Tornehave morphisms, II: The lifted Tornehave morphism and the dual of the Burnside functor

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We introduce the lifted Tornehave morphism $\text{torn}^\pi : K \to B^*$, an inflation Mackey morphism for finite groups, π being a set of primes, K the kernel of linearization, and B^* the dual of the Burnside functor. For p-groups, torn^p is unique up to scalar multiples. It induces two morphisms of biset functors, one with a codomain associated with a subgroup of the Dade group, the other with a codomain associated with a quotient of the Burnside unit group.

1. Introduction

The first paper [2] of a trilogy was concerned with the reduced Tornehave morphism $\overline{\text{torn}}^\pi$, which can be regarded as a kind of π-adic analogue of the reduced exponential morphism $\overline{\exp}$. Here, π is a set of rational primes. For both of these morphisms, the codomain is the Burnside unit functor $B^\times$. The present paper, the second in the trilogy, introduces the lifted Tornehave morphism torn^π, a kind of π-adic analogue of the lifted exponential morphism $\exp()$. For the two lifted morphisms, the codomain is the dual B^* of the Burnside functor B.

The defining formulas for the lifted morphisms $\exp()$ and torn^π are much the same as the defining formulas for the reduced morphisms $\overline{\exp}$ and $\overline{\text{torn}}^\pi$, except that the codomain B^* of the lifted morphisms is a biset functor over $\mathbb{Z}$ whereas the codomain $B^\times$ of the reduced morphisms is a biset functor over the field $\mathbb{F}_2$ with order 2. One advantage of working with coefficients in $\mathbb{Z}$ rather than coefficients in $\mathbb{F}_2$ is that it enables us to extend to coefficients in $\mathbb{Q}$ and then to characterize $\exp()$ and torn^π in terms of their actions on the primitive idempotents of the Burnside ring. That leads to some uniqueness theorems which characterize $\exp()$ for arbitrary finite groups and torn^π for finite p-groups. All the uniqueness theorems are in the form of assertions that, up to scalar multiples, $\exp()$ and torn^π are the only morphisms satisfying certain conditions.

The third paper [3] of the trilogy concerns an isomorphism of Bouc [9, 6.5] whereby, for finite 2-groups, a difference between real and rational representations is related to a difference between rhetorical and rational biset functors. The main result in [3] asserts that Bouc's isomorphism is induced by the morphism $\overline{\text{torn}}^\pi$ (in the case $2 \in \pi$). The difficulty in achieving that result lies in the fact that two different kinds of morphism are involved. Bouc's isomorphism is an isomorphism of biset functors; it commutes with isogation, induction, restriction, inflation and deflation. On the other hand, torn^π and $\overline{\text{torn}}^\pi$ are merely inflaky morphisms (inflation Mackey morphisms); they commute with isogation, induction, restriction and inflation but not with deflation. It is easy to see that, granted its existence, then Bouc's isomorphism is the unique morphism of biset functors with the specified domain and codomain. It is not hard to see that $\overline{\text{torn}}^\pi$ induces a non-zero inflaky morphism with that domain and codomain. The trouble is in proving that $\overline{\text{torn}}^\pi$ induces a morphism of biset functors. At the end of the present paper, we deal with that crucial part of the argument by passing to the lifted morphism torn^π.

Along the way, it transpires that, for finite p-groups, torn^π induces a morphism of biset functors whose codomain D^Ω is associated with the subgroup of the Dade group generated by the relative syzygies. The question as to the interpretation of that result is left open.

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2. Conclusions

We shall be concerned with the functors and morphisms that appear in the following two commutative diagrams. All of these functors are biset functors and all of the morphisms in the left-hand diagram are morphisms of biset functors but, as we noted above, torn^π and $\overline{\text{torn}}^\pi$ commute only with isogation, induction, restriction and inflation, not with deflation, so the right-hand diagram is only a commutative square of inflaky morphisms.
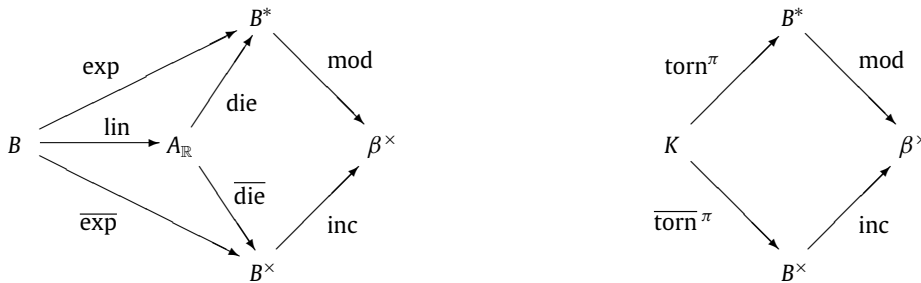
The lower and middle parts of the two diagrams have already been discussed in [2]. Let us review the notation. For a finite group G, the Burnside ring of G, denoted $B(G)$, is the coordinate module of the Burnside functor B. The unit group of $B(G)$, denoted $B^\times(G)$, is the coordinate module of the Burnside unit functor $B^\times$. The real representation ring of G, denoted $A_\mathbb{R}(G)$, is the coordinate module of the real representation functor $A_\mathbb{R}$. The linearization morphism lin() arises from the linearization map $\text{lin}_G : B(G) \to A_\mathbb{R}(G)$ whereby the isomorphism class $[X]$ of a finite G-set X is sent to the isomorphism class $[\mathbb{R}X]$ of the permutation $\mathbb{R}G$-module $\mathbb{R}X$. The reduced tom Dieck morphism $\overline{\text{die}}$ and the reduced exponential morphism $\overline{\exp}$ arise from the reduced tom Dieck map $\overline{\text{die}}_G : A_\mathbb{R}(G) \to B^\times(G)$ and the reduced exponential map $\overline{\exp}_G : B(G) \to B^\times(G)$. For the definitions of these two maps, see [2].

To make a study of the Burnside unit functor $B^\times$, we can extend to the ghost unit functor $\beta^\times$, whose coordinate module is the ghost unit group $\beta^\times(G) = \{x \in \mathbb{Q}B(G) : x^2 = 1\}$. We write inc to denote the morphism of biset functors whose coordinate map is the inclusion $\text{inc}_G : B^\times(G) \hookrightarrow \beta^\times(G)$. A further extension, introduced by Bouc [8, 7.2], is to realize $\beta^\times$ as the modulo 2 reduction of the dual B^* of the Burnside functor. We write mod to denote the morphism of biset functors whose coordinate map $\text{mod}_G : B^*(G) \to B^\times(G)$ is given by reduction from coefficients in $\mathbb{Z}$ to coefficients in $\mathbb{F}_2$. An explicit treatment of the lifted tom Dieck morphism die(), as a morphism of biset functors, appears in Bouc–Yalçin [11, Section 3]. Its coordinate map $\text{die}_G : A_\mathbb{R}(G) \to B^*(G)$ goes back to tom Dieck [12, Section III.5]. (Both of those sources refer to die_G as the "dimension function", denoted Dim.) Later in this section, we shall define the lifted exponential morphism exp() by means of a formula but, in Section 4, we shall find that $\exp = \text{die} \circ \text{lin}$. Thus, everything in the left-hand diagram above is already implicit in [11].

The biset functor $K = \text{Ker}(\text{lin})$ has seen an application to the study of Dade groups in Bouc [7, Sections 6, 7]. It also played an important role in the study of rational biset functors in [9, Section 6]. Its coordinate module $K(G) = \text{Ker}(\text{lin}_G)$ made an earlier appearance in connection with the reduced Tornehave map $\overline{\text{torn}}_G^\pi : K(G) \to B^\times(G)$ introduced by Tornehave [14]. In [2], it is shown that $\overline{\text{torn}}_G^\pi$ gives rise to an inflaky morphism $\overline{\text{torn}}^\pi$. Below, in this section, we shall introduce the lifted Tornehave map torn_G^π. In Section 4, we shall prove that torn_G^π gives rise to an inflaky morphism torn^π.

The morphisms exp() and torn^π are lifted from $\overline{\exp}$ and $\overline{\text{torn}}^\pi$ in the sense that we have commutative squares $\text{inc} \circ \overline{\exp} = \text{mod} \circ \exp$ and $\text{inc} \circ \overline{\text{torn}}^\pi = \text{mod} \circ \text{torn}^\pi$ as illustrated in the diagrams above. But those commutativity relationships between the two lifted morphisms and the two reduced morphisms will be examined only at the end of this paper, in Section 10. The rest of this paper is concerned with other features of exp() and torn^π.

Although exp() and torn^π are defined by means of formulas, we shall be presenting, in Section 5, some uniqueness theorems which characterize exp() and torn^π in a more structural way. The following result, an immediate consequence of Theorems 5.1 and 5.4, gives an indication of the kind of uniqueness properties that we shall be considering. In this section, for simplicity of discussion, we shall tend to confine our attention to p-biset functors, that is to say, biset functors whose coordinate modules are defined only for finite p-groups.

Theorem 2.1. *For finite p-groups, let M be a p-biset subfunctor of $_pB$, and let θ be a non-zero inflaky morphism $M \to {}_pB^*$. Then either $M = {}_pB$ and θ is a $\mathbb{Z}$-multiple of* exp()*, or else $M = {}_pK$ and $(p-1)\,\theta$ is a $\mathbb{Z}$-multiple of* torn^p.

Some of the notation, here, requires explanation. We sometimes write p-biset functors in the form $_pL$ just to emphasize the understanding that $_pL$ is indeed a p-biset functor and not a biset functor for arbitrary groups. When working with finite p-groups, we write $\text{torn}^p = \text{torn}^{\{p\}}$ and $\overline{\text{torn}}^p = \overline{\text{torn}}^{\{p\}}$. Actually, for finite p-groups, $\text{torn}^p = \text{torn}^\pi$ and $\overline{\text{torn}}^p = \overline{\text{torn}}^\pi$ for all π such that $p \in \pi$, while $\text{torn}^\pi = 0$ and $\overline{\text{torn}}^\pi = 0$ whenever $p \notin \pi$.

The kinship between exp() and torn^p becomes even more apparent upon comparing the next two theorems. The first of them, holding for arbitrary finite groups, follows immediately from Theorems 5.1 and 5.3.

Theorem 2.2. *If θ is an inflaky morphism or a deflaky morphism $B \to B^*$, then θ is a morphism of biset functors and, in fact, θ is a $\mathbb{Z}$-multiple of* exp()*.*

To state a closely analogous theorem for torn^p, we return to finite p-groups. We let $_pK^*$ be the quotient p-biset functor of $_pB^*$ such that the canonical projection $\pi^* : {_pB^*} \to {_pK^*}$ is the dual of the inclusion $_pK \hookrightarrow {_pB}$. Theorems 5.6 and 5.7 imply the next result.

Theorem 2.3. *For finite p-groups, if θ is an inflaky morphism or a deflaky morphism $_pK \to {_pK^*}$, then θ is a morphism of p-biset functors and, in fact, $(1 - p)\,\theta$ is a $\mathbb{Z}$-multiple of $\pi^* \circ \mathrm{torn}^p$.*

Already, the above results suggest that the morphisms $\exp()$ and torn^π are of some fundamental theoretical interest; the two morphisms seem to stand out and demand attention simply because of their uniqueness properties.

Putting aside the uniqueness properties now, the latest theorem tells us that torn^p induces a morphism of biset functors from $_pK$ to the quotient $_pK^*$ of $_pB^*$. Now, as we explained in Section 1, the crux of the proof of the main result in [3] is to show that torn^2 induces a morphism of biset functors from $_2K$ to a suitable quotient of $_2B^*$. Unfortunately, the quotient $_2K^*$ of $_2B^*$ turns out to be too coarse for the intended application. The next result has a stronger conclusion and, moreover, it holds in the context of functors defined for arbitrary finite groups.

Theorem 2.4. *Let $\pi^*_\mathbb{Q}$ be the canonical epimorphism of biset functors $B^* \to B^*/\exp(B)$. Then the composite $\pi^*_\mathbb{Q} \circ \mathrm{torn}^p/(1-p) : K \to B^*/\exp(B)$ is a morphism of biset functors.*

In Section 9, we shall prove Theorem 2.4 and we shall use it to deduce that, for finite p-groups, torn$^p/(1 - p)$ induces a morphism of p-biset functors $K \to D^\Omega$. In Section 10, using Theorem 2.4 again, we shall accomplish the crucial step towards the proof of the main result in the sequel paper [3].

3. Method

In this section, as well as introducing some notation, we shall make some comments on how we shall be proving the above theorems. This summary may be convenient for a casual reader who prefers not to delve into the details of the proofs.

The defining formulas for $\exp_G$ and torn^π_G are in terms of coordinate systems for $B(G)$ and $B^*(G)$ called the **square coordinate systems**. We define the **square basis** for $B(G)$ to be the $\mathbb{Z}$-basis $\{d^G_U : U \leq_G G\}$, where $d^G_U = [G/U]$ and the notation indicates that U runs over representatives of the conjugacy classes of subgroups of G. We define the **square basis** for $B^*(G)$ to be the corresponding dual $\mathbb{Z}$-basis, and we write it as $\{\delta^G_U : U \leq_G G\}$.

We define the **lifted exponential map** $\exp_G : B(G) \to B^*(G)$ to be the $\mathbb{Z}$-linear map such that, given a (finite) G-set X, then

$$\exp_G[X] = \sum_{U \leq_G G} |U \backslash X|\, \delta^G_U$$

where $U \backslash X$ denotes the set of U-orbits in X. We define the **lifted Tornehave map** $\mathrm{torn}^\pi_G : K(G) \to B^*(G)$ to be the restriction of the $\mathbb{Z}$-linear map $\widetilde{\mathrm{torn}}_G : B(G) \to B^*(G)$ such that

$$\widetilde{\mathrm{torn}}^\pi_G[X] = \sum_{U \leq_G G,\, \mathcal{U} \in U \backslash X} \log_\pi |\mathcal{U}|\, \delta^G_U.$$

Here, $\log_\pi$ is the function such that, given a positive integer n, and writing $n = p_1 \ldots p_r$ as a product of primes, then $\log_\pi(p_1 \ldots p_r) = |\{i : p_i \in \pi\}|$. Thus, the coefficient of δ^G_U in $\widetilde{\mathrm{torn}}^\pi_G[X]$ is a sum over the U-orbits in X, and the contribution from each U-orbit $\mathcal{U}$ is the number of prime factors of $|\mathcal{U}|$ that belong to π, counted up to multiplicity. To make it clear that the two defining formulas are matrix equations with respect to square coordinates, let us note that the formulas can be rewritten as

$$\exp_G(d^G_{U'}) = \sum_{U \leq_G G} |U \backslash G/U'|\, \delta^G_U, \qquad \widetilde{\mathrm{torn}}^\pi_G(d^G_{U'}) = \sum_{U \leq_G G,\, UgU' \subseteq G} \log_\pi |UgU'|\, \delta^G_U$$

where the notation indicates that UgU' runs over the elements of the set $U \backslash G/U'$ of double cosets of U and U' in G. In the next section, we shall show that $\exp_G$ and torn^π_G give rise to a morphism of biset functors $\exp : B \to B^*$ and an inflaky morphism $\mathrm{torn}^\pi : K \to B^*$.

By linear extension, we can regard $\exp_G$ and torn^π_G as $\mathbb{Q}$-linear maps $\exp_G : \mathbb{Q}B(G) \to \mathbb{Q}B^*(G)$ and $\mathrm{torn}^\pi_G : \mathbb{Q}K(G) \to \mathbb{Q}B^*(G)$. Hence – when we have checked the required commutativity properties in the next section – we shall obtain a morphism of biset functors $\exp : \mathbb{Q}B \to \mathbb{Q}B^*$ and an inflaky morphism $\mathrm{torn}^\pi : \mathbb{Q}K \to \mathbb{Q}B^*$. In Sections 7 and 8, we shall give formulas for $\exp_G$ and torn^π_G in terms of coordinate systems for $\mathbb{Q}B(G)$ and $\mathbb{Q}B^*(G)$ called the round coordinate systems.

Let us specify the bases associated with the round coordinate systems. For $I \leq G$, let ϵ^G_I be the algebra map $\mathbb{Q}B(G) \to \mathbb{Q}$ given by $[X] \mapsto |X^I|$, where X^I denotes the I-fixed subset of X. It is easy to show that, given $I' \leq G$, then $\epsilon^G_I = \epsilon^G_{I'}$ if and only if $I =_G I'$, moreover, $\{\epsilon^G_I : I \leq_G G\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}B^*(G)$. So there exists a unique element $e^G_I \in \mathbb{Q}B(G)$ such that $\epsilon^G_{I'}(e^G_I)$ is 1 of 0 depending on whether $I =_G I'$ or $I \neq_G I'$, respectively. Of course, $e^G_I = e^G_{I'}$ if and only if $I =_G I'$. The following easy remark is well-known.

Remark 3.1. Letting I run over representatives of the conjugacy classes of subgroups of G, then the elements e^G_I run over the primitive idempotents of $\mathbb{Q}B(G)$ without repetitions, furthermore, $\mathbb{Q}B(G) = \bigoplus_I \mathbb{Q}e^G_I$ as a direct sum of algebras $\mathbb{Q}e^G_I \cong \mathbb{Q}$. In particular, the set of primitive idempotents is a $\mathbb{Q}$-basis for $\mathbb{Q}B(G)$.

We define the **round bases** for $\mathbb{Q}B(G)$ to be the set $\{e_I^G : I \leq_G G\}$ of primitive idempotents of $\mathbb{Q}B(G)$. We define the **round basis** for $\mathbb{Q}B^*(G)$ to be the set $\{\epsilon_I^G : I \leq_G G\}$ of algebra maps $\mathbb{Q}B(G) \to \mathbb{Q}$.

Proposition 6.4 tells us that, allowing G to vary, then a family of maps $\theta_G : \mathbb{Q}B(G) \to \mathbb{Q}B^*(G)$ gives rise to a Mackey morphism $\theta : \mathbb{Q}B \to \mathbb{Q}B^*$ if and only if there is an isomorphism invariant $\Theta(G) \in \mathbb{Q}$ such that

$$\theta_G(e_I^G) = \frac{\Theta(I)}{|N_G(I) : I|} \epsilon_I^G$$

for all finite groups I and G with $I \leq G$. By first considering the case $I = G$, we shall evaluate the isomorphism invariant $\Theta^{\exp}(G)$ associated with the morphism $\exp()$. Proposition 7.3 says that $\exp_G(e_I^G)$ is non-zero if and only if I is cyclic, in which case

$$\exp_G(e_I^G) = \frac{\phi(|I|)}{|N_G(I)|} \epsilon_I^G$$

where ϕ is the Euler function. Let us note another way of expressing that formula in the case where I is a p-group. Given integers $d \geq c \geq 0$, we define

$$\beta_p(c, d) = \prod_{s=c-1}^{d-2} (1 - p^s)$$

with the understanding that $\beta(d, d) = 1$. If I is a p-group with rank d, then

$$\exp_G(e_I^G) = \frac{\beta_p(0, d)}{|N_G(I) : I|} \epsilon_I^G.$$

We shall apply a similar method to the morphism torn^p. The domain $\mathbb{Q}K(G)$ of torn^p has a $\mathbb{Q}$-basis consisting of those e_I^G such that I is non-cyclic. If I is a p-group with rank d, then I is non-cyclic if and only if $d \geq 2$, and in that case, Proposition 8.3 says that

$$\mathrm{torn}_G^p(e_I^G) = \frac{1-p}{p} \cdot \frac{\beta_p(2, d)}{|N_G(I) : I|} \epsilon_I^G.$$

The deflation map is not easy to describe in terms of the round coordinate system. Given $N \trianglelefteq G$ and writing $\overline{G} = G/N$, the **deflation number** for $\overline{G}$ and G is defined to be

$$\beta(\overline{G}, G) = \frac{1}{|G|} \sum_{S \leq G : SN = G} |S| \, \mu(S, G)$$

where μ denotes the Möbius function on the poset of subgroups of G. Bouc [4, page 706] showed that $\beta(\overline{G}, G)$ depends only on the isomorphism classes of G and $\overline{G}$. He also showed, in [4, Lemme 16], that the deflation map $\mathrm{def}_{\overline{G}, G} : \mathbb{Q}B(G) \to \mathbb{Q}B(\overline{G})$ is given by

$$\mathrm{def}_{\overline{G}, G}(e_I^G) = \frac{|N_{\overline{G}}(\overline{I}) : \overline{I}|}{|N_G(I) : I|} \beta(\overline{I}, I) \, e_{\overline{I}}^{\overline{G}}$$

where $\overline{I} = I/(I \cap N)$.

Some of the uniqueness theorems for $\exp()$ and torn^p, stated in Section 5, will be proved in Sections 6 and 8 by considering the constraints on Θ imposed by the condition that θ is an inflaky morphism or by the condition that θ is a deflaky morphism. Those two conditions are both characterized by the equation $\Theta(G) = \Theta(\overline{G}) \beta(\overline{G}, G)$. When we allow θ to have domain $\mathbb{Q}K$ or some other domain strictly contained in $\mathbb{Q}B$, the two conditions differ in the range of the pair of variables $(\overline{G}, G)$ for which the equation is required to hold. Nevertheless, both the inflaky morphisms and the deflaky morphisms are strongly constrained by the fact that, when $\Theta(G)$ and $\Theta(\overline{G})$ are defined, they determine each other unless $\beta(\overline{G}, G) = 0$.

In Appendix, we shall present a little application of the lifted Tornehave morphism. Using the round coordinate formulas for $\exp()$ and torn^p, we shall recover a result of Bouc–Thévenaz [10, 4.8, 8.1] which asserts that, if I is a p-subgroup of G, then

$$\mathrm{def}_{\overline{G}, G}(e_I^G) = \frac{|N_{\overline{G}}(\overline{I}) : \overline{I}|}{|N_G(I) : I|} \beta_p(c, d) \, e_{\overline{I}}^{\overline{G}}$$

where c and d are the ranks of $\overline{I}$ and I, respectively.

This paper does make much use of formulas and coordinates. No apology should be needed. The attraction of formulas, of course, is that they often speak back, saying more than one intended to put in; so they are likely to reveal more to some readers than they do to an author.

4. The lifted morphisms in square coordinates

Throughout, we shall be making use of the following variables. We always understand that H is a subgroup of G, that N is a normal subgroup of G and that $\phi : G \to F$ is a group isomorphism. We write $\overline{G} = G/N$ and, more generally, $\overline{H} = HN/N$. The groups H, $\overline{G}$, F will tend to be used when working with the five elemental maps: induction $\mathrm{ind}_{G,H}$, restriction $\mathrm{res}_{H,G}$,

inflation $\inf_{G,\overline{G}}$, deflation $\mathrm{def}_{\overline{G},G}$ and isogation $\mathrm{iso}_{F,G}^{\phi}$. We always understand that

$$U \leq G \geq I, \qquad V \leq H \geq J, \qquad N \leq W \leq G \geq K \geq N.$$

The subgroups $U \leq G$ and $V \leq H$ and $\overline{W} \leq \overline{G}$ will tend to be used when working with square coordinates. The subgroups $I \leq G$ and $J \leq H$ and $\overline{K} \leq \overline{G}$ will tend to be used when working with round coordinates.

We have good reason for making systematic use of variables and coordinates. Four coordinate systems will be coming into play: the square system for $\mathbb{Q}B(G)$, the round system for $\mathbb{Q}B(G)$, the square for $\mathbb{Q}B^{*}(G)$, the round for $\mathbb{Q}B^{*}(G)$. All four of the associated bases are indexed by the conjugacy classes of subgroups of G. For our purposes, it would no longer be convenient to continue with the notation in Bouc–Yalçın [11] whereby $B^{*}(G)$ is identified with the $\mathbb{Z}$-module $C(G)$ consisting of the $\mathbb{Z}$-valued functions on the set of conjugacy classes of subgroups of G. Indeed, our coordinate systems would yield four different identifications of $\mathbb{Q}B(G)$ or $\mathbb{Q}B^{*}(G)$ with $\mathbb{Q}C(G)$.

A scenario similar to ours is that of the canonical pairs of variables (p, q), as used in quantum mechanics, optics and signal processing. Where Dirac notation employs two bras $\langle p|$ and $\langle q|$ and two kets $|p\rangle$ and $|q\rangle$, the analogous notation in our context would be $\langle I|$ and $\langle U|$ and $|I\rangle$ and $|U\rangle$, respectively. But that formalism would require the reader to recognize the implied coordinate-system from the name of the variable. Such a device would be unsuitable in our context, so we shall make a compromise. We shall still make use of variables, but we shall explicitly indicate the coordinate system by using round or square brackets instead of angular brackets. Our notation is introduced below in a self-contained way, without any prerequisites concerning Dirac notation. But, for those who are familiar with Dirac notation, let us mention that the above bras and kets will be rendered as $\epsilon_{I}^{G} = (I @ -)$, $\delta_{U}^{G} = [U @ -]$, $e_{I}^{G} = (- @ I)$, $d_{U}^{G} = [- @ U]$, respectively.

Passing to coefficients in a commutative unital ring R, we replace the $\mathbb{Z}$-module $B^{*}(G) = \mathrm{Hom}_{\mathbb{Z}}(B(G), \mathbb{Z})$ with the R-module $RB^{*}(G) = R \otimes_{\mathbb{Z}} B^{*}(G)$, which can be identified with $\mathrm{Hom}_{\mathbb{R}}(RB(G), R)$. Let us write the duality between $RB^{*}(G)$ and $RB(G)$ as

$$RB^{*}(G) \times RB(G) \ni (\xi, x) \mapsto \langle \xi @ x \rangle \in R.$$

The expression $\langle \xi @ x \rangle$ may be read as: the value of ξ at x. The square bases $\{d_{U}^{G} : U \leq_{G} G\}$ and $\{\delta_{U}^{G} : U \leq_{G} G\}$ were introduced, in Section 3, as $\mathbb{Z}$-bases for $B(G)$ and $B^{*}(G)$, respectively. Of course, they are also R-bases for $RB(G)$ and $RB^{*}(G)$. The duality between then is expressed by the condition

$$\langle \delta_{U}^{G} @ d_{U'}^{G} \rangle = \lfloor U =_{G} U' \rfloor$$

where $U =_{G} U'$ means that U is G-conjugate to U', and the **logical delta symbol** $\lfloor P \rfloor$ is defined to be the integer 1 or 0 depending on whether a given statement P is true or false, respectively. The elements $\xi \in RB^{*}(G)$ and $x \in RB(G)$ have square coordinate decompositions

$$\xi = \sum_{U \leq_{G} G} [\xi @ U] \, \delta_{U}^{G}, \qquad x = \sum_{U \leq_{G} G} [U @ x] \, d_{U}^{G}$$

where $[\xi @ U] = \langle \xi @ d_{U}^{G} \rangle$ and $[U @ x] = \langle \delta_{U}^{G} @ x \rangle$. The elements $[\xi @ U] \in R \ni [U @ x]$ are called the **square coordinates** of ξ and x.

The isogation maps act on RB by transport of structure

$$\mathrm{iso}_{F,G}^{\phi}(d_{U}^{G}) = d_{\phi(U)}^{F}, \qquad [\phi(U) @ \mathrm{iso}_{F,G}^{\phi}(x)] = [U @ x].$$

The other four elemental maps act on RB by

$$\mathrm{res}_{H,G}(d_{U}^{G}) = \sum_{HgU \subseteq G} d_{H \cap {}^{g}U}^{H}, \qquad \mathrm{ind}_{G,H}(d_{V}^{H}) = d_{V}^{G}, \qquad \mathrm{def}_{\overline{G},G}(d_{U}^{G}) = d_{\overline{U}}^{\overline{G}}, \qquad \inf_{G,\overline{G}}(d_{\overline{W}}^{\overline{G}}) = d_{W}^{G}.$$

These four equations can be rewritten as

$$[V @ \mathrm{res}_{H,G}(x)] = \sum_{U \leq_{G} G, HgU \subseteq G : V =_{H} H \cap {}^{g}U} [U @ x], \qquad [U @ \mathrm{ind}_{G,H}(y)] = \sum_{V \leq_{H} H : V =_{G} U} [V @ y],$$

$$[\overline{W} @ \mathrm{def}_{\overline{G},G}(x)] = \sum_{U \leq_{G} G : \overline{U} =_{\overline{G}} \overline{W}} [U @ x], \qquad [U @ \inf_{G,\overline{G}}(z)] = \lfloor N \leq U \rfloor \, [\overline{U} @ z].$$

where $y \in RB(H)$ and $z \in RB(\overline{G})$. We mention that the deflation map $\mathrm{def}_{\overline{G},G}$ arises from the deflation functor which sends a G-set X to the $\overline{G}$-set of N-orbits $N \backslash X$.

The latest ten equations are the square-coordinate equations for the elemental maps on RB. Of course, there are really only five separate equalities here, each of them having been expressed in two different ways, as an action on basis elements and as an action on coordinates. We have recorded all of these equations because of the patterns that become apparent when comparing with the ten square-coordinate equations for the elemental maps on RB^{*}, which we shall record in a moment.

For a reason which will become clear in Section 10, we write the induction and deflation maps on RB^{*} as $\mathrm{jnd}_{G,H}$ and $\mathrm{jef}_{\overline{G},G}$. The action of a biset on a biset functor and the action of the opposite biset on the dual biset functor are related by

transposition; with respect to dual bases, the two matrices representing the two actions are the transposes of each other. So, in square coordinates, the matrices representing $\mathrm{res}_{H,G}, \mathrm{jnd}_{G,H}, \mathrm{jef}_{\overline{G},G}, \mathrm{inf}_{G,\overline{G}}, \mathrm{iso}_{G,F}^{\phi}$ on B^* are, respectively, the transposes of the matrices representing $\mathrm{ind}_{G,H}, \mathrm{res}_{H,G}, \mathrm{inf}_{G,\overline{G}}, \mathrm{def}_{\overline{G},G}, \mathrm{iso}_{F,G}^{\phi^{-1}}$ on B. We hence obtain another five pairs of equations,

$$\mathrm{iso}_{F,G}^{\phi}(\delta_U^G) = \delta_{\phi(U)}^F, \qquad [\mathrm{iso}_{F,G}^{\phi}(\xi) @ \phi(U)] = [\xi @ U],$$

$$\mathrm{res}_{H,G}(\delta_U^G) = \sum_{V \le_H H : V =_G U} \delta_V^G, \qquad [\mathrm{res}_{H,G}(\xi) @ V] = [\xi @ V],$$

$$\mathrm{jnd}_{G,H}(\delta_V^G) = \sum_{U \le_G G, HgU \subseteq G : V =_H H \cap {}^g U} \delta_U^G, \qquad [\mathrm{jnd}_{G,H}(\eta) @ U] = \sum_{HgU \subseteq G} [\eta @ H \cap {}^g U],$$

$$\mathrm{jef}_{\overline{G},G}(\delta_U^G) = \lfloor N \le U \rfloor \delta_{\overline{U}}^{\overline{G}}, \qquad [\mathrm{jef}_{\overline{G},G}(\xi) @ \overline{W}] = [\xi @ W],$$

$$\mathrm{inf}_{G,\overline{G}}(\delta_{\overline{W}}^{\overline{G}}) = \sum_{U \le_G G : \overline{U} =_{\overline{G}} \overline{W}} \delta_U^G, \qquad [\mathrm{inf}_{G,\overline{G}}(\zeta) @ U] = [\zeta @ \overline{U}].$$

Here, $\xi \in B^*(G)$ and $\eta \in B^*(H)$ and $\zeta \in B^*(\overline{G})$.

For a characteristic-zero field $\mathbb{K}$, the $\mathbb{K}$-representation functor $A_{\mathbb{K}}$ coincides with the $\mathbb{K}$-character functor. Its coordinate module $A_{\mathbb{K}}(G)$ is the $\mathbb{K}$-representation ring of G, which coincides with the $\mathbb{K}$-character ring; we mean to say, the ring of characters of $\mathbb{K}G$-modules. We shall neglect to distinguish between a $\mathbb{K}G$-character χ and the isomorphism class $[M]$ of a $\mathbb{K}G$-module M affording χ. Every $\mathbb{K}G$-character is a $\mathbb{C}G$-character, so $A_{\mathbb{K}}(G)$ is a subring of $A_{\mathbb{C}}(G)$. We write the inner product on $\mathbb{C}A_{\mathbb{C}}(G)$ as

$$\langle - | - \rangle^A : \mathbb{C}A_{\mathbb{C}}(G) \times \mathbb{C}A_{\mathbb{C}}(G) \to \mathbb{C}.$$

By restriction, we can regard $\langle - | - \rangle^A$ as a bilinear form on the real vector space $\mathbb{R}A_{\mathbb{R}}(G)$ or on the rational vector space $\mathbb{Q}A_{\mathbb{Q}}(G)$.

The induction, restriction and inflation maps on $A_{\mathbb{K}}$ are familiar to everyone and need no introduction. The isogation map comes from transport of structure in the evident way. In module-theoretic terms, deflation is given by $\mathrm{def}_{\overline{G},G}[M] = [M^N]$ where the $\mathbb{K}\overline{G}$-module M^N is the N-fixed subspace of M. We mention that, as $\mathbb{K}\overline{G}$-modules, M^N is isomorphic to the N-cofixed quotient space of M. In character-theoretic terms, $\mathrm{def}_{\overline{G},G}(\chi) = \chi^N$ where $\chi^N(gN)$ is the average value of $\chi(f)$ as f runs over the elements of the coset $gN \subseteq G$.

We can now start to discuss the morphisms. The **linearization map** $\mathrm{lin}_{\mathbb{K},G} : B(G) \to A_{\mathbb{K}}(G)$ is given by $\mathrm{lin}_{\mathbb{K},G}[X] = [\mathbb{K}X]$, where X is a G-set. The lifted **tom Dieck map** $\mathrm{die}_{\mathbb{K},G} : A_{\mathbb{K}}(G) \to B^*(G)$ is defined by

$$[\mathrm{die}_{\mathbb{K},G}[M] @ U] = \dim_{\mathbb{K}}(M^U)$$

for a $\mathbb{K}G$-module M. The dimension of M^U is the multiplicity of the trivial $\mathbb{K}G$-module in $\mathrm{res}_{U,G}(M)$. So, letting 1_U denote the trivial $\mathbb{K}$-character of U, the defining formula for $\mathrm{die}_{\mathbb{K},G}$ can be rewritten as

$$[\mathrm{die}_{\mathbb{K},G}(\chi) @ U] = \langle 1_U | \mathrm{res}_{U,G}(\chi) \rangle^A = \frac{1}{|U|} \sum_{g \in U} \chi(g)$$

for a $\mathbb{K}G$-character χ. Since $\mathrm{lin}_{\mathbb{K},G}$ and $\mathrm{die}_{\mathbb{K},G}$ are just restrictions of $\mathrm{lin}_{\mathbb{C},G}$ and $\mathrm{die}_{\mathbb{C},G}$, we can sometimes write lin_G and die_G without ambiguity. The **exponential map** $\exp_G : B(G) \to B^*(G)$, already defined in Section 2, is given by

$$[\exp_G[X] @ U] = |U \backslash X|$$

where $U \backslash X$ denotes the set of U-orbits in X. Plainly, $\exp_G = \mathrm{die}_G \circ \mathrm{lin}_G$.

The main content of the following result is the morphism property of $\mathrm{die}_{\mathbb{K}}$, which was established by Bouc–Yalçın [11, page 828]. Let us give a different proof.

Proposition 4.1 (*Bouc–Yalçın*). *The linearization map $\mathrm{lin}_{\mathbb{K},G}$, the lifted tom Dieck map $\mathrm{die}_{\mathbb{K},G}$ and the lifted exponential map $\exp_G$ give rise to morphisms of biset functors $\mathrm{lin}_{\mathbb{K}} : B \to A_{\mathbb{K}}$ and $\mathrm{die}_{\mathbb{K}} : A_{\mathbb{K}} \to B^*$ and $\exp : B \to B^*$.*

Proof. We must show that the three named maps commute with the five elemental maps. For lin_G, this commutativity property is easy and very well-known. Since $\exp_G = \mathrm{die}_G \circ \mathrm{lin}_G$, it suffices to deal with $\mathrm{die}_{\mathbb{K},G}$. For $\mathrm{die}_{\mathbb{K},G}$, the commutativity with restriction, inflation and isogation is obvious. The commutativity with deflation is easy. By Mackey Decomposition, Frobenius Reciprocity and the square-coordinate equation for induction on B^*, we have

$$[\mathrm{die}_{\mathbb{K},G}(\mathrm{ind}_{G,H}(\psi)) @ U] = \langle 1_U | \mathrm{res}_{U,G}(\mathrm{ind}_{G,H}(\psi)) \rangle^A = \sum_{UgH \subseteq G} \langle 1_{U \cap {}^g H} | \mathrm{res}_{U \cap {}^g H, {}^g H}({}^g \psi) \rangle^A$$

$$= \sum_{HgU \subseteq G} \langle 1_{H \cap {}^g U} | \mathrm{res}_{H \cap {}^g U, H}(\psi) \rangle^A = \sum_{HgU \subseteq G} [\mathrm{die}_{\mathbb{K},H}(\psi) @ H \cap {}^g U] = [\mathrm{jnd}_{G,H}(\mathrm{die}_{\mathbb{K},H}(\psi)) @ U]$$

for a $\mathbb{K}H$-character ψ. Therefore $\mathrm{die}_{\mathbb{K},G} \circ \mathrm{ind}_{G,H} = \mathrm{jnd}_{G,H} \circ \mathrm{die}_{\mathbb{K},H}$. $\square$

The morphisms $\mathrm{lin}_{\mathbb{K}}$ and $\mathrm{die}_{\mathbb{K}}$ and $\exp()$ are called the **linearization morphism**, the **lifted tom Dieck morphism**, the **exponential morphism**. Again, we may sometimes drop the subscript $\mathbb{K}$. Obviously, $\exp = \mathrm{die} \circ \mathrm{lin}$.

The biset functor $K = \mathrm{Ker}(\mathrm{lin}_{\mathbb{K}})$ is independent of $\mathbb{K}$. Its coordinate module $K(G)$ consists of those elements of $B(G)$ that can be written in the form $[X] - [X']$ where X and X' are G-sets satisfying the condition $\mathbb{K}X \cong \mathbb{K}X'$, or equivalently, $\mathbb{C}X \cong \mathbb{C}X'$. As in Section 2, we define the **Tornehave map** $\mathrm{torn}_G^\pi : K(G) \to B^*(G)$ to be the restriction of the $\mathbb{Z}$-linear map $\widetilde{\mathrm{torn}}_G^\pi : B(G) \to B^*(G)$ such that, given a G-set X, then

$$[\widetilde{\mathrm{torn}}_G^\pi[X] @ U] = \sum_{\mathcal{U} \in U \backslash X} \log_\pi |\mathcal{U}|.$$

We point out that the defining formulas for the exponential map and the Tornehave map differ only in the weighting assigned to each U-orbit $\mathcal{U}$. As a formal device, one could understand $\log_0$ to have constant value 1, and then one could write $\exp_G = \widetilde{\mathrm{torn}}_G^0$.

Lemma 4.2. *The maps $\widetilde{\mathrm{torn}}_G^\pi$ commute with restriction, inflation and isogation.*

Proof. Given an $\overline{G}$-set $\overline{X}$, then the $\overline{U}$-orbits of $\overline{X}$ can be identified with the U-orbits of the G-set inflated from $\overline{X}$. So the square-coordinate equation for inflation on B^* yields

$$[\widetilde{\mathrm{torn}}_G^\pi(\mathrm{inf}_{G,\overline{G}}[\overline{X}]) @ U] = \sum_{\overline{\mathcal{U}} \in \overline{U} \backslash \overline{X}} \log_\pi |\overline{\mathcal{U}}| = [\widetilde{\mathrm{torn}}_{\overline{G}}^\pi[\overline{X}] @ \overline{U}] = [\mathrm{inf}_{G,\overline{G}}(\widetilde{\mathrm{torn}}_{\overline{G}}^\pi[\overline{X}]) @ U].$$

The commutativity with restriction and isogation is even easier. $\square$

Theorem 4.3. *The Tornehave map torn_G^π gives rise to an inflaky morphism $\mathrm{torn}^\pi : K \to B^*$.*

Proof. Consider an element $\kappa \in K(H)$, and write $\kappa = [Y] - [Y']$ where Y and Y' are H-sets satisfying $\mathbb{C}Y \cong \mathbb{C}Y'$. Given $g \in G$, and writing $H(g) = H \cap {}^g U$, we have $|H(g) \backslash Y| = \langle 1_{H(g)} \mid \mathrm{res}_{H(g),G}[\mathbb{C}Y]\rangle^A$. So the integer

$$\Delta(H, U, g, Y) = |H(g) \backslash Y| \, \log_\pi |{}^g U : H(g)|$$

depends only on H, U, g and $[\mathbb{C}Y]$. That is to say, $\Delta(H, U, g, Y) = \Delta(H, U, g, Y')$. We have

$$\mathrm{res}_{U,G}(\mathrm{ind}_{G,H}[Y]) = \sum_{UgH \subseteq G, \, \mathcal{V} \in (U \cap {}^g H) \backslash {}^g Y} \mathrm{ind}_{U, U \cap {}^g H}[\mathcal{V}].$$

Since the maps $\widetilde{\mathrm{torn}}_G^\pi$ commute with restriction,

$$\begin{aligned}
[\widetilde{\mathrm{torn}}_G^\pi(\mathrm{ind}_{G,H}[Y]) @ U] &= [\mathrm{res}_{U,G}(\widetilde{\mathrm{torn}}_G^\pi(\mathrm{ind}_{G,H}[Y])) @ U] \\
&= \sum_{UgH, \mathcal{V}} \log_\pi(|U : U \cap {}^g H|.|\mathcal{V}|) = \sum_{HgU \subseteq G, \mathcal{U} \in H(g) \backslash Y} \log_\pi(|{}^g U : H(g)|.|\mathcal{U}|).
\end{aligned}$$

On the other hand, using a square-coordinate equation again,

$$[\mathrm{jnd}_{G,H}(\widetilde{\mathrm{torn}}_H^\pi[Y]) @ U] = \sum_{HgU}[\widetilde{\mathrm{torn}}_H^\pi[Y] @ H(g)] = \sum_{HgU, \mathcal{U}} \log_\pi |\mathcal{U}|.$$

It follows that

$$[\widetilde{\mathrm{torn}}_G^\pi(\mathrm{ind}_{G,H}[Y]) - \mathrm{jnd}_{G,H}(\widetilde{\mathrm{torn}}_H^\pi[Y]) @ U] = \sum_{HgU, \mathcal{U}} \log_\pi |{}^g U : H(g)| = \sum_{HgU} \Delta(H, U, g, Y).$$

Regrettably, since the terms $\Delta(H, U, g, Y)$ are non-negative and sometimes positive, the maps $\widetilde{\mathrm{torn}}_G^\pi$ do not commute with induction. However,

$$[\mathrm{torn}_G^\pi(\mathrm{ind}_{G,H}(\kappa)) - \mathrm{jnd}_{G,H}(\mathrm{torn}_H^\pi(\kappa)) @ U] = \sum_{HgU}(\Delta(H, U, g, Y) - \Delta(H, U, g, Y')) = 0$$

so the maps torn_G^π do commute with induction. $\square$

We shall end by giving an example to show that torn^π is not a deflaky functor, except in the trivial case $\pi = \emptyset$. First, we need a preliminary remark.

Remark 4.4. *If U is cyclic, then $[\mathrm{torn}_G^\pi(\xi) @ U] = 0$ for all $\xi \in K(G)$.*

Proof. We have $[\mathrm{torn}_G^\pi(\xi) @ U] = [\mathrm{res}_{U,G}(\mathrm{torn}_G^\pi(\xi)) @ U] = [\mathrm{torn}_U^\pi(\mathrm{res}_{U,G}(\xi)) @ U]$. But $K(U) = 0$, hence $\mathrm{res}_{U,G}(\xi) = 0$. $\square$

Example 4.5. Suppose that $G = C_p^2$, the elementary abelian p-group with rank 2. Let $A_0, \ldots, A_p$ be the subgroups of G with order p. Let $\kappa_2 = d_1^G - d_{A_0}^G - \cdots - d_{A_p}^G + p\, d_G^G$ and $\delta_2 = \delta_G^G$. Then $K(G)$ is the free cyclic $\mathbb{Z}$-module generated by κ_2, and $\mathrm{torn}_G^p(\kappa_2) = (1 - p)\delta_2$.

Proof. The equality $K(G) = \mathbb{Z}\kappa_2$ appears in Bouc [7, 6.5], and it can also be obtained very easily by examining the action of lin_G on the elements of the square basis of $B(G)$. If $U < G$, then U is cyclic and, by the latest remark, $[\mathrm{torn}_G^p(\kappa_2) @ U] = 0$. Therefore $\mathrm{torn}_G^p(\kappa_2) = [\mathrm{torn}_G^p(\kappa_2) @ G] d_G^G$. The transitive G-sets $G/1$, G/A_0, ..., G/A_p, G/G have sizes p^2, p, ..., p, 1, respectively. So $[\mathrm{torn}_G^p(\kappa_2) @ G] = \log_p(p^2) - (p+1)\log_p(p) + p\log_p(1) = 1 - p$. □

Corollary 4.6. *If $\pi \neq \emptyset$, then the inflaky morphism torn_G^π is not a deflaky morphism.*

Proof. Choosing $p \in \pi$ and putting $G = C_p^2$, we have $\mathrm{torn}_G^\pi = \mathrm{torn}_G^p$ and $\mathrm{jef}_{1,G}(\mathrm{torn}_G^\pi(\kappa_2)) = (1-p)\delta_1^1 \neq 0$, whereas $\mathrm{def}_{1,G}(\kappa_2) = 0$ and $\mathrm{torn}_1^\pi(\mathrm{def}_{1,G}(\kappa_2)) = 0$. □

5. The uniqueness theorems

In this section, we state five uniqueness theorems, and we give an entirely structuralistic proof ("conceptual", in the vernacular) for one of them, Theorem 5.1. The other four, Theorems 5.3, 5.4, 5.6, 5.7, will be proved in Sections 6 and 8 using techniques that are more formulaic (we mean, with an emphasis on designing notation that facilitates argument by manipulation of symbols).

Theorem 5.1 (*Uniqueness of* exp() *as an Inflaky Morphism*). *Let θ be an inflaky morphism $B \to B^*$. Then θ is a morphism of biset functors. In fact, θ is a $\mathbb{Z}$-multiple of* exp()*. Furthermore,* exp() *is the unique inflaky morphism $B \to B^*$ whose coordinate map from $B(1) = \mathbb{Z}d_1^1$ to $B^*(1) = \mathbb{Z}\delta_1^1$ is given by $d_1^1 \mapsto \delta_1^1$.*

Before proving this theorem, let us make some general comments. Recall that, over a field $\mathbb{F}$, the simple biset functors $S_{L,V}$ are parameterized by the pairs (L, V) where L is the minimal group such that $S_{L,V}(L) \neq 0$ and V is the simple $\mathbb{F}\mathrm{Out}(L)$-module, unique up to isomorphism, such that $S_{L,V}(L) \cong V$. The simple inflaky functors $S_{L,V}^{\mathrm{inf}}$, the simple deflaky functors $S_{L,V}^{\mathrm{def}}$ and the simple Mackey functors $S_{L,V}^{\mathrm{mac}}$ are parameterized in the same way. Yaraneri [17, 3.10] has shown that, if $\mathbb{F}$ has characteristic zero, then the simple inflaky functors and the simple deflaky functors restrict to the simple Mackey functors and, as Mackey functors, $S_{L,V}^{\mathrm{inf}} \cong S_{L,V}^{\mathrm{mac}} \cong S_{L,V}^{\mathrm{def}}$.

Directly from the definitions of duality for biset functors and for group algebras (mutually opposite elements act as mutually transpose maps), it is easy to show that $(S_{L,V})^* \cong S_{L,V^*}$. Evaluation at L yields a ring isomorphism $\mathrm{End}(S_{L,V}) \to \mathrm{End}(V)$. (The injectivity follows from the simplicity of $S_{L,V}$. The surjectivity holds because, by transport of structure, any $\mathbb{F}\mathrm{Out}(L)$-isomorphism $V \to V'$ extends to an isomorphism $S_{L,V} \to S_{L,V'}$.) Similar observations hold for the endomorphism algebras of the simple inflaky, simple deflaky and simple Mackey morphisms.

One more general comment is needed before we can prove the theorem. The following proposition has been known to experts for a long time. A proof for RB as a biset functor can be found in [1, 2.6], and a similar argument applies to RB as an inflaky functor. We mention that the inflaky functor case is also implicit in the proof of Yaraneri [17, 3.9].

Proposition 5.2. *As an inflaky functor and also as a biset functor, RB is projective. If R is a field, then the biset functor RB is the projective cover of $S_{1,R}$ and the inflation functor RB is the projective cover of $S_{1,R}^{\mathrm{inf}}$.*

We can now prove Theorem 5.1. Throughout the argument, we regard $\mathbb{Q}B$ and $\mathbb{Q}B^*$ as inflation functors. Since $\mathbb{Q}B(1)$ and $\mathbb{Q}B^*(1)$ are 1-dimensional, $S_{1,\mathbb{Q}}^{\mathrm{inf}}$ occurs exactly once as a composition factor of $\mathbb{Q}B$ and exactly once as a composition factor of $\mathbb{Q}B^*$. The latest proposition implies that, as hom-sets in the category of inflaky functors over $\mathbb{Q}$, we have $\mathrm{Mor}_{\mathrm{inf}}(\mathbb{Q}B, \mathbb{Q}B^*) \cong \mathrm{End}_{\mathrm{inf}}(S_{1,\mathbb{Q}}^{\mathrm{inf}}) \cong \mathbb{Q}$. Hence $\mathrm{Mor}_{\mathrm{inf}}(B, B^*) \cong \mathbb{Z}$. Finally, $\theta(d_1^1) = \lambda\delta_1^1$ for some $\lambda \in \mathbb{Z}$. If $\theta = \exp$, then $\lambda = 1$. So, in general, $\theta = \lambda \exp$. The proof of Theorem 5.1 is complete.

The proof of the following theorem, presented at the end of Section 6, will require some work using the round coordinate systems.

Theorem 5.3 (*Uniqueness of* exp() *as a Deflaky Morphism*). *Let D be a deflaky subfunctor of B. Then every deflaky morphism $D \to B^*$ is a $\mathbb{Q}$-multiple of the restriction of* exp()*. Furthermore,* exp() *is the unique deflaky morphism $B \to B^*$ such that $d_1^1 \mapsto \delta_1^1$.*

The next theorem, an analogue of Theorem 5.1, will be proved in Section 8.

Theorem 5.4 (*Uniqueness of* torn^p *as an Inflaky Morphism*). *For finite p-groups, the inflaky morphisms $_pK \to {}_pB^*$ are precisely the $\mathbb{Z}$-multiples of* torn$^p/(1-p)$*. Furthermore,* torn^p *is the unique inflaky morphism $_pK \to {}_pB^*$ such that, in the notation of Example 4.5, $\mathrm{torn}_{C_p^2}^p(\kappa_2) = (1-p)\delta_2$.*

Since torn^p is not a deflaky morphism, there can be no direct analogue of Theorem 5.3 for torn^p. However, as we shall see below, we can obtain a morphism of biset functors from torn^p by replacing the codomain $_pB^*$ with a suitable quotient p-biset functor. To introduce that quotient functor, we first need to review some results of Bouc concerning the structure of the p-biset functor $_pB$.

When G is a finite p-group, we define the pure $\mathbb{Z}$-sublattice $A_\mathbb{Q}^*(G) \leq B^*(G)$ to be the annihilator of the pure $\mathbb{Z}$-sublattice $K(G) \leq B(G)$ under the duality $\langle - | - \rangle : B^*(G) \times B(G) \to \mathbb{Z}$. In this way, we obtain a p-biset subfunctor $_pA_\mathbb{Q}^* \leq {}_pB^*$ and we can form the quotient p-biset functor $_pK^* = {}_pB^*/{}_pA_\mathbb{Q}^*$. The Ritter–Segal Theorem, recall, asserts that $\mathrm{lin} : {}_pB \to {}_pA_\mathbb{Q}^*$ is

an epimorphism. The dual of the inclusion $_pK \hookrightarrow {_pB}$ is the canonical epimorphism $\pi^* : {_pB^*} \to {_pK^*}$. Thus, we have two mutually dual short exact sequences of p-biset functors

$$\mathrm{Lin} : 0 \longrightarrow {_pK} \hookrightarrow {_pB} \overset{\mathrm{lin}}{\longrightarrow} {_pA_\mathbb{Q}} \longrightarrow 0, \qquad \mathrm{Lin}^* : 0 \longrightarrow {_pA_\mathbb{Q}^*} \overset{\mathrm{lin}^*}{\longrightarrow} {_pB^*} \overset{\pi^*}{\longrightarrow} {_pK^*} \longrightarrow 0.$$

We mention that the sequence Lin^* appears in Bouc [6, 1.8], where K^* is identified with $D^\Omega / D_{\mathrm{tors}}^\Omega$; for the notation, see Section 9. That paper gives a reference for Bouc's treatment of the Ritter–Segal Theorem in the context of p-biset functors. As a special case of Bouc–Thévenaz [10, 8.2, 8.3], we have isomorphisms of p-biset functors $_p\mathbb{Q}A_\mathbb{Q} \cong {_p\mathbb{Q}A_\mathbb{Q}^*} \cong S_{1,\mathbb{Q}}$ and $_p\mathbb{Q}K \cong {_p\mathbb{Q}K^*} \cong S_{C_p^2, \mathbb{Q}}$. So, extending to coefficients in $\mathbb{Q}$, we obtain the short exact sequences

$$\mathbb{Q}\mathrm{Lin} : 0 \to S_{C_p^2, \mathbb{Q}} \to {_p\mathbb{Q}B} \to S_{1,\mathbb{Q}} \to 0, \qquad \mathbb{Q}\mathrm{Lin}^* : 0 \to S_{1,\mathbb{Q}} \to {_p\mathbb{Q}B^*} \to S_{C_p^2, \mathbb{Q}} \to 0.$$

Proposition 5.5. *The morphisms of biset functors $_pK \to {_pK^*}$ are precisely the $\mathbb{Z}$-multiplies of $\pi^* \circ \mathrm{torn}^p/(1-p)$. Furthermore, the element $\overline{\delta}_2 = \pi_{C_p^2}^*(\delta_2)$ is a generator for the cyclic $\mathbb{Z}$-module $K^*(C_p^2)$, and $\pi^* \circ \mathrm{torn}^p/(1-p)$ is the unique morphism of p-biset functors $_pK \to {_pK^*}$ such that $\pi_{C_p^2}^*(\mathrm{torn}_{C_p^2}^p(\kappa_2)) = \overline{\delta}_2$.*

Proof. We have $\mathrm{Mor}_{\mathrm{bis}}(_p\mathbb{Q}K, {_p\mathbb{Q}K^*}) \cong \mathrm{End}_{\mathrm{bis}}(S_{C_p^2, \mathbb{Q}}, S_{C_p^2, \mathbb{Q}}) \cong \mathbb{Q}$ as hom-sets in the category of p-biset functors over $\mathbb{Q}$. Hence $\mathrm{Mor}_{\mathrm{bis}}(_pK, {_pK^*}) \cong \mathbb{Z}$. By Theorem 5.4, $\pi^* \circ \mathrm{torn}^p/(1-p) \in \mathrm{Mor}_{\mathrm{bis}}(_pK, {_pK^*})$. By Example 4.5, $\pi_G^*(\mathrm{torn}_G^p(\kappa_2)) = (1-p)\overline{\delta}_2$. Plainly, $\overline{\delta}_2$ generates the cyclic $\mathbb{Z}$-module $K^*(C_p^2)$. But, as we saw in Example 4.5, κ_2 generates the cyclic $\mathbb{Z}$-module $K(C_p^2)$. So $\pi^* \circ \mathrm{torn}^p/(1-p)$ generates the cyclic $\mathbb{Z}$-module $\mathrm{Mor}_{\mathrm{bis}}(_pK, {_pK^*})$. $\square$

The latest proposition is a uniqueness property of $\pi^* \circ \mathrm{torn}^p$ as a morphism of biset functors. In Section 8, we shall obtain the following two stronger results.

Theorem 5.6 (*Uniqueness of $\pi^* \circ \mathrm{torn}^p$ as an Inflaky Morphism*). *Every inflaky morphism $_pK \to {_pK^*}$ is a morphism of p-biset functors. Perforce, the inflaky morphisms $_pK \to {_pK^*}$ are precisely the $\mathbb{Z}$-multiplies of $\pi^* \circ \mathrm{torn}^p/(1-p)$. Moreover, $\pi^* \circ \mathrm{torn}^p/(1-p)$ is the unique inflaky morphism $_pK \to {_pK^*}$ such that $\kappa_2 \mapsto \overline{\delta}_2$.*

Theorem 5.7 (*Uniqueness of $\pi^* \circ \mathrm{torn}^p$ as a Deflaky Morphism*). *Let D be a deflaky subfunctor of $_pK$. Then every deflaky morphism $D \to {_pK^*}$ is the restriction of a $\mathbb{Q}$-multiple of $\pi^* \circ \mathrm{torn}^p$. Moreover, $\pi^* \circ \mathrm{torn}^p/(1-p)$ is the unique deflaky morphism $_pK \to {_pK^*}$ such that $\kappa_2 \mapsto \overline{\delta}_2$.*

Also in Section 8, we shall prove the following remark, which provides an explanation as to why there does not exist a non-zero deflaky morphism with the same domain and codomain as torn^p.

Remark 5.8. The short exact sequence $\mathbb{Q}\mathrm{Lin}$ splits as a sequence of deflaky morphisms but not as a sequence of inflaky morphisms. Equivalently, the dual sequence $\mathbb{Q}\mathrm{Lin}^*$ splits as inflaky morphisms but not as deflaky morphisms.

6. Round coordinates and diagonal invariants

We shall examine the Mackey morphisms having the form $\theta : M \to \mathbb{Q}B^*$ where $M \leq \mathbb{Q}B$. Corollary 6.5 describes how the pairs (M, θ) are parameterized by the pairs $(\mathcal{L}, \Theta)$ where $\mathcal{L}$ is a set of isomorphism classes of finite groups and $\Theta : \mathcal{L} \to \mathbb{Q}$ is a function. We shall give criteria, in terms of $\mathcal{L}$, for M to be an inflaky subfunctor and for M to be a deflaky subfunctor. We shall also give conditions, in terms of $(\mathcal{L}, \Theta)$, for θ to be an inflaky morphism and for θ to be a deflaky morphism. Towards the end of this section, we shall prove Theorem 5.3. All the material in this section generalizes easily to the case where $\mathbb{Q}$ is replaced by an arbitrary field with characteristic zero.

The round basis $\{e_I^G : I \leq_G G\}$ for $\mathbb{Q}B(G)$ and the round basis $\{\epsilon_I^G : I \leq G\}$ for $\mathbb{Q}B^*(G)$ were introduced in Section 3. The two bases are mutually dual in the sense that

$$\langle \epsilon_I^G @ e_{I'}^G \rangle = \lfloor I =_G I' \rfloor.$$

Given $\xi \in \mathbb{Q}B^*(G)$ and $x \in \mathbb{Q}B(G)$, then

$$\xi = \sum_{I \leq_G G} (\xi @ I)\, \epsilon_I^G, \qquad x = \sum_{I \leq_G G} (I @ x)\, e_I^G$$

where $(\xi @ I) = \langle \xi @ e_I^G \rangle$ and $(I @ x) = \langle \epsilon_I^G @ x \rangle$. We call $(\xi @ I)$ and $(I @ x)$ the **round coordinates** of ξ and x.

Recall that the transformation matrix from the round to the square coordinates of $\mathbb{Q}B(G)$ is the table of marks, whose (I, U)-entry is

$$m_G(I, U) = \langle \epsilon_I^G @ d_U^G \rangle = |\{gU \subseteq G : IgU = gU\}|.$$

Gluck's Idempotent Formula [13] expresses the (U, I)-entry of the inverse matrix as

$$m_G^{-1}(U, I) = \langle \delta_U^G @ e_I^G \rangle = \frac{|U|}{|N_G(I)|} \sum_{U' =_G U} \mu(U', I)$$

where the sum is over the subgroups U' that are G-conjugate to U, and μ denotes the Möbius function for the poset of subgroups of G. The defining equations for $m_G(I, U)$ and $m_G^{-1}(U, I)$ can be rewritten as

$$d_U^G = \sum_{I \leq_G G} m_G(I, U)\, e_I^G, \qquad \epsilon_I^G = \sum_{U \leq_G G} m_G(I, U)\, \delta_U^G,$$

$$e_I^G = \sum_{U \leq_G G} m_G^{-1}(U, I)\, d_U^G, \qquad \delta_U^G = \sum_{U \leq_G G} m_G^{-1}(U, I)\, \epsilon_I^G.$$

The round coordinate equations for the elemental maps on $\mathbb{Q}B$ were given by Bouc [4, Section 7]. They were reviewed in Bouc–Thévenaz [10, Section 8], but let us review them again, with a different notation. The deflation numbers $\beta(\overline{G}, G)$ were defined in Section 3. We write $\beta_G(\overline{G}, G) = \beta(\overline{G}, G)$ and, more generally,

$$\beta_G(\overline{I}, I) = \frac{|N_{\overline{G}}(\overline{I}) : \overline{I}|}{|N_G(I) : I|}\, \beta(\overline{I}, I).$$

The actions of the elemental maps are such that, for $x \in \mathbb{Q}B(G)$ and $y \in \mathbb{Q}B(H)$ and $z \in \mathbb{Q}B(\overline{G})$, we have

$$\mathrm{res}_{H,G}(e_I^G) = \sum_{J \leq_H H : J =_G I} e_J^H, \qquad (J @ \mathrm{res}_{H,G}(x)) = (J @ x),$$

$$\mathrm{ind}_{G,H}(e_J^H) = \frac{|N_G(J)|}{|N_H(J)|}\, e_J^G, \qquad (I @ \mathrm{ind}_{G,H}(y)) = \sum_{J \leq_H H : J =_G I} \frac{|N_G(J)|}{|N_H(J)|}\, (J @ y).$$

$$\mathrm{def}_{\overline{G},G}(e_I^G) = \beta_G(\overline{I}, I)\, e_{\overline{I}}^{\overline{G}}, \qquad (\overline{K} @ \mathrm{def}_{\overline{G},G}(x)) = \sum_{I \leq_G G : \overline{I} =_{\overline{G}} \overline{K}} \beta_G(\overline{I}, I)\, (I @ x),$$

$$\mathrm{inf}_{G,\overline{G}}(e_{\overline{K}}^{\overline{G}}) = \sum_{I \leq_G G : \overline{I} =_{\overline{G}} \overline{K}} e_I^G, \qquad (I @ \mathrm{inf}_{G,\overline{G}}(z)) = (\overline{I} @ z),$$

$$\mathrm{iso}_{F,G}^\phi(e_I^G) = e_{\phi(I)}^F, \qquad (\phi(I) @ \mathrm{iso}_{F,G}^\phi(x)) = (I @ x).$$

Dualizing by transposing the five matrices, we obtain the round-coordinate equations for the actions of the elemental maps on $\mathbb{Q}B^*$. Thus, for $\xi \in \mathbb{Q}B^*(G)$ and $\eta \in \mathbb{Q}B^*(H)$ and $\zeta \in \mathbb{Q}B^*(\overline{G})$, we have

$$\mathrm{res}_{H,G}(\epsilon_I^G) = \sum_{J \leq H : J =_G I} \frac{|N_G(J)|}{|N_H(J)|}\, \epsilon_J^H, \qquad (\mathrm{res}_{H,G}(\xi) @ J) = \frac{|N_G(J)|}{|N_H(J)|}\, (\xi @ J),$$

$$\mathrm{jnd}_{G,H}(\epsilon_J^H) = \epsilon_J^G, \qquad (\mathrm{jnd}_{G,H}(\eta) @ I) = \sum_{J \leq_H H : J =_G I} (\eta @ J).$$

$$\mathrm{jef}_{\overline{G},G}(\epsilon_I^G) = \epsilon_{\overline{I}}^{\overline{G}}, \qquad (\mathrm{jef}_{\overline{G},G}(\xi) @ \overline{K}) = \sum_{I \leq_G G : \overline{I} =_{\overline{G}} \overline{K}} (\xi @ I),$$

$$\mathrm{inf}_{G,\overline{G}}(\epsilon_{\overline{K}}^{\overline{G}}) = \sum_{I \leq_G G : \overline{I} =_{\overline{G}} \overline{K}} \beta_G(\overline{I}, I)\, \epsilon_I^G, \qquad (\mathrm{inf}_{G,\overline{G}}(\zeta) @ I) = \beta_G(\overline{I}, I)\, (\zeta @ \overline{I}),$$

$$\mathrm{iso}_{F,G}^\phi(\epsilon_I^G) = \epsilon_{\phi(I)}^F, \qquad (\mathrm{iso}_{F,G}^\phi(\xi) @ \phi(I)) = (\xi @ I).$$

Using the equality $m_G(G, U) = \lfloor G = U \rfloor$ together with the square coordinate formula for induction on $\mathbb{Q}B^*$, we obtain the following lemma.

Lemma 6.1. *We have $\epsilon_G^G = \delta_G^G$ and, more generally, $\epsilon_I^G = \mathrm{jnd}_{G,I}(\epsilon_I^I) = \mathrm{jnd}_{G,I}(\delta_I^I)$.*

Let us introduce a set $\mathcal{X}$ of isomorphism classes of finite groups such that $\mathcal{X}$ is closed under subquotients. Abusing notation, we write $G \in \mathcal{X}$ to mean that the isomorphism class of G belongs to $\mathcal{X}$, and we write $\bigoplus_{G \in \mathcal{X}}$ to indicate a direct sum where G runs over representatives of the isomorphism classes in $\mathcal{X}$. Equally well, we could understand $\mathcal{X}$ to be a class of finite groups that is closed under isomorphism and subquotients, in which case we would have to understand the notation $\mathcal{L} \subseteq \mathcal{X}$ to imply that $\mathcal{L}$ is a subclass of $\mathcal{X}$ that is closed under isomorphism. We write $\mathbb{Q}B^{\mathcal{X}}$ and $(\mathbb{Q}B^*)^{\mathcal{X}}$ to denote $\mathbb{Q}B$ and $\mathbb{Q}B^*$ regarded as functors whose coordinate modules are defined for groups in $\mathcal{X}$.

As in Section 5, we write the simple Mackey functors in the form $S_{L,V}^{\mathrm{mac}}$. The following lemma is clear from the round-coordinate equations for induction, restriction and isogation.

Lemma 6.2. *As Mackey functors, $\mathbb{Q}B^{\mathcal{X}} \cong \bigoplus_{L \in \mathcal{X}} S_{L,\mathbb{Q}}^{\mathrm{mac}} \cong (\mathbb{Q}B^*)^{\mathcal{X}}$. The copy of $S_{L,\mathbb{Q}}^{\mathrm{mac}}$ in $\mathbb{Q}B$ is such that, for each $G \in \mathcal{X}$, the $\mathbb{Q}$-vector space $S_{L,\mathbb{Q}}^{\mathrm{mac}}(G)$ has a $\mathbb{Q}$-basis consisting of those primitive idempotents e_I^G which satisfy $I \cong L$. The copy of $S_{L,\mathbb{Q}}^{\mathrm{mac}}$ in $\mathbb{Q}B^*$ is characterized similarly, with ϵ_I^G in place of e_I^G.*

The next result follows immediately.

Proposition 6.3. *There is a bijective correspondence $M \leftrightarrow \mathcal{L}$ between the Mackey subfunctors M of $\mathbb{Q}B^{\mathcal{X}}$ and the subsets $\mathcal{L} \subseteq \mathcal{X}$. The correspondence is characterized by the condition that $M \cong \bigoplus_{L \in \mathcal{L}} S_{L,\mathbb{Q}}^{\mathrm{mac}}$. This is equivalent to the condition that, for all $I \leq G \in \mathcal{X}$, we have $e_I^G \in M(G)$ if and only if $I \in \mathcal{L}$. Similar assertions hold for $\mathbb{Q}B^*$.*

Proposition 6.4. *Let $M \leftrightarrow \mathcal{L}$ as above. Then there is a bijective correspondence $\theta \leftrightarrow \Theta$ between the Mackey morphisms $\theta : M \to (\mathbb{Q}B^*)^{\mathcal{X}}$ and the functions $\Theta : \mathcal{L} \to \mathbb{Q}$. The correspondence is characterized by the condition that $\theta_I(e_I^I) = \Theta(I)\,\epsilon_I^I$ for all $I \in \mathcal{L}$. When that condition holds, θ determines Θ via the equality $\Theta(I) = (\theta_I(e_I^I) @ I) = [\theta_I(e_I^I) @ I]$ with $I \in \mathcal{L}$, while Θ determines θ via the equality*

$$\theta_G(e_I^G) = \frac{\Theta(I)}{|N_G(I) : I|}\,\epsilon_I^G.$$

Proof. For $I \in \mathcal{L}$, the copy of $S_{I,\mathbb{Q}}^{\mathrm{mac}}$ in $\mathbb{Q}B$ is such that $S_{I,\mathbb{Q}}^{\mathrm{mac}}(I) = \mathbb{Q}e_I^I$, while the copy of $S_{I,\mathbb{Q}}^{\mathrm{mac}}$ in $\mathbb{Q}B^*$ is such that $S_{I,\mathbb{Q}}^{\mathrm{mac}}(I) = \mathbb{Q}\epsilon_I^I$. So, given θ, then there exists an element $\Theta(I) \in \mathbb{Q}$ such that $\theta_I(e_I^I) = \Theta(I)\,\epsilon_I^I$. Then $\Theta(I) = (\theta_I(e_I^I) @ I)$ and, using Lemma 6.1, $\Theta(I) = [\theta_I(e_I^I) @ I]$. The round coordinate formula for induction now yields the formula for $\theta_G(e_I^G)$. $\square$

The latest two results together give a classification of the pairs (M, θ).

Corollary 6.5. *Consider the pairs (M, θ) where M is a Mackey subfunctor of $\mathbb{Q}B^{\mathcal{X}}$ and $\theta : M \to (\mathbb{Q}B^*)^{\mathcal{X}}$ is a Mackey morphism. Also consider the pairs $(\mathcal{L}, \Theta)$ where $\mathcal{L} \subseteq \mathcal{X}$ and Θ is a function $\mathcal{L} \to \mathbb{Q}$. These two kinds of pairs are in a bijective correspondence $(M, \theta) \leftrightarrow (\mathcal{L}, \Theta)$ whereby $M \leftrightarrow \mathcal{L}$ and $\theta \leftrightarrow \Theta$ as above.*

The benefit of the round coordinate system is now clear: under its auspices, the Mackey morphisms in question are represented by diagonal matrices. To choose a Mackey morphism θ, there is no constraint on the choices of the diagonal entries $\Theta(I)$ at the I-th coordinates of $\mathbb{Q}B(I)$ and $\mathbb{Q}B^*(I)$. Those diagonal entries then determine all the other diagonal entries. We call Θ the **diagonal invariant** of θ.

The next two results characterize the inflaky subfunctors $M \leq \mathbb{Q}B$ and the inflaky morphisms $\theta : M \to \mathbb{Q}B^*$.

Proposition 6.6. *Let $M \leftrightarrow \mathcal{L}$ as above. Then M is an inflaky subfunctor of $\mathbb{Q}B$ if and only if $G \in \mathcal{L}$ whenever $\overline{G} \in \mathcal{L}$.*

Proof. Suppose that $\overline{G} \in \mathcal{L}$ and M is an inflaky functor. Then $e_{\overline{G}}^{\overline{G}} \in M(\overline{G})$ and $\inf_{G,\overline{G}}(e_{\overline{G}}^{\overline{G}}) \in M(G)$. But $\inf_{G,\overline{G}}(e_{\overline{G}}^{\overline{G}}) = e_G^G + \sum_I e_I^G$ where I runs over representatives of the G-conjugacy classes of strict subgroups $I < G$ such that $\overline{I} = \overline{G}$. By an inductive argument on $|G|$, we may assume that each $I \in \mathcal{L}$, in other words, each $e_I^G \in M(G)$. We deduce that $e_G^G \in M(G)$, in other words, $G \in \mathcal{L}$. The converse is obtained similarly using the formula for $\inf_{G,\overline{G}}(e_{\overline{K}}^{\overline{G}})$. $\square$

Theorem 6.7. *Let $(M, \theta) \leftrightarrow (\mathcal{L}, \Theta)$ as above. Suppose that M is an inflaky subfunctor of $\mathbb{Q}B$. Then θ is an inflaky morphism if and only if $\Theta(G) = \Theta(\overline{G})\,\beta(\overline{G}, G)$ whenever $\overline{G} \in \mathcal{L}$.*

Proof. The assertion makes sense because, by the previous proposition, the condition $\overline{G} \in \mathcal{L}$ implies that $G \in \mathcal{L}$, hence $\Theta(G)$ is defined. By direct calculation,

$$\inf_{G,\overline{G}}(\theta(e_{\overline{K}}^{\overline{G}})) = \frac{\theta(\overline{K})}{|N_{\overline{G}}(\overline{K}) : \overline{K}|} \sum_{I \leq_G G : \overline{I} = _{\overline{G}} \overline{K}} \beta_G(\overline{I}, I)\,\epsilon_I^G,$$

$$\theta(\inf_{G,\overline{G}}(e_{\overline{K}}^{\overline{G}})) = \sum_{I \leq_G G : \overline{I} = _{\overline{G}} \overline{K}} \frac{\Theta(I)}{|N_G(I) : I|}\,\epsilon_I^G.$$

Again, the previous proposition guarantees that, for each index I in the sums, $I \in \mathcal{L}$ and $\Theta(I)$ is defined. Comparing coefficients, we find that θ commutes with inflation if and only if

$$\frac{\Theta(I)}{|N_G(I) : I|} = \frac{\Theta(\overline{I})}{|N_{\overline{G}}(\overline{I}) : \overline{I}|}\,\beta_G(\overline{I}, I)$$

whenever $\overline{I} \in \mathcal{L}$ (and perforce $I \in \mathcal{L}$). From the definition of $\beta_G(\overline{I}, I)$, the latest equation can be rewritten as $\Theta(I) = \Theta(\overline{I})\,\beta(\overline{I}, I)$. Replacing I with G, the required conclusion follows. $\square$

The deflaky subfunctors $M \leq \mathbb{Q}B$ and the deflaky morphisms $\theta : M \to \mathbb{Q}B^*$ are a little harder to characterize. We first need to recall some material from Bouc [4, 7.2.3]. Let us call G a **base group** (in Bouc's terminology, a b-group) provided $\beta(\overline{G}, G) = 0$ whenever $N \neq 1$. The following result is Bouc [4, Proposition 9, Lemme 18].

Theorem 6.8 (*Bouc*). *There exists a base group* base(G), *unique up to isomorphism, with the universal property that any base group isomorphic to a quotient of G is also isomorphic to a quotient of* base(G). *Furthermore, the following three conditions are equivalent:*
(a) $\overline{G}$ *is isomorphic to a quotient of* base(G),

(b) base$(\overline{G}) \cong$ base(G),
(c) $\beta(\overline{G}, G) \neq 0$.

The main significance of the notion of a base group lies in the next result, Bouc [4, Propositions 10, 12].

Theorem 6.9 (*Bouc*). *Regarding* $\mathbb{Q}B^{\mathcal{X}}$ *as a biset functor, its simple composition factors are those having the form* $S_{L,\mathbb{Q}}$, *where L is a base group in* $\mathcal{X}$. *Each simple composition factor occurs only once.*

Now we can characterize the deflaky subfunctors of $\mathbb{Q}B$.

Proposition 6.10. *Let M* $\leftrightarrow$ *$\mathcal{L}$ as above. Then M is a deflaky subfunctor of* $\mathbb{Q}B$ *if and only if* $\overline{G} \in \mathcal{L}$ *whenever* $G \in \mathcal{L}$ *and the three equivalent conditions in Theorem 6.8 hold.*

Proof. The argument is fairly similar to the proof of Proposition 6.6. Supposing that M is a deflaky subfunctor, that $G \in \mathcal{L}$ and that $\beta(\overline{G}, G) \neq 0$, then $\mathrm{def}_{\overline{G}, G}^G(e_G^G) \neq 0$ hence $\overline{G} \in \mathcal{L}$. The converse is obtained by considering the formula for $\mathrm{def}_{\overline{G}, G}^G(e_I^G)$. $\square$

We mention that Propositions 6.6 and 6.10 together recover the following result of Bouc [4, 7.2.4].

Corollary 6.11 (*Bouc*). *Let M* $\leftrightarrow$ *$\mathcal{L}$ as above. Then M is a biset subfunctor of* $\mathbb{Q}B^{\mathcal{X}}$ *if and only if, for all* $L \in \mathcal{L}$ *and* $G \in \mathcal{X}$ *such that* base(L) *is isomorphic to a quotient of* base(G), *we have* $G \in \mathcal{L}$.

For deflaky functors, we have the following analogue of Theorem 6.7.

Lemma 6.12. *Let* $(M, \theta) \leftrightarrow (\mathcal{L}, \Theta)$ *as above. Suppose that M is a deflaky subfunctor of* $\mathbb{Q}B$. *Then* θ *is an deflaky morphism if and only if, for all* $G \in \mathcal{L}$ *and* $N \trianglelefteq G$, *we have* $\Theta(G) = \Theta(\overline{G}) \beta(\overline{G}, G)$ *if* $\overline{G} \in \mathcal{L}$, *while* $\Theta(G) = 0$ *otherwise.*

Proof. If $I \in \mathcal{L}$, then

$$\mathrm{jef}_{\overline{G}, G}(\theta(e_I^G)) = \frac{\Theta(I)}{|N_G(I) : I|} \epsilon_{\overline{I}}^{\overline{G}}, \qquad \theta(\mathrm{def}(e_I^G)) = \frac{\Theta(\overline{I})}{|N_{\overline{G}}(\overline{I}) : \overline{I}|} \beta_G(\overline{I}, I) \epsilon_{\overline{I}}^{\overline{G}}.$$

The second equation makes sense even though the condition $I \in \mathcal{L}$ does not imply that $\overline{I} \in \mathcal{L}$. Indeed, the latest proposition guarantees that, if $\overline{I} \notin \mathcal{L}$, then $\beta_G(\overline{I}, I) = 0$ and, in that case, we can understand that $\Theta(\overline{I}) \beta_G(\overline{I}, I) = 0$ even though $\Theta(\overline{I})$ is undefined. Comparing coefficients, we find that θ commutes with deflation if and only if $\Theta(I) = \Theta(\overline{I}) \beta(\overline{I}, I)$, understanding that $\Theta(I) = 0$ when $\overline{I} \notin \mathcal{L}$. $\square$

Comparing Lemma 6.12 with Theorem 6.7, we observe that the inflaky morphisms $M \to \mathbb{Q}B^*$ and the deflaky morphisms $M \to \mathbb{Q}B^*$ are characterized by the same formula, the two criteria apparently differing only slightly. The next result reveals, however, that the criterion expressed in Lemma 6.12 is quite strong.

Theorem 6.13. *Let* $(M, \theta) \leftrightarrow (\mathcal{L}, \Theta)$ *as above. Suppose that M is a deflaky subfunctor of* $\mathbb{Q}B$ *and* θ *is a non-zero deflaky morphism. Then* $1 \in \mathcal{L}$ *and* θ *is the restriction of* $\Theta(1)$ exp *to M.*

Proof. By the latest lemma, $\Theta(G) = \Theta(1) \beta(1, G)$ for all $G \in \mathcal{L}$, understanding that $\Theta(G) = 0$ when $1 \notin \mathcal{L}$. But θ is non-zero, so we must have $1 \in \mathcal{L}$. Letting $\Theta^{\exp} : \mathcal{X} \to \mathbb{Q}$ be the diagonal invariant for exp(), it is easy to see that $\Theta^{\exp}(1) = 1$. So $\Theta^{\exp}(G) = \beta(1, G)$ and $\Theta(G) = \Theta(1) \Theta^{\exp}(G)$. $\square$

We have now proved Theorem 5.3. Indeed, it is clear that Theorems 5.3 and 6.13 are equivalent to each other.
The following recapitulation of Example 4.5 will be of use to us in Section 8.

Example 6.14. Supposing that $G = C_p^2$ then, in the notation of Example 4.5, $\kappa_2 = p e_G^G$ and $\delta_2 = \delta_G^G = \epsilon_G^G$, hence $p.\mathrm{torn}_G^p(e_G^G) = (1 - p) \delta_G^G = (1 - p) \epsilon_G^G$.

7. The diagonal invariant for exp()

Having introduced the notion of a diagonal invariant in the previous section, we shall determine the diagonal invariant $\Theta^{\exp} : \mathcal{X} \to \mathbb{Q}$ for the lifted exponential morphism exp : $\mathbb{Q}B \to \mathbb{Q}B^*$. That will yield a formula for exp(). We shall also note some similar formulas for the morphisms lin() and die(). Some of the material in this section will be used in later sections, but not in a very crucial way. After all, we have already established both of the uniqueness theorems for exp() that were stated in Section 5. Largely, this section will serve as a warm-up for the treatment of the morphism torn^p in Section 8.

We say that elements $g, g' \in G$ are $\mathbb{Q}G$-**conjugate**, and we write $g =_{\mathbb{Q}G} g'$, provided the cyclic groups that they generate, $\langle g \rangle$ and $\langle g' \rangle$, are G-conjugate. We let e_g^G denote the element of $\mathbb{Q}A_{\mathbb{Q}}(G)$ such that, regarding e_g^G as a character, $e_g^G(g') = \lfloor g =_{\mathbb{Q}G} g' \rfloor$. We have the following obvious analogue of Remark 3.1.

Remark 7.1. Letting g run over representatives of the $\mathbb{Q}G$-conjugacy classes in G, then the elements e_g^G run over the primitive idempotents of $\mathbb{Q}A_{\mathbb{Q}}(G)$ without repetitions, furthermore, $\mathbb{Q}A_{\mathbb{Q}}(G) = \bigoplus_g \mathbb{Q}e_g^G$ as a direct sum of algebras $\mathbb{Q}e_g^G \cong \mathbb{Q}$. Given a $\mathbb{Q}G$-character χ, then $\chi = \sum_g \chi(g) e_g^G$. In particular, the set of primitive idempotents is a $\mathbb{Q}$-basis for $\mathbb{Q}A_{\mathbb{Q}}(G)$.

Remark 7.2 (*Diagonal Formula for the Linearization Map*). If I is non-cyclic then $\mathrm{lin}_G(e_I^G) = 0$, while if I is cyclic with generator g, then $\mathrm{lin}_G(e_I^G) = e_g^I$.

Proof. The permutation character $\chi = [\mathbb{Q}X]$ associated with a G-set X is given by $\chi(g) = |X^{\langle g \rangle}| = \epsilon^G_{\langle g \rangle}[X]$. $\square$

Recall that, for a positive integer n, the classical Euler function $\phi(n)$ is given by

$$\phi(n)/n = \sum_d \mu(d)/d = \prod_p (1 - 1/p)$$

where μ is the classical Möbius function and d runs over the divisors of n, while p runs over the prime divisors of n. Defining $\phi(G) = |\{g \in G : \langle g \rangle = G\}|$, then $\phi(G) \neq 0$ if and only if G is cyclic, in which case, $\phi(G) = \phi(|G|)$.

Proposition 7.3 (*Diagonal Formula for the Exponential Map*). *We have*

$$\Theta^{\exp}(G) = \beta(1, G) = \phi(G)/|G|, \qquad \exp(e^G_I) = \frac{\phi(I)}{|N_G(I)|}\, \epsilon^G_I.$$

Proof. The equality $\Theta^{\exp}(G) = \beta(1, G)$ was established in the proof of Theorem 6.13. Gluck's Idempotent Formula and the defining formula for the deflation numbers have the special cases

$$e^G_G = \frac{1}{|G|} \sum_{U \leq G} |U|\, \mu(U, G)\, d^G_U, \qquad \beta(1, G) = \frac{1}{|G|} \sum_{U \leq G} |U|\, \mu(U, G).$$

Comparing the two formulas, we find that $\beta(1, G) = \langle 1_G \mid \lin_G(e^G_G) \rangle^A$. If G is non-cyclic, then $\lin_G(e^G_G) = 0$, hence $\beta(1, G) = 0 = \phi(G)/|G|$. If G is cyclic, then $\mu(U, G) = \mu(|G : U|)$, whence the above Möbius inversion formula for $\phi(n)$ yields, again, $\beta(1, G) = \phi(G)/|G|$. Proposition 6.4 now surrenders the asserted formula for $\exp(e^G_I)$. $\square$

Proposition 7.4 (*Diagonal Formula for the tom Dieck Map*). *Given an $\mathbb{R}G$-character χ, then* $\die_G(\chi) = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \chi(g)\, \epsilon^G_{\langle g \rangle}$. *In particular,* $\die_G(e^G_g) = \dfrac{\phi(\langle g \rangle)}{|N_G(\langle g \rangle)|}\, \epsilon^G_{\langle g \rangle}$.

Proof. The first asserted equality holds because

$$[\die_G(\chi) @ U] = \frac{1}{|U|} \sum_{g \in U} \chi(g) = \frac{1}{|G|.|U|} \sum_{g \in G} \chi(G)\, |\{x \in G : {}^x g \in U\}|$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(g)\, m_G(\langle g \rangle, U) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\, [\epsilon^G_{\langle g \rangle} @ U].$$

The rider holds because the number of $\mathbb{Q}G$-conjugates of g is $\phi(\langle g \rangle)\, |G : N_G(\langle g \rangle)|$. Alternatively, since $\exp = \die \circ \lin$, the rider follows from Remark 7.2 and Proposition 7.3. $\square$

Let us end this section with one last comment concerning our coordinate systems. Out of technical need, we introduced square and round coordinate systems for $\mathbb{Q}B$ and $\mathbb{Q}B^*$. There is no such need as regards $A_\mathbb{K}(G)$, so we have refrained from setting up a systematic notation for the evident square bases $\mathrm{Irr}(\mathbb{K}G)$ and the round basis $\{e^{\mathbb{K}G}_g : g \in_{\mathbb{K}G} G\}$, where g runs over the $\mathbb{K}G$-conjugacy classes of g in the sense of Berman–Witt Theorem, and $e^{\mathbb{K}G}_g$ is the unique primitive idempotent of $\mathbb{K}A_\mathbb{K}(G)$ such that $e^{\mathbb{K}G}_g(g) = 1$. Remark 7.2 and Propositions 7.3 and 7.4 tell us that, putting $\mathbb{K} = \mathbb{Q}$ then, with respect to the round coordinates, all three maps in the commutative triangle $\exp_G = \die_G \circ \lin_G$ are represented by diagonal matrices. Of course, with $\mathbb{K}A_\mathbb{K}(G)$ instead of $\mathbb{Q}B(G)$, the transformation matrix from square to round coordinates is the transpose of the $\mathbb{K}$-character table instead of the table of marks. When $\mathbb{K} = \mathbb{C}$ and G is cyclic with order n, these round and square coordinate systems are very closely related to the canonical pairs (p, q) that appear in discrete quantum systems with n sample points; the character table, as a transformation matrix, is precisely the discrete Fourier transform matrix. The connection with canonical pairs becomes even more striking when we replace $\mathbb{Q}B(G)$ with the $\mathbb{C}$-monomial Burnside algebra over $\mathbb{C}$, which has a $\mathbb{C}$-basis indexed by the G-conjugacy classes of irreducible complex characters of cyclic subquotients of G.

8. The diagonal invariant for torn^p

At this point, four of the results stated in Section 5 remain to be established: Theorems 5.4, 5.6, 5.7 and Remark 5.8. We shall supply their proofs in this section. We shall be needing a formula, in Proposition 8.3, for the diagonal invariant Θ^π for torn^p.

The biset subfunctor $\mathbb{Q}K$ of $\mathbb{Q}B$ is easy to describe in terms of the round coordinate system. Indeed, Remark 7.2 immediately yields the following well-known observation.

Remark 8.1. Letting I run over representatives of the conjugacy classes of non-cyclic subgroups of G, then the primitive idempotents e^G_I run, without repetitions, over the elements of a $\mathbb{Q}$-basis for $\mathbb{Q}K(G)$.

Let $\mathcal{X}^p$ denote the set of isomorphism classes of finite p-groups, and let $\mathcal{L}^p$ denote the set of isomorphism classes of non-cyclic finite p-groups. The latest remark tells us that, in the sense of Proposition 6.3, $\mathcal{L}^p$ is the subset of $\mathcal{X}^p$ corresponding to the p-biset subfunctor ${}_p\mathbb{Q}K$ of ${}_p\mathbb{Q}B$. So, linearly extending torn^p to an inflaky morphism ${}_p\mathbb{Q}K \to {}_p\mathbb{Q}B$, then the diagonal invariant of torn^p is a function $\Theta^p : \mathcal{L}^p \to \mathbb{Q}$. Thus, in the sense of Corollary 6.5, $(\mathcal{L}^p, \Theta^p)$ is the pair corresponding to $({}_p\mathbb{Q}K, \mathrm{torn}^p)$. Already, we can prove a large part of Theorem 5.4.

Lemma 8.2. *For finite p-groups, every inflaky morphism $_pK \to \, _pB^*$ is a $\mathbb{Z}$-multiple of* $\mathrm{torn}^p/(1 - p)$. *Furthermore,* torn^p *is the unique inflaky morphism $_pK \to \, _pB^*$ such that $\kappa_2 \mapsto (1 - p)\,\delta_2$.*

Proof. Let $\theta : \, _pK \to \, _pB^*$ be a non-zero inflaky morphism with diagonal invariant $\Theta : \mathcal{L}^p \to \mathbb{Q}$. Suppose that G is a noncyclic p-group. Then C_p^2 occurs as quotient group of G and the deflation number $\beta(C_p^2, G)$ is well-defined; indeed, as we noted in Section 3, $\beta(\overline{G}, G)$ depends only on the isomorphism classes of $\overline{G}$ and G. Theorem 6.7 tells us that $\Theta(G) = \Theta(C_p^2)\,\beta(C_p^2, G)$ and $\Theta^p(G) = \Theta^p(C_p^2)\,\beta(C_p^2, G)$. But the morphisms θ and torn^p are non-zero, so $\Theta(C_p^2)$ and $\Theta^p(C_p^2)$ are non-zero. Hence $\Theta(G)/\Theta(C_p^2) = \Theta^p(G)/\Theta^p(C_p^2)$. By the definition of diagonal invariants, we deduce that θ and torn^p are $\mathbb{Q}$-multiples of each other.

By Example 6.14, $\theta_{C_p^2}(\kappa_2) = p\,\Theta(C_p^2)\,\delta_2$ while $\mathrm{torn}_{C_p^2}(\kappa_2) = p\,\Theta^p(C_p^2)\,\delta_2 = (1-p)\delta_2$. Since κ_2 is a generator for $K(C_p^2)$ and δ_2 is an element of a $\mathbb{Z}$-basis for $B^*(C_p^2)$, we deduce that $p\,\Theta(C_p^2) \in \mathbb{Z}$. Moreover, $\Theta^p(C_p^2) = (1-p)/p$, so the rational number $\Theta(C_p^2)$ is a $\mathbb{Z}$-multiple of $\Theta^p(C_p^2)/(1-p)$. Therefore, the function Θ is a $\mathbb{Z}$-multiple of Θ^p. That is to say, θ is a $\mathbb{Z}$-multiple of $\mathrm{torn}^p/(1-p)$. $\square$

To complete the proof of Theorem 5.4, we must show that the morphism $\mathrm{torn}_p/(1-p) : \, _p\mathbb{Q}K \to \, _p\mathbb{Q}B^*$ restricts to a morphism $_pK \to \, _pB^*$. For that, we shall be needing the next result. Given an integer $d \geq 2$, we define $\beta_p(d)$ such that $p\,\beta_p(d) = (1-p)\,\beta_p(2, d)$. That is,

$$\beta_p(d + 1) = (1 - p^{d-1})\,\beta_p(d), \qquad \beta_p(2) = (1 - p)/p.$$

Proposition 8.3 (*Diagonal Formula for the Tornehave Map*). *Suppose that I is a non-cyclic p-group with rank d. Then*

$$\Theta^p(I) = \frac{1-p}{p}\,\beta_p(2, d) = \beta_p(d), \qquad \mathrm{torn}_G^p(e_I^G) = \frac{\beta_p(d)}{|N_G(I) : I|}\,\epsilon_I^G.$$

We shall prove the proposition using the quantum binomial formula. Recall that, for integers $d \geq 0$ and $d \geq r \geq 0$, the **q-binomial coefficients** in $\mathbb{Z}[q]$ are defined by

$$\binom{d}{r}_q = \frac{(q^d - 1)(q^d - q) \ldots (q^d - q^{d-1})}{(q^r - 1)(q^r - q) \ldots (q^r - q^{r-1})} = \frac{(q^d - 1)(q^{d-1} - 1) \ldots (q^{d-r+1} - 1)}{(q^r - 1)(q^{r-1} - 1) \ldots (q - 1)}.$$

It is easy to check that these polynomials in $\mathbb{Z}[q]$ are determined by the **q-Pascal Relation**

$$\binom{d}{r}_q = \binom{d-1}{r}_q + q^{d-r}\binom{d-1}{r-1}_q$$

for $0 < r < d$, with the initial conditions $\binom{d}{0}_q = \binom{d}{d}_q = 1$. Using the q-Pascal Relation, an inductive argument yields the **q-Binomial Formula**

$$\prod_{s=0}^{d-1}(1 + q^s t) = \sum_{r=0}^{d}\binom{d}{r}_q q^{r(r-1)/2}\,t^r.$$

Here and elsewhere, we understand a product over an empty indexing set to have value 1. We mention that the usual Binomial Formula is recovered by considering the limit $q \to 1$.

Let C_p^d denote the elementary abelian p-group with rank d. The integer $\binom{d}{r}_p = \binom{d}{d-r}_p$ is equal to the number of subgroups with size p^d in C_p^r, and it is also equal to the number of subgroups with index p^d in C_p^d. Recall that the **Möbius function** of G is defined to be $\mu(G) = \mu(1, G)$.

Theorem 8.4 (*Weisner's Theorem*). *Suppose that G is a p-group. Then $\mu(G) = 0$ unless $G \cong C_p^d$ for some $d \geq 0$, in which case, $\mu(G) = (-1)^d p^{d(d-1)/2}$. More generally, $\mu(U, G) = 0$ unless $U \lhd G$ and $G/U \cong C_p^r$ for some $r \geq 0$, in which case $\mu(U, G) = (-1)^r p^{r(r-1)/2}$.*

The theorem is very well-known – and there is a quick and easy proof using the chain-pairing method – but Weisner's original proof [15] is worth noting: he substituted $q = p$ and $t = -1$ into the q-Binomial Formula, then he compared the resulting equation with the recurrence relation for the Möbius function. We shall be making a similar application of the q-Binomial Formula. For $d \geq 2$, we differentiate with respect to t and then substitute $q = p$ and $t = -1/p$, arriving at

$$\beta_p(d) = \frac{1-p}{p}\prod_{s=1}^{d-2}(1 - p^s) = \sum_{r=1}^{d}\binom{d}{r}_p (-1)^r\,r\,p^{r(r-3)/2}.$$

Let $\Phi(G)$ denote the Frattini subgroup of G. Combining Weisner's Theorem with the equation for e_G^G appearing in the proof of Proposition 7.3, we obtain

$$e_G^G = \frac{1}{|G|} \sum_U |U| \, (-1)^r \, p^{r(r-1)/2} \, d_U^G = \sum_U (-1)^r \, p^{r(r-3)/2} \, d_U^G$$

where U runs over the intermediate subgroups $\Phi(G) \leq U \leq G$ and $r = \log_p |G : U|$. By the definition of the map $\widetilde{\text{torn}}_G^p = \widetilde{\text{torn}}_G^{\{p\}}$ in Section 4, we also have $r = [\widetilde{\text{torn}}_G^p (d_U^G) @ G]$. Applying Proposition 6.4,

$$\Theta^p(G) = [\text{torn}_G^p (e_G^G) @ G] = \sum_U (-1)^r \, p^{r(r-3)/2} \, [\widetilde{\text{torn}}_G^p (d_U^G) @ G] = \sum_U (-1)^r \, p^{r(r-3)/2} \, r.$$

The sum has precisely $\binom{d}{r}_p$ subgroups U associated with each r. Therefore $\Theta^p(G) = \beta_p(d)$. Applying Proposition 6.4 again, the proof of Proposition 8.3 is now complete.

We shall be making use of the following corollary, a generalization of Example 6.14.

Corollary 8.5. *Supposing that G is a non-cyclic p-group with rank d, then $d \geq 2$ and*

$$\text{torn}_G^p(e_G^G) = \beta_p(d) \, \epsilon_G^G = \beta_p(d) \, \delta_G^G.$$

Furthermore, $p \, e_G^G \in K(G)$ and $\text{torn}_G(p \, e_G^G) \equiv \delta_G^G$ modulo $p \, B^*(G)$.

Proof. The first equality is a special case of Proposition 8.3. The second equality holds by Lemma 6.1. Since $r(r-3)/2 \geq -1$ for all integers $r \geq 0$, a formula above for e_G^G implies that $p \, e_G^G \in K(G)$. The congruence holds because $p \, \beta_p(d) \equiv 1$ modulo p. $\square$

The next corollary is Bouc [4, page 709]. Let us give an alternative proof of it, making use of the exponential and Tornehave morphisms. The argument will be developed further in Appendix.

Corollary 8.6 (*Bouc*). *Suppose that G is a p-group. Then $\beta(\overline{G}, G) = 0$ if and only if $\overline{G}$ is cyclic and G is non-cyclic. That is to say, G is a base group if and only if G is trivial or $G \cong C_p^2$.*

Proof. First suppose that $\overline{G}$ is non-cyclic. Then G is non-cyclic. By Proposition 8.3, $\Theta^p(\overline{G})$ and $\Theta^p(G)$ are non-zero. But torn^p is an inflaky morphism so, by Theorem 6.7, $\beta(\overline{G}, G) \neq 0$. Now suppose that $\overline{G}$ is cyclic. By Proposition 7.3, $\Theta^{\exp}(\overline{G}) \neq 0$ while $\Theta^{\exp}(G) \neq 0$ if and only if G is cyclic. This time, Theorem 6.7 implies that $\beta(\overline{G}, G)$ is non-zero if and only if G is cyclic. The rider now follows via Theorem 6.8. $\square$

We can now finish the proof of Theorem 5.4. As we noted earlier in this section, it remains only to show that $\text{torn}^p/(1-p)$ restricts to a morphism $_pK \to {_pB^*}$. Actually, since $1 - p$ is coprime to p, it suffices to show that $\text{torn}^p/(1-p)$ restricts to a morphism $_pRK \to {_pRB^*}$ where $R = \mathbb{Z}[1/p]$ as a subring of $\mathbb{Q}$. Suppose that G is a p-group. By Theorem 8.4 and Gluck's Idempotent Formula (see Section 2), $m_G^{-1}(U, I) \in R$ for all subgroups U and I of G. Therefore $RK(G)$ has an R-basis consisting of the elements having the form e_I^G where I is non-cyclic. By the round-coordinate formula for induction, $_pRK$ is generated, as a Mackey functor – and perforce as an inflaky functor – by the elements having the form e_G^G where G is now assumed to be a non-cyclic p-group. So, with that assumption on G, it suffices to show that $\text{torn}_G^p(e_G^G)/(1-p) \in RB^*(G)$. By Lemma 6.1 and Proposition 8.3,

$$\text{torn}_G^p(e_G^G) = \beta_p(d) \, \epsilon_G^G = \beta_p(d) \, \delta_G^G$$

where d is the rank of G. But $\delta_G^G \in RB^*(G)$ and $\beta_p(d)$ is divisible by $(1-p)$ in R. We have now shown that $\text{torn}^p/(1-p)$ restricts to a morphism $_pRK \to {_pRB^*}$. The proof of Theorem 5.4 is complete.

The proofs of Theorems 5.6 and 5.7 will be fairly similar to the proofs of Theorems 5.3 and 5.4, but with some considerable simplifications. However, we shall first have to modify the theory of diagonal invariants, because we shall be working with $_pK^*$ and $_p\mathbb{Q}K^*$ as the codomains, rather than $_pB^*$ and $_p\mathbb{Q}B^*$. It will turn out that the modified theory is easier than before, because all of the relevant deflation numbers are non-zero.

Consider the exact sequence of p-biset functors

$$0 \longrightarrow {_p\mathbb{Q}A_\mathbb{Q}^*} \xrightarrow{\text{lin}^*} {_p\mathbb{Q}B^*} \xrightarrow{\pi^*} {_p\mathbb{Q}K^*} \longrightarrow 0.$$

For convenience, let us assume that G is a p-group (although some of the following comments generalize easily to arbitrary finite groups). By Remarks 7.2 and 8.1, $\mathbb{Q}A_\mathbb{Q}^*(G)$ has a $\mathbb{Q}$-basis consisting of the elements ϵ_I^G where I runs over representatives of the conjugacy classes of cyclic subgroups of G, while $\mathbb{Q}K^*(G)$ has a $\mathbb{Q}$-basis consisting of the elements $\overline{\epsilon}_I^G = \pi_G^*(\epsilon_I^G)$ where I now runs over representatives of the conjugacy classes of non-cyclic subgroups of G. With one exception, the elemental maps for $_p\mathbb{Q}K^*$ act on the basis elements $\overline{\epsilon}_I^G$ in just the same way that the elemental maps for $_p\mathbb{Q}B^*$ act on the basis elements ϵ_I^G. The exception is the deflation map, which is given by

$$\text{jef}_{\overline{G}, G}(\overline{\epsilon}_I^G) = \begin{cases} 0 & \text{if } \overline{I} \text{ is cyclic,} \\ \overline{\epsilon}_{\overline{I}}^G & \text{if } \overline{I} \text{ is non-cyclic.} \end{cases}$$

Recall that $\mathcal{L}^p$ denotes the set of isomorphism classes of non-cyclic finite p-groups.

Proposition 8.7. *Consider the pairs* (M, θ) *where M is a Mackey subfunctor of* $_p\mathbb{Q}K$ *and* θ *is a Mackey morphism* $M \to {}_p\mathbb{Q}K^*$. *Also consider the pairs* $(\mathcal{L}, \Theta)$ *where* $\mathcal{L} \subseteq \mathcal{L}^p$ *and* Θ *is a function* $\mathcal{L} \to \mathbb{Q}$. *These two kinds of pairs are in a bijective correspondence* $(M, \theta) \leftrightarrow (\mathcal{L}, \Theta)$ *characterized by the condition that, whenever G is a p-group and I is non-cyclic, we have* $I \in \mathcal{L}$ *if and only if* $\bar{\epsilon}_I^G \in M(G)$, *in which case,*

$$\theta_G(e_I^G) = \frac{\Theta(I)}{|N_G(I) : I|} \bar{\epsilon}_I^G.$$

Proof. The correspondence $M \leftrightarrow \mathcal{L}$ is a special case of Proposition 6.3. As a direct sum of Mackey functors, $_p\mathbb{Q}B^* = {}_p\mathbb{Q}A_\mathbb{Q}^* \oplus {}_p\mathbb{Q}K^*$. So, confining our attention to those Mackey morphisms $_p\mathbb{Q}K \to {}_p\mathbb{Q}B^*$ whose image is contained in $_p\mathbb{Q}K^*$, we see that the correspondence $\theta \leftrightarrow \Theta$ is a special case of Proposition 6.4. □

Theorem 8.8. *Let* $(M, \theta) \leftrightarrow (\mathcal{L}, \Theta)$ *as in Proposition 8.7. Then:*
(1) θ *is an inflaky morphism if and only if M is an inflaky subfunctor and* $\Theta(G) = \Theta(\overline{G}) \beta(\overline{G}, G)$ *whenever G and* $\overline{G}$ *belong to* $\mathcal{L}$.
(2) θ *is a deflaky morphism if and only if M is a deflaky subfunctor and* $\Theta(G) = \Theta(\overline{G}) \beta(\overline{G}, G)$ *whenever G and* $\overline{G}$ *belong to* $\mathcal{L}$.

Proof. For part (1), the proof of Theorem 6.7 carries over almost verbatim, except that $\overline{K}$ and $\overline{I}$ are now assumed to be non-cyclic and the term ϵ_I^G is to be replaced by $\bar{\epsilon}_I^G$. For part (2), the proof of Lemma 6.12 carries over in a similar way, and it actually simplifies slightly because, as we noted in Corollary 8.6, $\beta_G(\overline{I}, I)$ is non-zero whenever $\overline{I}$ is non-cyclic. □

Putting $M = {}_p\mathbb{Q}K$, we deduce that if θ is an inflaky morphism or a deflaky morphism, then θ is a morphism of biset functors. Proposition 5.5 now completes the proof of Theorem 5.6, and it almost completes the proof of Theorem 5.7.

To finish the proof of Theorem 5.7, consider a deflaky morphism $\theta : D \to {}_pK^*$ where D is a deflaky subfunctor of $_pK$. It remains only to show that θ is a restriction of a $\mathbb{Q}$-multiple of $\pi^* \circ \mathrm{torn}^p$. We may assume that θ is non-zero, hence D is non-zero and $\mathcal{L} \neq \emptyset$. Suppose that $G \in \mathcal{L}$. Since G is non-cyclic, C_p^2 is isomorphic to a quotient group of G. As we noted in the proof of Lemma 8.2, the deflation number $\beta(C_p^2, G)$ is well-defined. Corollary 8.6 implies that $\beta(C_p^2, G) \neq 0$. By considering the round-coordinate formula for deflation, we deduce that $C_p^2 \in \mathcal{L}$. Letting Θ be as in Theorem 8.8, then $\Theta(G) = \Theta(C_p^2) \beta(C_p^2, G)$. Thus, θ is determined by $\Theta(C_p^2)$, and we deduce that θ is the restriction to D of some $\mathbb{Q}$-multiple of $\pi^* \circ \mathrm{torn}^p$. The proof of Theorem 5.7 is now finished.

All of the theorems stated in Section 5 have now been established. We must not forget to deal with Remark 5.8. But Remark 5.8 is clear from Remarks 7.2 and 8.1, together with the round-coordinate formulas for the elemental maps on $\mathbb{Q}B(G)$.

9. Reduction to the functor D^Ω

All of results stated in earlier sections have now been established, except for Theorem 2.4. We shall prove that theorem and then we shall note a corollary concerning the biset functor D^Ω.

To prove Theorem 2.4, we must show that $\mathrm{jef}_{\overline{G}, G}(\mathrm{torn}_G^\pi(\kappa)) - \mathrm{torn}_{\overline{G}}^\pi(\mathrm{def}_{\overline{G}, G}(\kappa)) \in \exp_G(B)$ for all $\kappa \in K(G)$. We shall show, in fact, that

$$\mathrm{jef}_{\overline{G}, G}(\widetilde{\mathrm{torn}}_G^\pi(x)) - \widetilde{\mathrm{torn}}_{\overline{G}}^\pi(\mathrm{def}_{\overline{G}, G}(x)) \in \exp_G(B)$$

for all $x \in B(G)$. We may assume that $x = d_U^G$. So Theorem 2.4 follows from the next lemma.

Lemma 9.1. *We have* $\mathrm{jef}_{\overline{G}, G}(\widetilde{\mathrm{torn}}_G^\pi(d_U^G)) - \widetilde{\mathrm{torn}}_{\overline{G}}^\pi(\mathrm{def}_{\overline{G}, G}(d_U^G)) = \log_\pi |NU : U| \exp_{\overline{G}}(d_{\overline{U}}^{\overline{G}})$.

Proof. We have $\mathrm{def}_{\overline{G}, G}(d_U^G) = [G/NU]$ where G/NU is regarded as a $\overline{G}$-set in the evident way. The $\overline{W}$-orbit of the element $gNU \in G/NU$ consists of the elements having the form $wgNU$ where $w \in W$. Fixing g, then the number of such elements $wgNU$ is $|WgNU|/|NU|$. But W contains the normal subgroup N, so $WgNU = WgU$. Therefore

$$[\widetilde{\mathrm{torn}}_{\overline{G}}^\pi(\mathrm{def}_{\overline{G}, G}(d_U^G)) @ \overline{W}] = \sum_{WgNU \subseteq G} \log_\pi(|WgNU|/|NU|) = \sum_{WgU \subseteq G} \log_\pi(|WgU|/|NU|).$$

On the other hand,

$$[\mathrm{jef}_{\overline{G}, G}(\widetilde{\mathrm{torn}}_G^\pi(d_U^G)) @ \overline{W}] = [\widetilde{\mathrm{torn}}_G^\pi(d_U^G) @ W] = \sum_{WgU \subseteq G} \log_\pi(|WgU|/|U|).$$

Since $\log_\pi(|WgU|/|U|) - \log_\pi(|WgU|/|NU|) = \log_\pi(|NU : U|)$, we have

$$[\mathrm{jef}_{\overline{G}, G}(\widetilde{\mathrm{torn}}_G^\pi(d_U^G)) @ \overline{W}] - [\widetilde{\mathrm{torn}}_{\overline{G}}^\pi(\mathrm{def}_{\overline{G}, G}(d_U^G)) @ \overline{W}] = \log_\pi(|NU : U|) |W \backslash G/U|$$

$$= \log_\pi(|NU : U|) |\overline{W} \backslash \overline{G}/\overline{U}| = \log_\pi(|NU : U|)[\exp_{\overline{G}}(d_{\overline{U}}^{\overline{G}}) @ \overline{W}]. \quad \square$$

For the rest of this section, we shall be discussing the *p*-biset functor D^Ω. Throughout the discussion, G is assumed to be a *p*-group, and all the biset functors are understood to be *p*-biset functors. Let us briefly recall some material from Bouc [5–7], Bouc–Yalçın [11]. Consider a non-empty G-set X and a field $\mathbb{F}$ with characteristic p. The **relative syzygy** of $\mathbb{F}$ with respect to X, denoted $\Omega_X(\mathbb{F})$, is defined to be the kernel of the augmentation map $\mathbb{F}X \to \mathbb{F}$. The $\mathbb{F}G$-module $\Omega_X(\mathbb{F})$ is a capped endo-permutation module if and only if $|X^G| \neq 1$. In that case, we let Ω_X be the image of $\Omega_X(\mathbb{F})$ in the Dade group $D_\mathbb{F}(G)$. When $|X^G| = 1$, we define $\Omega_X = 0$ in $D_\mathbb{F}(G)$. The subgroup of $D_\mathbb{F}(G)$ generated by the elements Ω_X, denoted $D^\Omega(G)$, is independent of $\mathbb{F}$. Actually, [5, 3.1.2] says that if $|X^G| \geq 2$ then $\Omega_X = 0$. So $D^\Omega(G)$ is generated by those elements Ω_X such that $X^G = \emptyset$.

Bouc [5, Section 6] showed that, letting G vary, the groups $D^\Omega(G)$ give rise to a p-biset functor D^Ω. In [7, 10.2], he showed that the groups $D_\mathbb{F}(G)/D^\Omega(G)$ give rise to a p-biset functor $D_\mathbb{F}/D^\Omega$ which is isomorphic to a subfunctor of $\mathbb{F}_2 A_\mathbb{Q}$. The groups $D_\mathbb{F}(G)$ do not always give rise to a p-biset functor but, in [7, 7.7], he showed that $D_\mathbb{F}(G) = D^\Omega(G)$ when $p \neq 2$. Thus, in a matter of speaking, most of $D_\mathbb{F}(G)$ is contained in $D^\Omega(G)$.

Let ω_X be the element of $B^*(G)$ such that

$$[\omega_X @ U] = \lfloor X^U \neq \emptyset \rfloor = \lfloor [U @ X] \neq 0 \rfloor.$$

Note that, letting X run over the transitive G-sets up to isomorphism, then ω_X runs over the elements of a $\mathbb{Z}$-basis for $B^*(G)$. Bouc [6, 1.7] constructed a morphism of p-biset functors $\Psi : B^* \to D^\Omega$ such that

$$\Psi_G(\omega_X) = \Omega_X$$

for all X. His result [6, 1.8] asserts that Ψ induces an isomorphism of p-biset functors

$$K^* \cong D^\Omega/D^\Omega_{\text{tors}}$$

where $D^\Omega_{\text{tors}}(G)$ is the torsion subgroup of $D^\Omega(G)$. Combining this with Proposition 5.5 and Theorems 5.6 and 5.7, we obtain the next result.

Proposition 9.2. *For p-groups, the inflaky morphisms $K \to D^\Omega/D^\Omega_{\text{tors}}$ and the deflaky morphisms $K \to D^\Omega/D^\Omega_{\text{tors}}$ are all morphisms of p-biset functors. Furthermore, the free cyclic $\mathbb{Z}$-module $\mathrm{Mor}_{\text{bis}}(K, D^\Omega/D^\Omega_{\text{tors}}) \cong \mathbb{Z}$ is generated by the morphism $\pi \circ \Psi \circ \mathrm{torn}^p/(1-p)$, where $\pi : D^\Omega \to D^\Omega/D^\Omega_{\text{tors}}$ is the canonical epimorphism.*

Thus far, we have not made use of Theorem 2.4. But we shall be needing it in a moment, in connection with a theorem of Bouc–Yalçin [11, 1.2] which refines the isomorphism $\overline{\Psi}$. Some of the results in [11] are stated in terms of a biset functor denoted C_b which is defined in terms of the Borel–Smith relations. But, [11, 3.3] asserts that $C_b = \text{die}(A_\mathbb{R})$. By Proposition 7.4, $\text{die}(A_\mathbb{R}) \leq A^*_\mathbb{Q}$; this inclusion was already observed in [11, 4.3]. Meanwhile, by the Ritter–Segal Theorem, the morphism of p-biset functors $\text{lin} : B \to A_\mathbb{Q}$ is surjective and $\text{die}(A_\mathbb{Q}) = \exp(B)$. Thus,

$$\exp(B) = \text{die}(A_\mathbb{Q}) \leq \text{die}(A_\mathbb{R}) \leq A^*_\mathbb{Q}$$

as a chain of p-biset functors. The theorem [11, 1.2] asserts that the morphism $\Psi : B^* \to D^\Omega$ induces an isomorphism of p-biset functors

$$B^*/\text{die}(A_\mathbb{R}) \cong D^\Omega.$$

Since $\exp(B) \leq \text{die}(A_\mathbb{R})$, Theorem 2.4 yields the following result.

Theorem 9.3. *The composite $\Psi \circ \mathrm{torn}^p/(1-p) : K \to D^\Omega$ is a morphism of p-biset functors.*

The morphism $\Psi \circ \mathrm{torn}^p/(1-p)$ seems to merit further study. In particular, we ask as to whether it can be characterized in a more direct or more structuralistic way, and as to whether there might be a more direct or more structuralistic proof of the morphism property.

10. Reduction to the unit functor

In Section 2, we sketched some relationships between the lifted morphisms discussed in the present paper and the reduced morphisms discussed in [2]. In this final section, we explain these relationships in detail, and then we establish two results that will be needed in [3].

We begin with a review of some material in Yoshida [18], Yalçin [16], Bouc [8] concerning the biset functors $B^\times$ and $\beta^\times$. Our notation is taken from [2]. In abstract, the ghost unit functor $\beta^\times$ is easy to characterize. Recall that the **ghost ring** $\beta(G)$ is defined to be the subring of $\mathbb{Q}B(G)$ spanned over $\mathbb{Z}$ by the primitive idempotents of $\mathbb{Q}B(G)$. The unit group of $\beta(G)$ is the elementary abelian 2-group $\beta^\times(G) = \{x \in B(G) : x^2 = 1\}$. Regarding $\beta^\times(G)$ as an $\mathbb{F}_2$-vector space, it has a basis $\{\partial_I^G : I \leq_G G\}$ where $\partial_I = 1 - 2e_I^I$. With respect to this basis, we write $[x @ I] \in \mathbb{F}_2$ for the coordinate of $x \in \beta^\times(G)$ at I. Thus,

$$x = \sum_{I \leq_G G} [x @ I] \, \partial_I^G.$$

To make $\beta^\times$ become a biset functor, we define the actions of the elemental maps by the equations

$$\text{iso}_{F,G}^\phi(\partial_U^G) = \partial_{\phi(U)}^F, \qquad [\text{iso}_{F,G}^\phi(x) @ \phi(U)] = [x @ U],$$

$$\text{res}_{H,G}(\partial_U^G) = \sum_{V \leq_H V : V =_G U} \partial_V^G, \qquad [\text{res}_{H,G}(x) @ V] = [x @ V],$$

$$\text{jnd}_{G,H}(\partial_V^G) = \sum_{U \leq_G G, HgU \subseteq G : V =_H H \cap {}^g U} \partial_U^G, \qquad [\text{jnd}_{G,H}(y) @ U] = \sum_{HgU \subseteq G} [y @ H \cap {}^g U],$$

$$\text{jef}_{\overline{G},G}(\partial_U^G) = \lfloor N \leq U \rfloor \, \partial_{\overline{U}}^{\overline{G}}, \qquad [\text{jef}_{\overline{G},G}(x) @ \overline{W}] = [x @ W],$$

$$\text{inf}_{G,\overline{G}}(\partial_{\overline{W}}^{\overline{G}}) = \sum_{U \leq_G G : \overline{U} =_{\overline{G}} \overline{W}} \partial_U^G, \qquad [\text{inf}_{G,\overline{G}}(z) @ U] = [z @ \overline{U}]$$

where x, y, z are elements of $\beta^\times(G)$, $\beta^\times(H)$, $\beta^\times(\overline{G})$, respectively. Comparing with the square-coordinate equations for the elemental maps on B^*, we see that the actions we have defined on the coordinate modules $\beta^\times(G)$ do indeed make $\beta^\times$ become a biset functor and, furthermore, there is an epimorphism of biset functors

$$\mathrm{mod} : B^* \to \beta^\times$$

given by $\delta_I^G \mapsto \partial_I^G$. Thus, we have realized $\beta^\times$ as the modulo 2 reduction of B^*.

Yoshida [18, Section 2] introduced two functions $\mathrm{jnd}_{G,H} : \mathbb{Q}B(H) \to \mathbb{Q}B(G)$ and $\mathrm{jef}_{\overline{G},G} : \mathbb{Q}B(G) \to \mathbb{Q}B(\overline{G})$ given by the round coordinate equations

$$(I @ \mathrm{jnd}_{G,H}(y)) = \prod_{HgI \subseteq G} (H \cap {}^g I @ y), \qquad (\overline{K} @ \mathrm{jef}_{\overline{G},G}(x)) = (K @ x)$$

where $x \in \mathbb{Q}B(G)$ and $y \in \mathbb{Q}B(H)$. He explained how the function $\mathrm{jnd}_{G,H}$ arises from the functor $\mathrm{Map}_G(G, -) : H\text{-set} \to G\text{-set}$ (variously called Japanese induction, tensor induction or multiplicative induction), while the function $\mathrm{jef}_{\overline{G},G}$ arises from the functor $-^N : G\text{-set} \to \overline{G}\text{-set}$ (called Japanese deflation or multiplicative deflation). Yoshida showed that the functions $\mathrm{jnd}_{G,H}$ and $\mathrm{jef}_{\overline{G},G}$ have the following three properties: they preserve products of elements of the Burnside algebras $\mathbb{Q}B(H)$, $\mathbb{Q}B(G)$, $\mathbb{Q}B(\overline{G})$; they restrict to functions between the ghost rings $\beta(H)$, $\beta(G)$, $\beta(\overline{G})$; they further restrict to functions on the Burnside rings $B(H), B(G), B(\overline{G})$. The third property, by far the most difficult, was established by Yoshida using a theory of polynomial maps; another proof, given by Yalçın [16], made use of Lefschetz invariants of G-posets. On the other hand, it is easy to see (without using any theory of polynomial maps or Lefschetz invariants) that the usual isogation, restriction and inflation maps on the Burnside algebras $\mathbb{Q}B(-)$ have the same three properties. It follows that the five kinds of maps iso(), res(), jnd(), jef(), inf() restrict to maps on the ghost unit groups $\beta^\times$ and maps on the Burnside unit groups $B^\times(-)$. Yoshida's formulas [18, Section 3b] coincide with the formulas we gave above for the actions of the elemental maps on $\beta^\times$. We deduce that the elemental maps on $\beta^\times$ preserve the subspaces $B^\times(-) \leq \beta^\times(-)$, hence $B^\times$ becomes a biset subfunctor of $\beta^\times$. We write the inclusion morphism as

$$\mathrm{inc} : B^\times \to \beta^\times.$$

Comparing the definitions of exp(), die(), torn^π given in Section 5 with the definitions of $\overline{\exp}$, $\overline{\mathrm{die}}$, $\overline{\mathrm{torn}}^\pi$ given in [2], we see that the two diagrams depicted in Section 2 are commutative. We can now tie up a loose end from [2], where the proof that $\overline{\mathrm{torn}}^\pi$ is an inflaky morphism depended on the fact that the zombie morphism zom^α is an inflaky morphism. A much easier argument is simply to observe that $\overline{\mathrm{torn}}^\pi$ is an inflaky morphism because torn^π is an inflaky morphism.

The next two results will be needed in [3]. The first of them is due to Tornehave [14], who obtained it by direct calculation. We point out that it also follows immediately from Corollary 8.5.

Proposition 10.1 (*Tornehave*). *If G is a non-cyclic 2-group, then $2e_G^G \in K(G)$ and*

$$\overline{\mathrm{torn}}_G^2(2e_G^G) = \partial_G^G.$$

The second of the two results contains much of the content of the main theorem in [3].

Proposition 10.2. *Let $\frac{\mathbb{Q}}{2}B^\times = \overline{\exp}(B) = \overline{\mathrm{die}}(A_\mathbb{Q})$, and let $\pi_\mathbb{Q}^\times : {}_2B^\times \to {}_2B^\times/\frac{\mathbb{Q}}{2}B^\times$ be the canonical epimorphism of 2-biset functors. Then the composite $\pi_\mathbb{Q}^\times \circ \overline{\mathrm{torn}}^2 : {}_2K \to {}_2B^\times/\frac{\mathbb{Q}}{2}B^\times$ is non-zero morphism of 2-biset functors.*

Proof. Theorem 2.4 implies that $\pi_\mathbb{Q}^\times \circ \overline{\mathrm{torn}}^2$ is a morphism of 2-biset functors. Proposition 10.1 implies that $\pi_\mathbb{Q}^\times \circ \overline{\mathrm{torn}}^2 \neq 0$. $\square$

In [3, Section 6], we shall show that, in fact, $\pi_B^\times \circ \overline{\mathrm{torn}}^2$ is an epimorphism of 2-biset functors. The proof of that stronger conclusion will be based on a reexamination of an argument of Tornehave [14].

Bouc [9, 6.10] remarked that the crucial difficulty in obtaining his theorem [9, 6.5] was in showing that there exists a non-zero morphism of 2-biset functors ${}_2K/\frac{\mathbb{Q}}{2}K \to {}_2B^\times/\frac{\mathbb{Q}}{2}B^\times$, where $\frac{\mathbb{Q}}{2}K$ is the 2-biset subfunctor of ${}_2K$ generated by the coordinate module $K(D_8)$. But that assertion follows quickly from Proposition 10.2. Indeed, a direct calculation shows $\pi_\mathbb{Q}^\times \circ \overline{\mathrm{torn}}^2$ annihilates $K(D_8)$ and hence also annihilates $\frac{\mathbb{Q}}{2}K$. Therefore, thanks to Proposition 10.2 $\pi_\mathbb{Q}^\times \circ \overline{\mathrm{torn}}^2$ induces a non-zero morphism of 2-biset functors ${}_2K/\frac{\mathbb{Q}}{2}K \to {}_2B^\times/\frac{\mathbb{Q}}{2}B^\times$.

Appendix. Deflation numbers for *p*-groups

In this appendix, we give a little application of the lifted Tornehave morphism torn^p. Or rather, we make a comment concerning a connection between the deflation numbers and the diagonal formula for torn^p.

Of the five round-coordinate formulas for the elementary maps on $\mathbb{Q}B$, the most difficult, the one for deflation, involves the deflation numbers $\beta(\overline{G}, G)$, which are defined in terms of the Möbius function on the poset of subgroups of G. For arbitrary finite groups, no very explicit formula for the deflation numbers is known (and there seems to be no good reason

to expect that a very explicit formula exists). For finite p-groups, though, we have the following theorem of Bouc–Thévenaz [10, 4.8, 8.1]. Observe that

$$\beta_p(c, d) = \prod_{s=c-1}^{d-2} (1 - p^s) = \begin{cases} 1 & \text{if } c = d, \\ (p-1)/p & \text{if } c = 0 \text{ and } d = 1, \\ 0 & \text{if } c \leq 1 \text{ and } 2 \leq d, \\ \beta_p(d)/\beta_p(c) & \text{if } 2 \leq c. \end{cases}$$

Theorem A.1 (*Bouc–Thévenaz*). *Suppose that G is a p-group. Let d and c be the ranks of G and $\overline{G}$, respectively. Then $\beta(\overline{G}, G) = \beta_p(c, d)$.*

We give a quick alternative proof of Theorem A.1 using properties of exp() and torn^p. (But the original proof of Theorem A.1 in Bouc–Thévenaz [10] is quite short anyway.) In the notation of Theorem 6.7, $\beta(\overline{G}, G) = \Theta(G)/\Theta(\overline{G})$ when $\Theta(\overline{G})$ is defined and non-zero. Since exp() is an inflaky morphism, we can put $\Theta = \Theta^{\exp}$, whereupon Proposition 7.3 implies that, if $\overline{G}$ is cyclic, then $\beta(\overline{G}, G) = (\phi(G)/|G|)/(\phi(\overline{G})/|\overline{G}|)$. But $\phi(G)/|G|$ is 1 or $(p-1)/p$ or 0 depending on whether $d = 0$ or $d = 1$ or $d \geq 2$, respectively. We deduce Theorem A.1 in the case $c \leq 1$. Meanwhile, since torn^p is an inflaky morphism, we can put $\Theta = \Theta^p$, whereupon Proposition 8.3 implies that, if $\overline{G}$ is non-cyclic, then $\beta(\overline{G}, G) = \beta_p(d)/\beta_p(c)$. We now proved Theorem A.1 in general.

The argument can be put in reverse, supplying an alternative proof Proposition 8.3 using Theorem A.1 together with Theorem 6.7 and the $d = 2$ case of Corollary 8.5.

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