \right]. \quad (3.33)$$

where $B_n = \frac{1}{n!}$ is the normalization constant [15]. Then, y_n is given by the Jacobi polynomials

$$y_n(z) = P_n^{(2\epsilon, 2\sqrt{\epsilon^2 - \beta^2 + \gamma^2})}(1-2z), \quad (3.34)$$

where

$$P_n^{(\alpha, \beta)}(1-2z) = \frac{1}{n!} z^{-\alpha} (1-z)^{-\beta} \frac{d^n}{dz^n} \left[z^{n+\alpha} (1-z)^{n+\beta} \right].$$

The corresponding $u_{nl}(z)$ radial wave functions are found to be

$$u_{nl}(z) = C_{nl} z^\epsilon (1-z) \sqrt{\epsilon^2 - \beta^2 + \gamma^2} P_n^{(2\epsilon, 2\sqrt{\epsilon^2 - \beta^2 + \gamma^2})}(1-2z), \quad (3.35)$$

where C_{nl} is the normalization constants determined using $\int_0^\infty [u_{nl}(r)]^2 dr = 1$ constraint, i.e.

$$a C_{nl}^2 \int_0^1 z^{2\epsilon-1} (1-z)^{2\sqrt{\epsilon^2 - \beta^2 + \gamma^2} - 1} \left[P_n^{(2\epsilon, 2\sqrt{\epsilon^2 - \beta^2 + \gamma^2})}(1-2z) \right]^2 dz = 1$$

IV. CONCLUSION

In this paper, we have analytically calculated energy eigenvalues of the bound states and corresponding eigenfunctions in the new exactly solvable Woods-Saxon potential. The energy eigenvalue expression for Woods-Saxon potentials is given by Eq.(3.28). As it should be expected, for any given set of parameters V_0, R_0 and a , the energy levels of standard Woods-Saxon potential are positive. The obtained results are interesting for both theoretical and experimental physicists, because they provide exact expression for energy eigenvalues and corresponding eigenfunctions.

Appendix A

After some simple transformations Eq.(3.27) takes the form

$$E_{nl}^{(R)} = -\frac{V_0}{2} \left(1 - \frac{l(l+1)(C_1+C_2)}{\alpha^2 \left(n'^2 + \frac{V_0^2 a^2}{\hbar^2 c^2} \right)} \right) + m_0 c^2 \left\{ 1 + \frac{\hbar^2 l(l+1)C_0}{m_0^2 c^2 R_0^2} - \frac{V_0^2 a^2}{\hbar^2 c^2 \left(n'^2 + \frac{V_0^2 a^2}{\hbar^2 c^2} \right)} - \frac{4\hbar^2 l(l+1)C_0 V_0^2 a^2}{\hbar^2 c^4 m_0^2 R_0^2 \left(n'^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right)} - \frac{\hbar^2 n'^2}{4m_0^2 c^2 a^2} \left(1 - \frac{l(l+1)(C_1+C_2)}{\alpha^2 \left(n'^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right)} \right)^2 \right\}^{\frac{1}{2}} \quad (\text{A1})$$

Expanding, finally, the above eigenvalue in a series of powers of $\frac{1}{c^2}$ yields

$$\begin{aligned} E_{nl}^{(R)} &= -\frac{V_0}{2} \left[1 - \frac{4l(l+1)(C_1+C_2)}{\alpha^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} \right] + m_0 c^2 \left\{ 1 + \frac{\hbar^2 l(l+1)C_0}{2m_0^2 c^2 R_0^2} - \frac{4V_0^2 a^2}{2\hbar^2 c^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} - \frac{\hbar^2}{32m_0^2 c^2 a^2} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2 \times \right. \\ &\left. \left[1 - \frac{4l(l+1)(C_1+C_2)}{\alpha^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} \right]^2 \right\} + \dots = -\frac{V_0}{2} + \frac{2l(l+1)(C_1+C_2)V_0}{\alpha^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} - \\ &m_0 c^2 + \frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{2m_0 V_0^2 a^2}{\hbar^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} - \frac{\hbar^2}{32m_0 a^2} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2 + \\ &\frac{\hbar^2 l(l+1)(C_1+C_2)}{4m_0 a^2 \alpha^2} - \frac{\hbar^2 l^2 (l+1)^2 (C_1+C_2)^2}{2m_0 a^2 \alpha^4 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} + \dots = m_0 c^2 + \frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{V_0}{2} \quad (\text{A2}) \\ &- \frac{1}{16} \frac{\hbar^2}{2m_0 a^2} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2 - \frac{\hbar^2}{2m_0 a^2} \frac{\left(\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2} \right)^2}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} + \\ &\frac{\hbar^2 l(l+1)(C_1+C_2)}{4m_0 a^2 \alpha^2} + \dots = m_0 c^2 + \frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{\hbar^2}{2m_0 a^2} \left[\frac{1}{16} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2 + \right. \\ &\left. \frac{1}{2} \left(\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2} \right) + \frac{\left(\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2} \right)^2}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} \right] + \dots = m_0 c^2 + \\ &\frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{\hbar^2}{2m_0 a^2} \left[\frac{1}{4} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right) + \frac{\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2}}{\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1} \right]^2 + o\left(\frac{1}{c^4}\right) \end{aligned}$$

Denoting, as usual, the energy eigenvalues in the non-relativistic case by $E_{nl}^{(R)} - m_0 c^2 \rightarrow E_{nl}^{(NR)}$, we have

$$E_{nl}^{(NR)} = \frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{\hbar^2}{2m_0 a^2} \left[\frac{1}{4} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right) + \frac{\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2}}{\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1} \right]^2 \quad (\text{A3})$$

Thus, substituting the expressions of C_0, C_1 and C_2 into the above energy eigenvalues $E_{nl}^{(NR)}$, we obtain

$$E_{nl}^{(NR)} = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2} \left(1 - \frac{4a}{R_0} + \frac{12a^2}{R_0^2} \right) - \frac{\hbar^2}{2m_0 a^2} \left[\frac{1}{4} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right) + \frac{2 \left(\frac{m_0 V_0 a^2}{\hbar^2} - \frac{4l(l+1)a^3}{R_0^3} \right)}{\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1} \right]^2 \quad (\text{A4})$$

or

$$E_{nl}^{(NR)} = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2} \left(1 + \frac{12a^2}{R_0^2} \right) - \frac{\hbar^2}{2m_0 a^2} \left[\frac{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right)^2}{16} + \frac{4 \left(\frac{m_0 V_0 a^2}{\hbar^2} - \frac{4l(l+1)a^3}{R_0^3} \right)^2}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right)^2} + \frac{m_0 V_0 a^2}{\hbar^2} \right] \quad (\text{A5})$$

-
- [1] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251** (1995) 267.
- [2] D. A. Morales, Chem. Phys. Lett. **394** (2004) 68.
- [3] C. L. Pekeris, Phys. Rev. **45** (1934) 98.
- [4] S. Flügge, Practical Quantum Mechanics, Vol.1 (Springer, Berlin, 1994)
- [5] O. Bayrak and I. Boztosun, J. Phys. A **39** (2006) 6955 (arxiv: nucl-th / 0604042 V1).
- [6] O. Bayrak, G. Kocak and I. Boztosun, J. Phys. A: Math. Gen. **39** (2006) 11521 (arxiv: math-ph / 0609010 V1).
- [7] V. H. Badalov, H. I. Ahmadov and S. V. Badalov, News of Baku University, N2 (2008) 157.
- [8] V. H. Badalov, H. I. Ahmadov and A. I. Ahmadov, Int.J.Mod.Phys. E **18** (2009) 631 (arxiv: math-ph / 0905.273 V1).
- [9] S. M. Ikhdair and R. Sever,(arxiv: quant-ph /0610183 V1).
- [10] H. Egrifes and R. Sever, Int. J. Theor. Phys. **46** (2007) 935 (arxiv: quant-ph / 0609231).
- [11] R. D. Woods and D. S. Saxon , Phys. Rev. **95** (1954) 577.
- [12] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics (Birkhäuser, Basel, 1988).
- [13] W. Greiner, Relativistic Quantum Mechanics (Springer, Berlin, 1990).
- [14] C. M. Perey, F. G. Perey, J. K. Dickens and R. J. Silva, Phys. Rev. **175** (1968) 1460.
- [15] H. Bateman and A. Erdelyi, Higher Transcendental functions, Vol.2 (McGraw-Hill, New York, 1953)

A	R_0, fm	V_0, MeV	n	l	E_b, MeV
40	4.3946	45.70	0	1	-107.8777
56	4.9162	47.78	0	1	-127.5238
56	4.9162	47.78	0	2	-17.5985
66	5.1930	49.08	0	2	-50.3359
92	5.8010	52.46	0	2	-101.8967
140	6.6724	58.70	0	3	-92.5327
208	7.6136	67.54	0	4	-105.0865
208	7.6136	67.54	0	5	-33.6014

TABLE I: Energies of the bound states for the Woods-Saxon potential for different values of n, l calculated using Eqs.(3.25), (3.26), (3.28) and (3.29).

Any l-state analytical solutions of the Klein-Gordon equation for the Woods-Saxon potential

V. H. Badalov^{1,*}, H. I. Ahmadov², and S. V. Badalov³

¹*Institute for Physical Problems, Baku State University,*

Z. Khalilov st. 23, AZ-1148 Baku, Azerbaijan

²*Department of Mathematical Physics,*

Faculty of Applied Mathematics and Cybernetics,

Baku State University, Z. Khalilov st. 23, AZ-1148 Baku, Azerbaijan

³*Department of Physics, Bilkent University, TR-06800 Ankara, Turkey*

Abstract

The radial part of the Klein-Gordon equation for the Woods-Saxon potential is solved. In our calculations, we have applied the Nikiforov-Uvarov method by using the Pekeris approximation to the centrifugal potential for any l states. The exact bound state energy eigenvalues and the corresponding eigenfunctions are obtained on the various values of the quantum numbers n and l . The non-relativistic limit of the bound state energy spectrum was also found.

PACS numbers: 03.65.-w; 03.65.Fd; 03.65.Ge; 03.65.Pm

Keywords: Klein-Gordon equation, Nikiforov-Uvarov method, Pekeris approximation, Exact solutions

*Electronic address: E-mail:badalovvatan@yahoo.com

I. INTRODUCTION

An analytical solution of the radial part of the Klein-Gordon equation is of high importance spinless in relativistic quantum mechanics, because the wave function contains all necessary information for full description of a quantum system. There are only few potentials for which the radial part of the Klein-Gordon equation can be solved explicitly for all n and l . So far, many methods were developed, such as supersymmetry (SUSY) [1,2] and the Pekeris approximation [3-8], to solve radial part of the Klein-Gordon equation exactly or quasi-exactly for $l \neq 0$ within these potentials.

The one-dimensional Klein-Gordon (KG) equation is investigated for the PT -symmetric generalized Woods-Saxon (WS) potential [9] and Hulthén [10] and is solved by using the Nikiforov-Uvarov (NU) method which is based on solving the second-order linear differential equations by reduction to a generalized equation of hypergeometric type.

The radial part of the Klein-Gordon equation for the Woods-Saxon potential [11] cannot be solved exactly for $l \neq 0$. But in Refs.[9, 12-14] authors as in Refs.[15, 16] because of errors made by them in application of the NU method to investigate the Woods-Saxon potential obtained wrong results. It is well known that the Woods-Saxon potential is one of the important short-range potentials in physics. And this potential was applied to numerous problems, in nuclear and particle physics, atomic physics, condensed matter and chemical physics. Therefore, it would be interesting and important to solve the radial of the Klein - Gordon equation for Woods-Saxon potential for $l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interacting systems.

In this work, we solve the radial part of the Klein-Gordon equation for the standard Woods-Saxon potential using NU method [17], and obtain the energy eigenvalues and corresponding eigenfunctions for any l states.

II. NIKIFOROV-UVAROV METHOD

The Nikiforov-Uvarov (NU) method is based on the solutions of general second-order linear equations with special orthogonal functions. It has been extensively used to solve the nonrelativistic Schrödinger equation and other Schrödinger-like equations. The one-dimensional Schrödinger equation or similar second-order differential equations can be writ-

ten using NU method in the following form:

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0, \quad (2.1)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials, at most second-degree, and $\tilde{\tau}(z)$ is a first-degree polynomial.

Using Eq(2.1) the transformation

$$\psi(z) = \Phi(z)y(z) \quad (2.2)$$

one reduces it to the hypergeometric-type equation

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0. \quad (2.3)$$

The function $\Phi(z)$ is defined as the logarithmic derivative [10]

$$\frac{\Phi'(z)}{\Phi(z)} = \frac{\pi(z)}{\sigma(z)}, \quad (2.4)$$

where $\pi(z)$ is at most the first-degree polynomial.

The another part of $\psi(z)$, namely $y(z)$, is the hypergeometric-type function, that for fixed n is given by the Rodriguez relation:

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \quad (2.5)$$

where B_n is the normalization constant and the weight function $\rho(z)$ must satisfy the condition [12]

$$\frac{d}{dz} (\sigma(z)\rho(z)) = \tau(z)\rho(z), \quad (2.6)$$

with $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$.

For accomplishment of the conditions imposed on function $\rho(z)$ of the classical orthogonal polynomials, it is necessary, that polynomial $\tau(z)$ becomes equal to zero in some point of an interval (a, b) and derivative of this polynomial for this interval at $\sigma(z) > 0$ will be negative, i.e., $\tau'(z) < 0$.

The function $\pi(z)$ and the parameter λ required for this method are defined as follows:

$$\pi(z) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \quad (2.7)$$

$$\lambda = k + \pi'(z). \quad (2.8)$$

On the other hand, in order to find the value of k , the expression under the square root must be the square of a polynomial. This is possible only if its discriminant is zero. Thus, the new eigenvalue equation for the Eq.(2.1) is [17]:

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \quad (n = 0, 1, 2, \dots), \quad (2.9)$$

After the comparison of Eq.(2.8) with Eq.(2.9), we obtain the energy eigenvalues.

III. SOLUTIONS OF THE KLEIN-GORDON EQUATION WITH THE WOODS-SAXON POTENTIAL

The standard Woods-Saxon potential [11] is defined by

$$V(r) = -\frac{V_0}{1 + \exp\left(\frac{r-R_0}{a}\right)}, \quad a \ll R_0, \quad 0 \leq r < \infty. \quad (3.1)$$

This potential was used for description of interaction of a neutron with a heavy nucleus. The parameter R_0 is interpreted as radius of a nucleus, the parameter a characterizes thickness of the superficial layer inside, which the potential falls from value $V = 0$ outside of a nucleus up to value $V = -V_0$ inside of a nucleus. At $a = 0$, one gets the simple potential well with jump of potential on the surface of a nucleus.

In the spherical coordinates, the stationary Klein-Gordon equation with Woods-Saxon potential is [18]

$$-\hbar^2 c^2 \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \left[\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \right\} \psi = [(E - V(r))^2 - m_0^2 c^4] \psi. \quad (3.2)$$

where m_0 is the rest mass of a scalar particle.

The terms in the square brackets with the overall minus sign are the dimensionless angular momentum squared operator, \widehat{L}^2 . Defining $\psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$, we obtain the radial part of the Klein-Gordon equation [18]

$$\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[\frac{(E - V)^2 - m_0^2 c^4}{\hbar^2 c^2} - \frac{l(l+1)}{r^2} \right] R(r) = 0. \quad (3.3)$$

where l is the angular momentum quantum number.

After introducing the new function:

$$u(r) = rR(r),$$

Eq.(3.3) takes the form:

$$\frac{d^2u(r)}{dr^2} + \left[\frac{(E - V)^2 - m_0^2c^4}{\hbar^2c^2} - \frac{l(l+1)}{r^2} \right] u(r) = 0. \quad (3.4)$$

The Eq.(3.4) has the same form as the equation for a particle in one dimension, except for two important differences. First, there is a repulsive effective potential proportional to the eigenvalue of $\hbar^2l(l+1)$. Second, the radial function must satisfy the boundary conditions $u(0) = 0$ and $u(\infty) = 0$.

If in Eq.(3.4) introduce the notations:

$$x = \frac{r - R_0}{R_0}, \quad \alpha = \frac{R_0}{a},$$

then the Woods-Saxon potential is given by the expression:

$$V_{WS} = -\frac{V_0}{1 + \exp(\alpha x)}.$$

It is known that the radial part Klein-Gordon equation cannot be solved exactly for this potential at the value $l \neq 0$ using the standard methods as SUSY and NU. From Eq.(3.4) it is seen that the centrifugal potential $V_l(r) = \frac{\hbar^2l(l+1)}{2m_0r^2}$ is inverse square potentials and as a result, this equation (3.4) cannot be solved analytically. Therefore, in order to solve this problem we can take the most widely used and convenient for our purposes Pekeris approximation. This approximation is based on the expansion of the centrifugal barrier in a series of exponentials depending on the internuclear distance, taking into account terms up to second order, so that the effective l dependent potential preserves the original form. It should be pointed out, however, that this approximation is valid only for low vibrational energy states. By changing the coordinates $x = \frac{r-R_0}{R_0}$ or $r = R_0(1+x)$, the centrifugal potential is expanded in the Taylor series around the point $x = 0$ ($r = R_0$)

$$V_l(r) = \frac{\hbar^2l(l+1)}{2m_0r^2} = \frac{\hbar^2l(l+1)}{2m_0R_0^2} \frac{1}{(1+x)^2} = \delta (1 - 2x + 3x^2 - 4x^3 + \dots), \quad (3.5)$$

where $\delta = \frac{\hbar^2l(l+1)}{2m_0R_0^2}$.

According to the Pekeris approximation, we shall replace potential $V_l(r)$ with expression [7,8]:

$$V_l^*(r) = \delta \left(C_0 + \frac{C_1}{1 + \exp \alpha x} + \frac{C_2}{(1 + \exp \alpha x)^2} \right), \quad (3.6)$$

where C_0 , C_1 and C_2 are some constants.

In order to define the constants C_0 , C_1 and C_2 , we also expand this potential in the Taylor series around the point $x = 0$ ($r = R_0$):

$$V_l^*(x) = \delta \left[\left(C_0 + \frac{C_1}{2} + \frac{C_2}{4} \right) - \frac{\alpha}{4} (C_1 + C_2)x + \frac{\alpha^2}{16} C_2 x^2 + \frac{\alpha^3}{48} (C_1 + C_2)x^3 - \frac{\alpha^4}{96} C_2 x^4 + \dots \right]. \quad (3.7)$$

Comparing equal powers of x Eqs.(3.5) and (3.7), we obtain the constants C_0 , C_1 and C_2 [7,8]:

$$C_0 = 1 - \frac{4}{\alpha} + \frac{12}{\alpha^2}, \quad C_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2}, \quad C_2 = \frac{48}{\alpha^2}.$$

Instead of solving the radial part Klein-Gordon equation for the centrifugal potential $V_l(r)$ given by Eq.(3.5), we now solve the radial part Klein-Gordon equation for the new centrifugal potential $V_l^*(r)$ given by Eq.(3.6) obtained using the Pekeris approximation. Having inserted this new centrifugal potential into Eq.(3.4), we obtain:

$$\frac{d^2 u}{dr^2} + \left[\left(\frac{E^2 - m_0^2 c^4}{\hbar^2 c^2} - \frac{l(l+1)C_0}{\alpha^2 a^2} \right) + \frac{\left(\frac{2EV_0}{\hbar^2 c^2} - \frac{l(l+1)C_1}{\alpha^2 a^2} \right)}{1 + e^{\frac{r-R_0}{a}}} + \frac{\left(\frac{V_0^2}{\hbar^2 c^2} - \frac{l(l+1)C_2}{\alpha^2 a^2} \right)}{\left(1 + e^{\frac{r-R_0}{a}} \right)^2} \right] u = 0. \quad (3.8)$$

We use the following dimensionless notations:

$$\epsilon^2 = - \left(\frac{(E^2 - m_0^2 c^4) a^2}{\hbar^2 c^2} - \frac{l(l+1)C_0}{\alpha^2} \right); \quad \beta^2 = \frac{2EV_0 a^2}{\hbar^2 c^2} - \frac{l(l+1)C_1}{\alpha^2}; \quad \gamma^2 = - \left(\frac{V_0^2 a^2}{\hbar^2 c^2} - \frac{l(l+1)C_2}{\alpha^2} \right), \quad (3.9)$$

with real $\epsilon > 0$ ($E^2 \leq m_0^2 c^2$) for bound states; β is real, γ is real and positive.

If we rewrite Eq.(3.8) by using a new variable of the form [4]:

$$z = \left(1 + \exp \left(\frac{r - R_0}{a} \right) \right)^{-1},$$

we obtain:

$$u''(z) + \frac{1 - 2z}{z(1 - z)} u'(z) + \frac{-\epsilon^2 + \beta^2 z - \gamma^2 z^2}{(z(1 - z))^2} u(z) = 0, \quad (0 \leq z \leq 1), \quad (3.10)$$

with $\tilde{\tau}(z) = 1 - 2z$; $\sigma(z) = z(1 - z)$; $\tilde{\sigma}(z) = -\epsilon^2 + \beta^2 z - \gamma^2 z^2$.

In the NU-method the new function $\pi(z)$ is:

$$\pi(z) = \pm \sqrt{\epsilon^2 + (k - \beta^2) z - (k - \gamma^2) z^2}. \quad (3.11)$$

The constant parameter k can be found employing the condition that the expression under the square root has a double zero, i.e., its discriminant is equal to zero. So, there are

two possible functions for each k :

$$\pi(z) = \pm \begin{cases} \left(\epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) z - \epsilon, & \text{for } k = \beta^2 - 2\epsilon^2 + 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}, \\ \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) z - \epsilon, & \text{for } k = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \end{cases} \quad (3.12)$$

According to the NU-method, from the four possible forms of the polynomial $\pi(z)$ we select the one for which the function $\tau(z)$ has the negative derivative and root lies in the interval $(0,1)$. Therefore, the appropriate functions $\pi(z)$ and $\tau(z)$ have the following forms:

$$\pi(z) = \epsilon - \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) z, \quad (3.13)$$

$$\tau(z) = 1 + 2\epsilon - 2 \left(1 + \epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) z, \quad (3.14)$$

and

$$k = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \quad (3.15)$$

Then, the constant $\lambda = k + \pi'(z)$ is written as:

$$\lambda = \beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2} - \epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2}. \quad (3.16)$$

An alternative definition of λ_n (Eq.(2.9)) is:

$$\lambda = \lambda_n = 2 \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) n + n(n+1). \quad (3.17)$$

Having compared Eqs.(3.16) and (3.17):

$$\beta^2 - 2\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \beta^2 + \gamma^2} - \epsilon - \sqrt{\epsilon^2 - \beta^2 + \gamma^2} = 2 \left(\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} \right) n + n(n+1), \quad (3.18)$$

we obtain

$$\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} + n + \frac{1}{2} - \frac{\sqrt{1+4\gamma^2}}{2} = 0, \quad (3.19)$$

or

$$\epsilon + \sqrt{\epsilon^2 - \beta^2 + \gamma^2} - n' = 0. \quad (3.20)$$

Here

$$n' = -n + \frac{\sqrt{1+4\gamma^2} - 1}{2}, \quad (3.21)$$

n being the radial quantum number ($n = 0, 1, 2, \dots$).

After substituting α, γ, C_2 into Eq.(3.21), we obtain:

$$n' = -n + \frac{\sqrt{1 + \frac{192a^4l(l+1)}{R_0^4} - \frac{4V_0^2a^2}{\hbar^2c^2}} - 1}{2}. \quad (3.22)$$

From Eq.(3.20), we find:

$$\epsilon = \frac{1}{2} \left(n' + \frac{\beta^2 - \gamma^2}{n'} \right). \quad (3.23)$$

Because for the bound states $\epsilon > 0$, we get:

$$n' > 0. \quad (3.24)$$

This physically interesting point imposes that $n' > 0$, which certainly determines the number of physically meaningful bound states for a deep potential appearing near the surface. If $n' > 0$, there exist bound states, otherwise, there are no bound states at all. By using Eq.(3.22) this relation can be recast into the form:

$$0 \leq n < \frac{\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2a^2}{\hbar^2c^2}} - 1}{2} \quad (3.25)$$

i.e. it gives the finite coupling value.

The condition $\gamma > 0$ gives the define coupling value for the potential depth V_0 :

$$0 < V_0 < \frac{4\hbar ca\sqrt{3l(l+1)}}{R_0^2}. \quad (3.26)$$

From Eqs. (3.23) and (3.9), we obtain:

$$E_{nl} = -\frac{V_0}{2} \left(1 - \frac{l(l+1)(C_1+C_2)}{\alpha^2 \left(n'^2 + \frac{4V_0^2a^2}{\hbar^2c^2} \right)} \right) \pm cn' \sqrt{\frac{m_0^2c^2 + \frac{\hbar^2l(l+1)}{R_0^2}C_0}{n'^2 + \frac{4V_0^2a^2}{\hbar^2c^2}} - \frac{\hbar^2}{4a^2} \left(1 - \frac{l(l+1)(C_1+C_2)}{\alpha^2 \left(n'^2 + \frac{4V_0^2a^2}{\hbar^2c^2} \right)} \right)^2} \quad (3.27)$$

Thus, substituting the expressions of α, C_0, C_1, C_2 and n' into Eq.(3.27), one can find the

energy eigenvalues E_{nl}

$$\begin{aligned}
E_{nl} = & -\frac{V_0}{2} \left(1 - \frac{32l(l+1)a^3}{R_0^3 \left[\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right]} \right) \pm \\
& c \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n - 1 \right) \left\{ \frac{m_0^2 c^2 + \frac{\hbar^2 l(l+1)}{R_0^2} \left(1 - \frac{4a}{R_0} + \frac{12a^2}{R_0^2} \right)}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2}} - \right. \\
& \left. \frac{\hbar^2}{16a^2} \left(1 - \frac{32l(l+1)a^3}{R_0^3 \left[\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - \frac{4V_0^2 a^2}{\hbar^2 c^2}} - 2n - 1 \right)^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right]} \right)^2 \right\}^{\frac{1}{2}} \quad (3.28)
\end{aligned}$$

If all two conditions (3.25) and (3.26) are satisfied simultaneously, the bound states exist. From Eq.(3.25) is seen that if $l = 0$, then one gets $n < 0$. Hence, the Klein-Gordon equation for the standard Woods-Saxon potential with zero angular momentum has no bound states. For larger values of V_0 ($V_0 > \frac{4\hbar a c \sqrt{3l(l+1)}}{R_0^2}$) the condition (3.25) is not satisfied. Therefore, no bound states exist for these values of V_0 .

According to Eq.(3.28) the energy eigenvalues depend on the depth of the potential V_0 , the width of the potential R_0 , and the surface thickness a . Any energy eigenvalue must be less than V_0 . If constraints imposed on n and V_0 are satisfied, the bound states appear. From Eq.(3.26) is seen that the potential depth increases when the parameter a increases, but the parameter R_0 is decreasing for given l quantum number and vice versa. Therefore, one can say that the bound states exist within this potential. Thus, the energy spectrum Eq.(3.28) are limited, i.e., we have only the finite number of energy eigenvalues.

The binding energy of a bound Klein-Gordon particle is defined as [18]:

$$E_b = E_{nl} - m_0 c^2. \quad (3.29)$$

By using the empirical values $r_0 = 1.285 fm$ and $a = 0.65 fm$ taken from Ref.[19] the potential depth $V_0 = (40.5 + 0.13A) MeV$ and the radius of the nucleus $R_0 = r_0 A^{1/3} fm$ are calculated for the atomic mass number of target nucleus $A = 40; 56; 66; 92; 140; 208$ and pions with mass $m_0 c^2 = 139.570 MeV$. In Table 1, energies of the bound states obtained numerically for the spherical standard Woods-Saxon potential for some values of l and n are given. From Table 1 is seen that for fixed n the energy of the bound states increases with increase of l . This means that due to the centrifugal potential $V_l(r)$ in the system the

repulsive forces appear. Therefore, in order to compensate this potential the energy of the bound state must increase [20].

In addition, we have seen that there are some restrictions on the potential parameters in order to obtain bound state solutions. We also point out that the exact results obtained for the standard Woods-Saxon potential may have some interesting applications for studying different quantum mechanical and nuclear scattering problems.

However, in the nonrelativistic limit with the mapping $E_{nl}^{(R)} - m_0c^2 \rightarrow E_{nl}^{(NR)}$, according to Appendix A for bound state energy eigenvalues, we obtain [8]:

$$E_{nl} = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2} \left(1 + \frac{12a^2}{R_0^2} \right) - \frac{\hbar^2}{2m_0 a^2} \left[\frac{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right)^2}{16} + \frac{4 \left(\frac{m_0 V_0 a^2}{\hbar^2} - \frac{4l(l+1)a^3}{R_0^3} \right)^2}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right)^2} + \frac{m_0 V_0 a^2}{\hbar^2} \right] \quad (3.30)$$

Let us note, that in work [12] the Klein-Gordon and Schrödinger equations with Woods-Saxon potential for $l \neq 0$ states was solved, the bound state energy eigenvalues and the corresponding eigenfunctions were found. But in the work [12] errors were made in application of the NU method to Schrödinger equation, which led, as in papers [15, 16] to wrong predictions for the bound energy eigenvalues and the corresponding eigenfunctions. Such errors were made also in the works [9, 13, 14].

Now, we are going to determine the radial eigenfunctions of this potential. Having substituted $\pi(z)$ and $\sigma(z)$ into Eq.(2.4) and then solving first-order differential equation, one can find the finite function $\Phi(z)$ in the interval $(0, 1)$

$$\Phi(z) = z^\epsilon (1 - z)^{\sqrt{\epsilon^2 - \beta^2 + \gamma^2}}. \quad (3.31)$$

It is easy to find the second part of the wave function from the definition of weight function:

$$\rho(z) = z^{2\epsilon} (1 - z)^{2\sqrt{\epsilon^2 - \beta^2 + \gamma^2}}, \quad (3.32)$$

and substituting into Rodrigues relation (2.5), we get

$$y_n(z) = B_n z^{-2\epsilon} (1 - z)^{-2\sqrt{\epsilon^2 - \beta^2 + \gamma^2}} \frac{d^n}{dz^n} \left[z^{n+2\epsilon} (1 - z)^{n+2\sqrt{\epsilon^2 - \beta^2 + \gamma^2}} \right]. \quad (3.33)$$

where $B_n = \frac{1}{n!}$ is the normalization constant [21]. Then, y_n is given by the Jacobi polynomials

$$y_n(z) = P_n^{(2\epsilon, 2\sqrt{\epsilon^2 - \beta^2 + \gamma^2})}(1 - 2z), \quad (3.34)$$

where

$$P_n^{(\alpha,\beta)}(1-2z) = \frac{1}{n!} z^{-\alpha} (1-z)^{-\beta} \frac{d^n}{dz^n} \left[z^{n+\alpha} (1-z)^{n+\beta} \right].$$

The corresponding $u_{nl}(z)$ radial wave functions are found to be

$$u_{nl}(z) = C_{nl} z^\epsilon (1-z)^{\sqrt{\epsilon^2-\beta^2+\gamma^2}} P_n^{(2\epsilon, 2\sqrt{\epsilon^2-\beta^2+\gamma^2})}(1-2z), \quad (3.35)$$

where C_{nl} is the normalization constants determined using $\int_0^\infty [u_{nl}(r)]^2 dr = 1$ constraint, i.e.,

$$aC_{nl}^2 \int_0^1 z^{2\epsilon-1} (1-z)^{2\sqrt{\epsilon^2-\beta^2+\gamma^2}-1} \left[P_n^{(2\epsilon, 2\sqrt{\epsilon^2-\beta^2+\gamma^2})}(1-2z) \right]^2 dz = 1.$$

IV. CONCLUSION

In this paper, we have analytically calculated energy eigenvalues of the bound states and corresponding eigenfunctions in the new exactly solvable Woods-Saxon potential. The energy eigenvalue expression for Woods-Saxon potentials is given by Eq.(3.28). The nonrelativistic limit (3.30) of the bound state energy spectrum was also obtained. As it should be expected, for any given set of parameters V_0 , R_0 and a , the energy levels of standard Woods-Saxon potential are positive. The obtained results are interesting for both theoretical and experimental physicists, because they provide exact expression for energy eigenvalues and corresponding eigenfunctions.

Appendix A

After some simple transformations Eq.(3.27) takes the form:

$$E_{nl}^{(R)} = -\frac{V_0}{2} \left(1 - \frac{l(l+1)(C_1+C_2)}{\alpha^2 \left(n'^2 + \frac{V_0^2 a^2}{\hbar^2 c^2} \right)} \right) + m_0 c^2 \left\{ 1 + \frac{\hbar^2 l(l+1)C_0}{m_0^2 c^2 R_0^2} - \frac{V_0^2 a^2}{\hbar^2 c^2 \left(n'^2 + \frac{V_0^2 a^2}{\hbar^2 c^2} \right)} - \frac{4\hbar^2 l(l+1)C_0 V_0^2 a^2}{\hbar^2 c^4 m_0^2 R_0^2 \left(n'^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right)} - \frac{\hbar^2 n'^2}{4m_0^2 c^2 a^2} \left(1 - \frac{l(l+1)(C_1+C_2)}{\alpha^2 \left(n'^2 + \frac{4V_0^2 a^2}{\hbar^2 c^2} \right)} \right)^2 \right\}^{\frac{1}{2}} \quad (\text{A1})$$

Expanding, finally, the above eigenvalue in a series of powers of $\frac{1}{c^2}$ yields:

$$\begin{aligned} E_{nl}^{(R)} &= -\frac{V_0}{2} \left[1 - \frac{4l(l+1)(C_1+C_2)}{\alpha^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} \right] + m_0 c^2 \left\{ 1 + \frac{\hbar^2 l(l+1)C_0}{2m_0^2 c^2 R_0^2} - \frac{4V_0^2 a^2}{2\hbar^2 c^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} - \frac{\hbar^2}{32m_0^2 c^2 a^2} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2 \times \right. \\ &\left. \left[1 - \frac{4l(l+1)(C_1+C_2)}{\alpha^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} \right]^2 \right\} + \dots = -\frac{V_0}{2} + \frac{2l(l+1)(C_1+C_2)V_0}{\alpha^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} - \\ &m_0 c^2 + \frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{2m_0 V_0^2 a^2}{\hbar^2 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} - \frac{\hbar^2}{32m_0 a^2} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2 + \\ &\frac{\hbar^2 l(l+1)(C_1+C_2)}{4m_0 a^2 \alpha^2} - \frac{\hbar^2 l^2 (l+1)^2 (C_1+C_2)^2}{2m_0 a^2 \alpha^4 \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} + \dots = m_0 c^2 + \frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{V_0}{2} \\ &- \frac{1}{16} \frac{\hbar^2}{2m_0 a^2} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2 - \frac{\hbar^2}{2m_0 a^2} \frac{\left(\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2} \right)^2}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} + \\ &\frac{\hbar^2 l(l+1)(C_1+C_2)}{4m_0 a^2 \alpha^2} + \dots = m_0 c^2 + \frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{\hbar^2}{2m_0 a^2} \left[\frac{1}{16} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2 + \right. \\ &\left. \frac{1}{2} \left(\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2} \right) + \frac{\left(\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2} \right)^2}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right)^2} \right] + \dots = m_0 c^2 + \\ &\frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{\hbar^2}{2m_0 a^2} \left[\frac{1}{4} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right) + \frac{\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2}}{\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1} \right]^2 + o\left(\frac{1}{c^4}\right) \end{aligned} \quad (\text{A2})$$

Denoting, as usual, the energy eigenvalues in the nonrelativistic case by $E_{nl}^{(R)} - m_0 c^2 \rightarrow E_{nl}^{(NR)}$, we have:

$$E_{nl}^{(NR)} = \frac{\hbar^2 l(l+1)C_0}{2m_0 R_0^2} - \frac{\hbar^2}{2m_0 a^2} \left[\frac{1}{4} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1 \right) + \frac{\frac{2m_0 V_0 a^2}{\hbar^2} - \frac{l(l+1)(C_1+C_2)}{\alpha^2}}{\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n-1} \right]^2 \quad (\text{A3})$$

Thus, substituting the expressions of C_0, C_1 and C_2 into the above energy eigenvalues $E_{nl}^{(NR)}$, we obtain:

$$E_{nl}^{(NR)} = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2} \left(1 - \frac{4a}{R_0} + \frac{12a^2}{R_0^2} \right) - \frac{\hbar^2}{2m_0 a^2} \left[\frac{1}{4} \left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right) + \frac{2 \left(\frac{m_0 V_0 a^2}{\hbar^2} - \frac{4l(l+1)a^3}{R_0^3} \right)}{\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4} - 2n - 1}} \right]^2 \quad (\text{A4})$$

OR

$$E_{nl}^{(NR)} = \frac{\hbar^2 l(l+1)}{2m_0 R_0^2} \left(1 + \frac{12a^2}{R_0^2} \right) - \frac{\hbar^2}{2m_0 a^2} \left[\frac{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right)^2}{16} + \frac{4 \left(\frac{m_0 V_0 a^2}{\hbar^2} - \frac{4l(l+1)a^3}{R_0^3} \right)^2}{\left(\sqrt{1 + \frac{192l(l+1)a^4}{R_0^4}} - 2n - 1 \right)^2} + \frac{m_0 V_0 a^2}{\hbar^2} \right] \quad (\text{A5})$$

-
- [1] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251** (1995) 267.
- [2] D. A. Morales, Chem. Phys. Lett. **394** (2004) 68.
- [3] C. L. Pekeris, Phys. Rev. **45** (1934) 98.
- [4] S. Flügge, Practical Quantum Mechanics, Vol.1 (Springer, Berlin, 1994)
- [5] O. Bayrak and I. Boztosun, J. Phys. A **39** (2006) 6955 (arxiv: nucl-th / 0604042 V1).
- [6] O. Bayrak, G. Kocak and I. Boztosun, J. Phys. A: Math. Gen. **39** (2006) 11521 (arxiv: math-ph / 0609010 V1).
- [7] V. H. Badalov, H. I. Ahmadov and S. V. Badalov, News of Baku University, N2 (2008) 157.
- [8] V. H. Badalov, H. I. Ahmadov and A. I. Ahmadov, Int.J.Mod.Phys. E **18** (2009) 631 (arxiv: math-ph / 0905.273 V1).
- [9] S. M. Ikhdair and R. Sever, (arxiv: quant-ph /0610183 V1).
- [10] H. Egrifes and R. Sever, Int. J. Theor. Phys. **46** (2007) 935 (arxiv: quant-ph / 0609231).
- [11] R. D. Woods and D. S. Saxon , Phys. Rev. **95** (1954) 577.
- [12] A. Arda and R. Sever, Int. J. Mod. Phys. C **20** (2009) 651 (arxiv: math-ph /0901.2773 V1).
- [13] A. Arda and R. Sever, Int.J.Mod.Phys. A **24** (2009) 3985 (arxiv: quant-ph/0902.2088 V1).
- [14] A.Berkdemir, C.Berkdemir and R.Sever, Mod.Phys.Lett. A **21**, 2087 (2006) (arxiv: quant-ph/0410153 V3).
- [15] C. Berkdemir, A. Berkdemir and R. Sever, Phys. Rev. C **72** (2005) 027001.
- [16] Editorial Note to Phys. Rev. C **72**, (2005) 027001; Phys. Rev. C **74** (2006) 039902(E).
- [17] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics (Birkhäuser, Basel, 1988).
- [18] W. Greiner, Relativistic Quantum Mechanics (Springer, Berlin, 1990).
- [19] C. M. Perey, F. G. Perey, J. K. Dickens and R. J. Silva, Phys. Rev. **175** (1968) 1460.
- [20] L.E. Ballentine, Quantum Mechanics (World Scientific Publishing, Singapore, 1998).
- [21] H. Bateman and A. Erdelyi, Higher Transcendental functions, Vol.2 (McGraw-Hill, New York, 1953)

A	R_0, fm	V_0, MeV	n	l	E_b, MeV
40	4.3946	45.70	0	1	-107.8777
56	4.9162	47.78	0	1	-127.5238
56	4.9162	47.78	0	2	-17.5985
66	5.1930	49.08	0	2	-50.3359
92	5.8010	52.46	0	2	-101.8967
140	6.6724	58.70	0	3	-92.5327
208	7.6136	67.54	0	4	-105.0865
208	7.6136	67.54	0	5	-33.6014

TABLE I: Energies of the bound states for the Woods-Saxon potential for different values of n, l calculated using Eqs.(3.25), (3.26), (3.28) and (3.29).