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# On the solvability of the discrete second Painlevé equation

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**Abstract**
The inverse monodromy method for studying the Riemann–Hilbert problem associated with classical Painlevé equations is applied to the discrete second Painlevé equation.

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## 1. Introduction

The inverse monodromy method (IMM), an extension of the inverse spectral method (ISM) to ordinary differential equations (ODE), was introduced in [1–6] for studying the initial value problem for certain nonlinear ODEs. This method can be thought as a nonlinear analog of the Laplace transform. Solving such an initial value problem is essentially equivalent to solving an inverse problem for a certain isomonodromic linear equation.

Rigorous investigation of the six continuous Painlevé transcendants, $P_I$–$P_{VI}$ [7] using this method has been carried out in [8–10]. The isomonodromy method is based on the fact that every Painlevé equation can be written as the compatibility condition of two linear equations (Lax Pair). Using this Lax pair, it is possible to reduce the solution of the Cauchy problem for a given Painlevé equation to the solution of a Riemann–Hilbert (RH) problem. This RH-problem is formulated in terms the so-called monodromy data which can be calculated in terms of the two initial data.

The IMM consists of the following two basic steps. (i) The direct problem: one of the two equations of the Lax pair is a linear ODE in the variable $\lambda$ for an eigenfunction $Y(\lambda, t)$. The essence of the direct problem is to establish the analytic structure of $Y(\lambda, t)$ in the entire complex $\lambda$-plane. Analytic structure of the eigenfunction $Y(\lambda, t)$ is characterized by the monodromy data. An important part of the direct problem is to establish that the set of all monodromy data can be written in terms of two of them. (ii) The inverse problem: the result obtained in part (i) can be used to formulate a continuous and regular RH-problem on a self-intersecting contour with the jump matrices defined in terms of the monodromy data. The RH-problem is equivalent to a certain Fredholm integral equation. Having established the solvability of the RH-problem, it can be shown that $Y(\lambda, t)$, defined as the solution of the RH-problem, satisfies the original Lax pair and hence can be used to derive solutions of the given Painlevé equation. Since the RH-problem is defined in terms of the monodromy data, which is calculated in terms of initial data, this step provides the solution of the Cauchy problem.

Recently, nonlinear integrable discrete equations among which the discrete Painlevé (dP) equations play a fundamental role, have attracted much attention. The difference relations related with the Painlevé equations, and discrete equations associated with $P_{VI}$ were first given by Jimbo and Miwa [3]. The so-called singularity confinement method has been an important tool to derive integrable discrete Painlevé equations [11, 12]. A systematic derivation of the dP equations by using the Bäcklund transformations of the continuous Painlevé equations was given by Fokas, Grammaticos and Ramani [13]. Besides the rich mathematical structures of dP equations, such as the existence of Lax pairs, Bäcklund transformations, singularity confinement properties [14–19], the relation of dP equations to the continuous ones has been extensively investigated in the literature.

By exploiting the relation between the continuous and discrete Painlevé equations, in this paper we apply the IMM to the discrete second Painlevé, $dP_{II}$. In the case of the $dP_{II}$, the singularity structure of the monodromy problem is more complicated (regular singular points at $\lambda = \pm 1$ and irregular singular points at $\lambda = 0, \infty$ of rank $r = 2$) with respect to the monodromy problem of $P_{II}$.

The discrete second Painlevé equation, $dP_{II}$:

$$2c_3(x_{n+1} + x_{n-1})\left(1 - x_n^2\right) = -x_n(2c_2 + 2n + 1) + c_0, \qquad c_3 \neq 0, \tag{1}$$

can be written as the compatibility condition of the Joshi–Nijhoff Lax pair [20]

$$\frac{\partial Y_n}{\partial \lambda} = M_n(\lambda)Y_n(\lambda), \tag{2a}$$

$$Y_{n+1} = L_n(\lambda)Y_n(\lambda), \tag{2b}$$

where

$$M_n(\lambda) = M_1\lambda + M_2 + M_3\frac{1}{\lambda} + M_4\frac{1}{\lambda^2} + M_5\frac{1}{\lambda^3} + M_6\frac{1}{\lambda^2 - 1}, \qquad L_n = \begin{pmatrix} \lambda & x_n \\ x_n & 1/\lambda \end{pmatrix}, \tag{3}$$

and

$$\begin{gathered} M_1 = M_5 = c_3\sigma_3, \quad M_2 = \begin{pmatrix} 0 & 2c_3x_n \\ 2c_3x_{n-1} & 0 \end{pmatrix}, \quad M_3 = (c_2 + n - 2c_3x_nx_{n-1})\sigma_3, \\ M_4 = -\sigma_1 M_2 \sigma_1, \qquad M_6 = c_0\sigma_1, \\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{gathered} \tag{4}$$

and $Y_n(\lambda)$ is $2 \times 2$ matrix-valued function in $\mathbb{C} \times \mathbb{C}$, $c_0$, $c_2$ and $c_3$ are constant parameters.

Entries $(1, 1)$ and $(2, 2)$ of the compatibility condition $\frac{\partial L_n}{\partial \lambda} + L_nM_n = M_{n+1}L_n$ are identically satisfied and entries $(1, 2)$ and $(2, 1)$ give the $dP_{II}$ .

The $dP_{II}$ equation (1) first appeared in the papers [21, 22]. In [22], Nijhoff and Papageorgious derived it as similarity reduction of an integrable lattice. We remark that, before the discovery of the Lax pair (2)–(4), due to Joshi and Nijhoff (an unpublished work), other examples of Lax pairs for $dP_{II}$ were known in the literature. In [22], such $2 \times 2$ Lax

Pair was already written, albeit with one less parameter. In [23], isomonodromic deformation problems for $dP_I$, $dP_{II}$, and $dP_{III}$ were obtained starting from the isospectral problems of two-dimensional integral mappings by using the procedure of de-autonomization of the spectral problems for mappings, obtaining $2 \times 2$ and $3 \times 3$ Lax matrices for $dP_I$ and for an alternative version of $dP_I$ respectively, and $4 \times 4$ linear problems associated with $dP_{II}$ and $dP_{III}$. In [24] a $2 \times 2$ Lax pair of the Ablowitz–Ladik type was constructed, not producing $dP_{II}$ directly, but rather a derivative form of it.

The IMM has been first applied to dP equations in [25, 26], where $dP_I$ was studied. In [26], $dP_I$ was obtained by using the similarity reduction of the Kac–Moerbeke (KM) equation, which is a discrete analog of the Korteweg–deVries (KdV) equation. Moreover, the associated Lax pair for $dP_I$ was derived and IMM was applied. In the case of $dP_I$, the singularity structure of Lax pair is much simpler, and hence the contours of the RH-problem is less complicated with respect to the case of $dP_{II}$.

## 2. Direct problem

In this section, we establish the analytic structure of $Y_n(\lambda)$ in the entire complex $\lambda$-plane by solving the linear problem (2a) which implies the existence of irregular singular points at the origin and at infinity with rank $r = 2$, and regular singular points at $\lambda = \pm 1$.

*Solution near $\lambda = 0$.* Since $\lambda = 0$ is an irregular singular point, the solution $Y_n^{(0)}(\lambda) = \big(Y_{n,1}^{(0)}(\lambda), Y_{n,2}^{(0)}(\lambda)\big)$, of (2a) has unique asymptotic expansion $\tilde{Y}_n^{(0)}(\lambda) = \big(\tilde{Y}_{n,1}^{(0)}(\lambda), \tilde{Y}_{n,2}^{(0)}(\lambda)\big)$ in certain sectors $S_j^{(0)}$ of the complex $\lambda$-plane. That is, $Y_{n(j)}(\lambda) \sim \tilde{Y}_n^{(0)}(\lambda) = \big(\tilde{Y}_{n,1}^{(0)}(\lambda), \tilde{Y}_{n,2}^{(0)}(\lambda)\big)$, as $\lambda \to 0$, in certain sectors $S_j^{(0)}$, $j = 1, \ldots, 4$ in the $\lambda$-plane. The formal expansion $\tilde{Y}_n^{(0)}(\lambda)$ near $\lambda = 0$ is given by

$$\tilde{Y}_n^{(0)}(\lambda) = \hat{Y}_n^{(0)}(\lambda)\left(\frac{1}{\lambda}\right)^{D_n^{(0)}} \mathrm{e}^{Q^{(0)}(\lambda)} = \big(I + \hat{Y}_{n,1}^{(0)}\lambda + \hat{Y}_{n,2}^{(0)}\lambda^2 + \cdots\big)\left(\frac{1}{\lambda}\right)^{D_n^{(0)}} \mathrm{e}^{Q^{(0)}(\lambda)}, \tag{5}$$

where

$$\begin{gathered}\hat{Y}_{n,1}^{(0)} = \begin{pmatrix} 0 & x_{n-1} \\ -x_n & 0 \end{pmatrix}, \qquad \hat{Y}_{n,2}^{(0)} = \begin{pmatrix} y_{11}^{(0)} & 0 \\ 0 & y_{22}^{(0)} \end{pmatrix}, \\ y_{11}^{(0)} = \tfrac{1}{2}\big[2c_3\big(x_n^2 x_{n-1}^2 - x_{n-1}^2 - x_n^2\big) - 2\big(c_2 + n + \tfrac{1}{2}\big)x_n x_{n-1} + c_0(x_n + x_{n-1}) + c_3\big], \\ y_{22}^{(0)} = \tfrac{1}{2}\big[2c_3\big(x_{n-1}^2 + x_n^2 - x_n^2 x_{n-1}^2\big) + 2\big(c_2 + n - \tfrac{1}{2}\big)x_n x_{n-1} - c_0(x_n + x_{n-1}) - c_3\big],\end{gathered} \tag{6}$$

and

$$D_n^{(0)} = -(c_2 + n)\sigma_3, \qquad Q^{(0)}(\lambda) = -\frac{c_3}{2\lambda^2}\sigma_3. \tag{7}$$

The relevant sectors $S_j^{(0)}$, $j = 1, \ldots, 4$ are determined by $\mathrm{Re}[\frac{c_3}{2\lambda^2}] = 0$ and given in figure 1. The non-singular matrices $Y_{n(j)}^{(0)}(\lambda)$, $j = 1, \ldots, 4$ satisfy

$$\begin{aligned} &Y_{n(j+1)}^{(0)}(\lambda) = Y_{n(j)}^{(0)}(\lambda)G_j^{(0)}, \qquad \lambda \in S_{j+1}^{(0)}, \qquad j = 1, 2, 3, \\ &Y_{n(1)}^{(0)}(\lambda) = Y_{n(4)}^{(0)}(\lambda\,\mathrm{e}^{2\mathrm{i}\pi})G_4^{(0)}M^{(0)}, \qquad \lambda \in S_1^{(0)}, \end{aligned} \tag{8}$$

where the Stokes matrices $G_j^{(0)}$ and the monodromy matrix $M^{(0)}$ are given as

$$\begin{aligned} &G_1^{(0)} = \begin{pmatrix} 1 & a^{(0)} \\ 0 & 1 \end{pmatrix}, \quad G_2^{(0)} = \begin{pmatrix} 1 & 0 \\ b^{(0)} & 1 \end{pmatrix}, \quad G_3^{(0)} = \begin{pmatrix} 1 & c^{(0)} \\ 0 & 1 \end{pmatrix}, \\ &G_4^{(0)} = \begin{pmatrix} 1 & 0 \\ d^{(0)} & 1 \end{pmatrix}, \qquad M^{(0)} = \mathrm{e}^{-2\mathrm{i}\pi c_2\sigma_3}, \end{aligned} \tag{9}$$

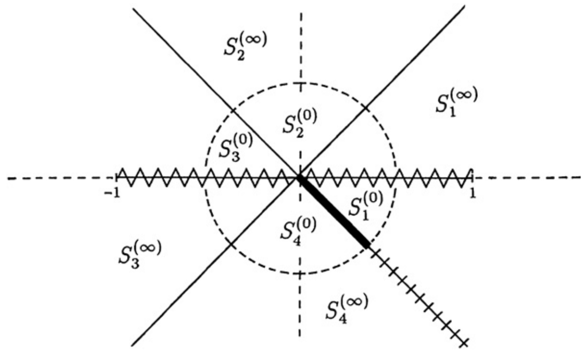


**Figure 1.** Sectors for the sectionally analytic function $Y_n(\lambda)$.

and the sectors are

$$\begin{aligned} &S_1^{(0)}: -\frac{\pi}{4} \leqslant \arg\lambda < \frac{\pi}{4}, \qquad S_2^{(0)}: \frac{\pi}{4} \leqslant \arg\lambda < \frac{3\pi}{4}, \\ &S_3^{(0)}: \frac{3\pi}{4} \leqslant \arg\lambda < \frac{5\pi}{4}, \qquad S_4^{(0)}: \frac{5\pi}{4} \leqslant \arg\lambda < \frac{7\pi}{4}, \qquad |\lambda| < 1. \end{aligned} \tag{10}$$

The entries $a^{(0)}$, $b^{(0)}$, $c^{(0)}$ and $d^{(0)}$ of the Stokes matrices $G_j^{(0)}$ are constants with respect to $\lambda$.

*Solution near* $\lambda = \infty$. The solution of (2*a*) possesses a formal expansion of the form $Y_{n(j)}(\lambda) \sim \tilde{Y}_n^{(\infty)}(\lambda) = \big(\tilde{Y}_{n,1}^{(\infty)}(\lambda), \tilde{Y}_{n,2}^{(\infty)}(\lambda)\big)$, as $\lambda \to \infty$, in certain sectors $S_j^{(\infty)}$, $j = 1, \ldots, 4$ in the $\lambda$-plane. The formal expansion $\tilde{Y}_n^{(\infty)}(\lambda)$ near $\lambda = \infty$ is given by

$$\tilde{Y}_n^{(\infty)}(\lambda) = \hat{Y}_n^{(\infty)}(\lambda)\lambda^{D_n^{(\infty)}} \mathrm{e}^{Q^{(\infty)}(\lambda)} = \big(I + \hat{Y}_{n,1}^{(\infty)}\lambda^{-1} + \hat{Y}_{n,2}^{(\infty)}\lambda^{-2} + \cdots\big)\lambda^{D_n^{(\infty)}} \mathrm{e}^{Q^{(\infty)}(\lambda)}, \tag{11}$$

where

$$\begin{aligned} &\hat{Y}_{n,1}^{(\infty)} = \begin{pmatrix} 0 & -x_n \\ x_{n-1} & 0 \end{pmatrix}, \qquad \hat{Y}_{n,2}^{(\infty)} = \begin{pmatrix} y_{11}^{(\infty)} & 0 \\ 0 & y_{22}^{(\infty)} \end{pmatrix}, \\ &y_{11}^{(\infty)} = \tfrac{1}{2}\big[2c_3\big(x_{n-1}^2 + x_n^2 - x_n^2 x_{n-1}^2\big) + 2\big(c_2 + n - \tfrac{1}{2}\big)x_n x_{n-1} - c_0(x_n + x_{n-1}) - c_3\big], \\ &y_{22}^{\infty)} = \tfrac{1}{2}\big[2c_3\big(x_n^2 x_{n-1}^2 - x_n^2 - x_{n-1}^2\big) - 2\big(c_2 + n + \tfrac{1}{2}\big)x_n x_{n-1} + c_0(x_n + x_{n-1}) + c_3\big], \end{aligned} \tag{12}$$

and

$$D_n^{(\infty)} = (c_2 + n)\sigma_3, \qquad Q^{(\infty)}(\lambda) = \frac{c_3}{2}\lambda^2\sigma_3. \tag{13}$$

The relevant sectors $S_j^{(\infty)}$, $j = 1, \ldots, 4$ are determined by $\mathrm{Re}\big[\frac{c_3}{2}\lambda^2\big] = 0$ and given in figure 1. The non-singular matrices $Y_{n(j)}^{(\infty)}(\lambda)$, $j = 1, \ldots, 4$ satisfy

$$\begin{aligned} &Y_{n(j+1)}^{(\infty)}(\lambda) = Y_{n(j)}^{(\infty)}(\lambda)G_j^{(\infty)}, \qquad \lambda \in S_j^{(\infty)}, \qquad j = 1, 2, 3, \\ &Y_{n(1)}^{(\infty)}(\lambda) = Y_{n(4)}^{(\infty)}(\lambda\,\mathrm{e}^{2\mathrm{i}\pi})G_4^{(\infty)}M^{(\infty)}, \qquad \lambda \in S_1^{(\infty)}, \end{aligned} \tag{14}$$

where the Stokes matrices $G_j^{(\infty)}$ and the monodromy matrix $M^{(\infty)}$ are given as

$$\begin{aligned} &G_1^{(\infty)} = \begin{pmatrix} 1 & 0 \\ a^{(\infty)} & 1 \end{pmatrix}, \quad G_2^{(\infty)} = \begin{pmatrix} 1 & b^{(\infty)} \\ 0 & 1 \end{pmatrix}, \quad G_3^{(\infty)} = \begin{pmatrix} 1 & 0 \\ c^{(\infty)} & 1 \end{pmatrix}, \\ &G_4^{(\infty)} = \begin{pmatrix} 1 & d^{(\infty)} \\ 0 & 1 \end{pmatrix}, \qquad M^{(\infty)} = \mathrm{e}^{-2\mathrm{i}\pi c_2\sigma_3}, \end{aligned} \tag{15}$$

and the sectors are

$$S_1^{(\infty)}: -\frac{\pi}{4} \leqslant \arg\lambda < \frac{\pi}{4}, \qquad S_2^{(\infty)}: \frac{\pi}{4} \leqslant \arg\lambda < \frac{3\pi}{4},$$
$$S_3^{(\infty)}: \frac{3\pi}{4} \leqslant \arg\lambda < \frac{5\pi}{4}, \qquad S_4^{(\infty)}: \frac{5\pi}{4} \leqslant \arg\lambda < \frac{7\pi}{4}, \qquad |\lambda| > 1. \tag{16}$$

The entries $a^{(\infty)}$, $b^{(\infty)}$, $c^{(\infty)}$ and $d^{(\infty)}$ of the Stokes matrices $G_j^{(\infty)}$ are constants with respect to $\lambda$.

*Solution near* $\lambda = 1$. Since $\lambda = 1$ is a regular singular point of (2*a*), the solution in the neighborhood of $\lambda = 1$ can be obtained via a convergent power series. For $\lambda = 1$, the solution $Y_n^{(1)}(\lambda) = \left(Y_{n,1}^{(1)}(\lambda), Y_{n,2}^{(1)}(\lambda)\right)$, for $c_0 \neq k$, $k \in \mathbb{Z}$ has the form

$$Y_n^{(1)}(\lambda) = \hat{Y}_n^{(1)}(\lambda)(\lambda-1)^{D^{(1)}} = \hat{Y}_{n,0}^{(1)}\left\{I + \hat{Y}_{n,1}^{(1)}(\lambda-1) + \hat{Y}_{n,2}^{(1)}(\lambda-1)^2 + \cdots\right\}(\lambda-1)^{D^{(1)}}, \tag{17}$$

$|\lambda - 1| < 1$, where

$$\hat{Y}_{n,0}^{(1)} = \begin{pmatrix} \mu_n^{(1)} & \nu_n^{(1)} \\ \mu_n^{(1)} & -\nu_n^{(1)} \end{pmatrix}, \qquad \det \hat{Y}_{n,0}^{(1)} = 1, \qquad D^{(1)} = \frac{c_0}{2}\sigma_3, \tag{18}$$

$$\mu_n^{(1)}\nu_n^{(1)} = -\frac{1}{2}, \qquad \mu_n^{(1)} = \mu_0^{(1)} \prod_{i=1}^{n-1}(1 + x_i). \tag{19}$$

It should be noted that $\mu_n^{(1)}$ and $\nu_n^{(1)}$ are constants with respect to $\lambda$, and $\mu_0^{(1)}$ is independent of $n$. $\hat{Y}_{n,1}^{(1)}$ satisfies the following equation:

$$\hat{Y}_{n,1}^{(1)} + \left[\hat{Y}_{n,1}^{(1)}, D^{(1)}\right] = \left(\hat{Y}_{n,0}^{(1)}\right)^{-1} M_0^{(1)} \hat{Y}_{n,0}^{(1)}, \tag{20}$$

where

$$M_0^{(1)} = \sum_{k=1}^{5} M_k - \frac{1}{4} M_6. \tag{21}$$

Equation (19) follows from the fact that $\det \hat{Y}_{n,0}^{(1)} = 1$, and $Y_n^{(1)}(\lambda)$ solves (2*b*). If $c_0 = k$, $k \in \mathbb{Z}$, the solution $Y_n^{(1)}(\lambda)$ may or may not contain the $\log(\lambda - 1)$ term. Monodromy matrix $M^{(1)}$ about $\lambda = 1$ is defined as

$$Y_n^{(1)}(\lambda\, \mathrm{e}^{2\mathrm{i}\pi}) = Y_n^{(1)}(\lambda) M^{(1)}, \qquad M^{(1)} = \mathrm{e}^{\mathrm{i}\pi c_0 \sigma_3}. \tag{22}$$

*Solution near* $\lambda = -1$. The solution $Y_n^{(-1)}(\lambda)$ in the neighborhood of the regular singular point $\lambda = 1$ can be obtained via a convergent power series. For $c_0 \neq k$, $k \in \mathbb{Z}$,

$$Y_n^{(-1)}(\lambda) = \hat{Y}_n^{(-1)}(\lambda)(\lambda+1)^{D^{(-1)}} = \hat{Y}_{n,0}^{(-1)}\left\{I + \hat{Y}_{n,1}^{(-1)}(\lambda+1) + \hat{Y}_{n,2}^{(-1)}(\lambda+1)^2 + \cdots\right\}(\lambda+1)^{D^{(-1)}} \tag{23}$$

$|\lambda + 1| < 1$, where

$$\hat{Y}_{n,0}^{(-1)} = \begin{pmatrix} \mu_n^{(-1)} & \nu_n^{(-1)} \\ -\mu_n^{(-1)} & \nu_n^{(-1)} \end{pmatrix}, \qquad \det \hat{Y}_{n,0}^{(-1)} = 1, \qquad D^{(-1)} = \frac{c_0}{2}\sigma_3, \tag{24}$$

$$\mu_n^{(-1)}\nu_n^{(-1)} = \frac{1}{2}, \qquad \mu_n^{(-1)} = (-1)^n \mu_0^{(-1)} \prod_{i=1}^{n-1}(1 + x_i), \qquad \mu_0^{(-1)} = \text{constant}, \tag{25}$$

and $\hat{Y}_{n,1}^{(1)}$ satisfies

$$\hat{Y}_{n,1}^{(-1)} + \left[\hat{Y}_{n,1}^{(-1)}, D^{(-1)}\right] = \left(\hat{Y}_{n,0}^{(-1)}\right)^{-1} M_0^{(-1)} \hat{Y}_{n,0}^{(-1)}, \tag{26}$$

where

$$M_0^{(-1)} = \sum_{k=1}^{5} (-1)^k M_k - \frac{1}{4} M_6. \tag{27}$$

Equation (25) follows from the fact that $\det \hat{Y}_{n,0}^{(-1)} = 1$ and $Y_n^{(-1)}(\lambda)$ solves (2*b*). If $c_0 = k$, $k \in \mathbb{Z}$, the solution $Y_n^{(-1)}(\lambda)$ may or may not contain the $\log(\lambda + 1)$ term.

Monodromy matrix about $\lambda = -1$ is defined as

$$Y_n^{(-1)}(\lambda\, e^{2i\pi}) = Y_n^{(-1)}(\lambda) M^{(-1)}, \qquad M^{(-1)} = e^{i\pi c_0 \sigma_3}. \tag{28}$$

Since $Y_n^{(\infty)}, Y_n^{(0)}, Y_n^{(1)}, Y_n^{(-1)}$ are locally analytic solutions of the linear equation (2*a*), they are related with constant (with respect to $\lambda$) matrices $E^{(0)}, E^{(1)}, E^{(-1)}$ which are called connection matrices:

$$Y_{n(1)}^{(\infty)}(\lambda) = Y_n^{(1)}(\lambda) E^{(1)}, \qquad Y_{n(3)}^{(\infty)}(\lambda) = Y_n^{(-1)}(\lambda) E^{(-1)}, \qquad Y_{n(1)}^{(\infty)}(\lambda) = Y_{n(1)}^{(0)}(\lambda) E^{(0)}, \tag{29}$$

where

$$E^{(j)} = \begin{pmatrix} \alpha^{(j)} & \beta^{(j)} \\ \gamma^{(j)} & \delta^{(j)} \end{pmatrix}, \qquad \det E^{(j)} = 1 \qquad j = -1, 0, 1. \tag{30}$$

The condition on the determinant of $E^{(j)} = 1$, $j = -1, 0, 1$ follows from the fact that the normalization of $Y_n^{(1)}, Y_n^{(-1)}$ and $Y_n^{(0)}, Y_n^{(\infty)}$ gives unit determinants. Branch cuts associated with the branch points $\lambda = \pm 1,\ 0,\ \infty$ are chosen along the real axis $-1 \leqslant |\lambda| < 0$, $0 < |\lambda| \leqslant 1$ for $\lambda = -1$ and $\lambda = 1$ respectively, and $0 \leqslant |\lambda| < 1$ and $1 < |\lambda| < \infty$, $\arg \lambda = -\pi/4$ for $\lambda = 0$ and $\lambda = \infty$, respectively, indicated in figure 1.

Clearly, the Stokes matrices $G_j^{(\infty)}, G_j^{(0)}$, $j = 1, \ldots, 4$, and the connection matrices $E^{(0)}, E^{(1)}$ and $E^{(-1)}$ are constants matrices with respect to $\lambda$, but they are also independent of $n$. Since, if we assume that $G_j^{(\infty)}$ depend on $n$, i.e. $G_j^{(\infty)} = G_{n,(j)}^{(\infty)}$, then by the definition of the Stokes matrices one can write $Y_{n+1,(j+1)}^{(\infty)} = Y_{n+1,(j)}^{(\infty)} G_{n+1,(j)}^{(\infty)}$, and using equation (2*b*), one gets $G_{n+1,(j)}^{(\infty)} = G_{n,(j)}^{(\infty)}$. Similar calculations hold for $G_j^{(0)}$, $j = 1, \ldots, 4$ and the connection matrices $E^{(0)}, E^{(1)}$ and $E^{(-1)}$.

*Symmetries of the differential equation.* The matrices $M_n(\lambda)$ and $L_n(\lambda)$ defined in (3) and (4) satisfy

$$\sigma_1 M_n\left(\frac{1}{\lambda}\right)\sigma_1 = -\lambda^2 M_n(\lambda), \qquad \sigma_1 L_n\left(\frac{1}{\lambda}\right)\sigma_1 = L_n(\lambda), \tag{31}$$

and

$$\sigma_3 M_n(\lambda\, e^{-i\pi})\sigma_3 = -M_n(\lambda), \qquad \sigma_3 L_n(\lambda\, e^{-i\pi})\sigma_3 = -L_n(\lambda). \tag{32}$$

Hence, if $Y_n(\lambda)$ solve the linear differential equation (2), $\sigma_1 Y_n\left(\frac{1}{\lambda}\right)\sigma_1$ also solves the linear differential equations, and if $\lambda \in S_j^{(0)}$, then $\lambda^{-1} \in S_j^{(\infty)}$. So we have the following symmetry for the sectionally analytic functions $Y_j^{(\infty),(0)}(\lambda)$ :

$$\sigma_1 Y_j^{(\infty)}\left(\frac{1}{\lambda}\right)\sigma_1 = Y_j^{(0)}(\lambda), \qquad j = 1, 2, \ldots, 5. \tag{33}$$

The symmetry relations (33) imply that

$$\sigma_1 G_j^{(\infty)}\sigma_1 = G_j^{(0)}, \qquad j = 1, 2, \ldots, 5, \qquad \sigma_1 E^{(0)}\sigma_1 = [E^{(0)}]^{-1}. \tag{34}$$

That is,

$$a^{(\infty)} = a^{(0)}, \quad b^{(\infty)} = b^{(0)}, \quad c^{(\infty)} = c^{(0)}, \quad d^{(\infty)} = d^{(0)}, \quad \gamma^{(0)} = -\beta^{(0)}. \tag{35}$$

Similarly, (32) implies that, if $Y(\lambda)$ solves the linear differential equations (2), then $\sigma_3 Y(\lambda\,\mathrm{e}^{-\mathrm{i}\pi})\sigma_3$ also solves (2), and if $\lambda\,\mathrm{e}^{-\mathrm{i}\pi} \in S_j^{(0),(\infty)}$, then $\lambda \in S_{j+2}^{(0),(\infty)}$, $j = 1, 2$. So we have the following symmetry relation for the sectionally analytic functions $Y_j^{(\infty),(0)}(\lambda)$ and $Y^{(-1),(1)}(\lambda)$ :

$$\begin{aligned} &Y_{j+2}^{(\infty)}(\lambda) = \sigma_3 Y_j^{(\infty)}(\lambda\,\mathrm{e}^{-\mathrm{i}\pi})\sigma_3, \qquad Y_{j+2}^{(0)}(\lambda) = \sigma_3 Y_j^{(0)}(\lambda\,\mathrm{e}^{-\mathrm{i}\pi})\sigma_3, \quad j = 1, 2, \\ &Y^{(-1)}(\lambda) = \sigma_3 Y^{(1)}(\lambda\,\mathrm{e}^{-\mathrm{i}\pi})\sigma_3, \end{aligned} \tag{36}$$

and (36) imply that

$$G_{j+2}^{(\infty)} = \sigma_3 G_j^{(\infty)}\sigma_3, \qquad G_{j+2}^{(0)} = \sigma_3 G_j^{(0)}\sigma_3, \qquad j = 1, 2, \qquad \sigma_3 E^{(-1)}\sigma_3 = E^{(1)}. \tag{37}$$

That is,

$$\begin{aligned} &a^{(\infty),(0)} = -c^{(\infty),(0)}, \quad b^{(\infty),(0)} = -d^{(\infty),(0)} \\ &\alpha^{(1)} = \alpha^{(-1)}, \quad \beta^{(1)} = -\beta^{(-1)}, \quad \gamma^{(1)} = -\gamma^{(-1)}, \quad \delta^{(1)} = \delta^{(-1)}. \end{aligned} \tag{38}$$

Therefore, the analytic structure of the solution matrix $Y_n(\lambda)$ of (2) is characterized by the monodromy data $MD = \{a^{(\infty)}, b^{(\infty)}, \alpha^{(0)}, \beta^{(0)}, \delta^{(0)}, \alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}, \delta^{(1)}\}$. The monodromy data, $MD$ satisfy the following product consistency condition around all singular points:

$$G_1^{(\infty)} G_2^{(\infty)} J^{(-1)} G_3^{(\infty)} G_4^{(\infty)} M^{(\infty)} J^{(1)} = (E^{(0)})^{-1} \prod_{j=1}^{4} G_j^{(0)} M^{(0)} E^{(0)}, \tag{39}$$

where

$$J^{(-1)} = (E^{(-1)})^{-1} M^{(-1)} E^{(-1)}, \qquad J^{(1)} = (E^{(1)})^{-1} M^{(1)} E^{(1)}. \tag{40}$$

If $Y_n$ solves (2) with $x_n$ satisfying $\mathrm{dP_{II}}$, then $\bar{Y}_n = R^{-1} Y_n R$ where $R = \mathrm{d}iag(r^{1/2}, r^{-1/2})$ and $r$ is non-zero complex constant, also solves (2) with $x_n$ satisfying $\mathrm{dP_{II}}$. But, the connection matrices $\bar{E}^{(0,1,-1)}$ and the Stokes matrices $\bar{G}_j^{(0,\infty)}$ for $\bar{Y}_n$ are $\bar{E}^{(0,1,-1)} = R^{-1} E^{(0,1,-1)} R$, and $\bar{G}_j^{(0,\infty)} = R^{-1} G_j^{(0,\infty)} R$. Thus, $r$ may be chosen to eliminate one of parameter, e.g. $r = \beta^{(0)}$. Also, changing the arbitrary integration constant $\mu_0^{(-1)}$ (see equation (19)) amounts to multiply $Y_{n,1}^{(1)}$ and $Y_{n,2}^{(1)}$ by an arbitrary nonzero complex constants $\kappa$ and $\kappa^{-1}$, respectively. This maps $E^{(1)}$ to $\mathrm{diag}(\kappa, \kappa^{-1})E^{(1)}$. Thus, $\kappa$ may be chosen to eliminate one of the entries of the connection matrix $E^{(1)}$. The freedom in choosing $E^{(1)}$ has no effect on the solution of the RH-problem. Equation (29) and the transformation $E^{(1)} \to \mathrm{diag}(\kappa, \kappa^{-1})E^{(1)}$ change $Y_n^{(1)}$ to $Y_n^{(1)}\mathrm{diag}(\kappa, \kappa^{-1})$, but the $\det Y_n^{(1)}$ remains the same. Therefore, together with the consistency condition (39), and $\det E^{(0)} = \det E^{(1)} = 1$, only two of the monodromy data are arbitrary.

## 3. One parameter family of solutions

If $c_0 \in \mathbb{Z}_+$, then the second linearly independent solutions about $\lambda = \pm 1$ may contain the $\log(\lambda \mp 1)$ terms. For $c_0 \in \mathbb{Z}_+$, two linearly independent solutions $Y_{n,1}^{(1)}(\lambda)$, and $Y_{n,2}^{(1)}(\lambda)$ about $\lambda = 1$ are

$$Y_{n,1}^{(1)}(\lambda) = (\lambda - 1)^{\frac{c_0}{2}} \hat{Y}_{n,1}^{(1)}(\lambda), \qquad Y_{n,2}^{(1)}(\lambda) = K \log(\lambda - 1) Y_{n,1}^{(1)}(\lambda) + (\lambda - 1)^{-\frac{c_0}{2}} \psi(\lambda), \tag{41}$$

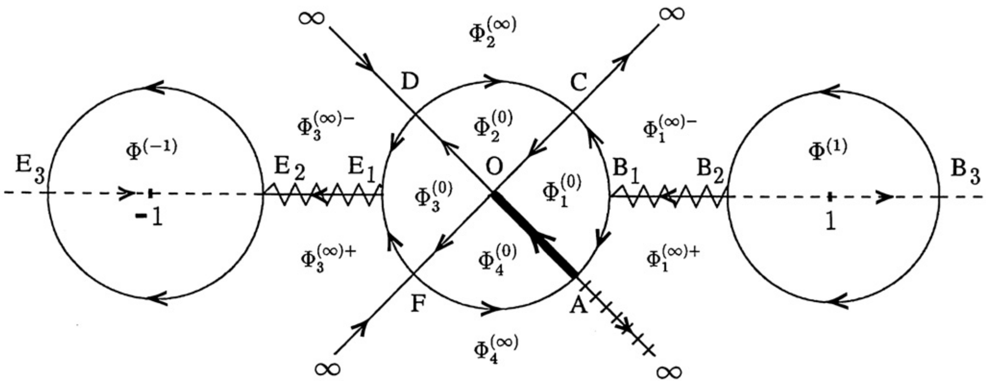


**Figure 2.** The contours for the RH-problem.

where $K$ is a constant, $\hat{Y}_{n,1}^{(1)}(\lambda)$ and $\psi(\lambda)$ are holomorphic at $\lambda = 1$. For $c_0 = 1$, the constant $K$ satisfies

$$2\mu^{(1)}K = \tau\,[2c_3(1 - x_n)(1 - x_{n-1}) + (c_2 + n)]\,, \tag{42}$$

where $\tau$ is an arbitrary non-zero constant. If $K = 0$, then (42) gives the following discrete Riccati equation for $x_n$ [17]:

$$x_n = 1 + \frac{\kappa_n}{1 - x_{n-1}}, \tag{43}$$

where $\kappa_n = (c_2 + n)/(2c_3)$. By using the solution about $\lambda = -1$, the same discrete Riccati equation is obtained for $c_0 = 1$.

By using the similar procedure, one obtains the discrete Riccati equations for $x_n$ which gives the one parameter family of solutions of $\mathrm{dP_{II}}$ for any positive integer value of $c_0$.

## 4. Inverse problem

In this section, we formulate a regular, continuous RH-problem over the intersecting contours for the sectionally analytic function $\Phi(\lambda)$. $\Phi(\lambda)$ depends on $n$, for simplicity in the notation we dropped the subscript $n$. We let $0 \leqslant c_0 < 1$ and $0 \leqslant c_2 < 1$, in order to have a regular RH-problem. The Schlesinger transformations for $\mathrm{dP_{II}}$ [18] allow one to completely cover the parameter space. Since $\hat{Y}_n^{(-1)}$, $\hat{Y}_n^{(0)}$, $\hat{Y}_n^{(1)}$ and $\hat{Y}_n^{(\infty)}$ are holomorphic at $\lambda = -1, 0, 1, \infty$, in order to formulate a continuous RH-problem, we insert the circles $C_{(-1)}$, $C_{(0)}$ and $C_{(1)}$ with radius $r < 1/4$ about the points $\lambda = -1, 0, 1$(see figure 2). Moreover, we apply a small clockwise rotation on the contours $A\infty$, $A0$, $C\infty$, $C0$, $D\infty$, $D0$, and $F\infty$, $F0$, in order to have decaying jump matrices as $\lambda \to \infty$, and $\lambda \to 0$, respectively, along these contours. Along these modified contours, we have a RH-problem which is analytic at $\lambda = 0, \infty$ and $\lambda = \pm 1$. The new RH-problem is equivalent to a certain Fredholm integral equation. The solution of the original RH-problem can be obtained once the solution of the new RH-problem is known. RH-problems appearing in the IMM were rigorously studied in [8, 27].

The jump matrices across the contours can be obtained from the definition of the Stokes matrices $G_j^{(0)}$, $G_j^{(\infty)}$ (equations (8) and (14), respectively) and the definition of the connection matrices $E^{(k)}$, $k = -1, 0, 1$ (equation (29)).

The jumps different than unity across the contours as indicated in figure 2 are given by

$$
\begin{aligned}
&CB_1 : Y_{n(1)}^{(\infty)} = Y_{n(1)}^{(0)} E_{(0)},\\
&C\infty : Y_{n(2)}^{(\infty)} = Y_{n(1)}^{(\infty)} G_1^{(\infty)},\\
&OC : Y_{n(2)}^{(0)} = Y_{n(1)}^{(0)} G_1^{(0)},\\
&DC : Y_{n(2)}^{(\infty)} = Y_{n(2)}^{(0)} \big(G_1^{(0)}\big)^{-1} E^{(0)} G_1^{(\infty)},\\
&D\infty : Y_{n(3)}^{(\infty)} = Y_{n(2)}^{(\infty)} G_2^{(\infty)},\\
&OD : Y_{n(3)}^{(0)} = Y_{n(2)}^{(0)} G_2^{(0)},\\
&C_{(-1)} : Y_{n(3)}^{(\infty)} = Y_n^{(-1)} E^{(-1)},\\
&E_1E_2 : Y_{n(3)}^{(\infty)}(\lambda\, e^{2i\pi}) = Y_{n(3)}^{(\infty)}(\lambda) J^{(-1)},\\
&DE_1 : Y_{n(3)}^{(\infty)} = Y_{n(3)}^{(0)} \big(G_1^{(0)} G_2^{(0)}\big)^{-1} E^{(0)} G_1^{(\infty)} G_2^{(\infty)},\\
&E_1F : Y_{n(3)}^{(\infty)}(\lambda\, e^{2i\pi}) = Y_{n(3)}^{(0)}(\lambda) \big(G_1^{(0)} G_2^{(0)}\big)^{-1} E^{(0)} G_1^{(\infty)} G_2^{(\infty)} J^{(-1)},\\
&F\infty : Y_{n(4)}^{(\infty)}(\lambda\, e^{2i\pi}) = Y_{n(3)}^{(\infty)}(\lambda\, e^{2i\pi}) G_3^{(\infty)},\\
&FO : Y_{n(4)}^{(0)} = Y_{n(3)}^{(0)} G_3^{(0},\\
&FA : Y_{n(4)}^{(\infty)}(\lambda\, e^{2i\pi}) = Y_{n(4)}^{(0)}(\lambda) \big(G_1^{(0)} G_2^{(0)} G_3^{(0)}\big)^{-1} E^{(0)} G_1^{(\infty)} G_2^{(\infty)} J^{(-1)} G_3^{(\infty)},\\
&OA : Y_{n(1)}^{(0)}(\lambda) = Y_{n(4)}^{(0)}(\lambda\, e^{2i\pi}) G_4^{(0)} M^{(0)},\\
&A\infty : Y_{n(1)}^{(\infty)}(\lambda) = Y_{n(4)}^{(\infty)}(\lambda\, e^{2i\pi}) G_4^{(\infty)} M^{(\infty)},\\
&B_1B_2 : Y_{n(1)}^{(\infty)}(\lambda\, e^{2i\pi}) = Y_{n(1)}^{(\infty)}(\lambda) J^{(1)},\\
&C_{(1)} : Y_{n(1)}^{(\infty)} = Y^{(1)} E^{(1)},\\
&B_1A : Y_{n(1)}^{(\infty)}(\lambda\, e^{2i\pi}) = Y_{n(1)}^{(0)}(\lambda) \left(\prod_{j=1}^{4} G_j^{(0)} M^{(0)}\right)^{-1} E^{(0)} G_1^{(\infty)} G_2^{(\infty)} J^{(-1)} G_3^{(\infty)} G_4^{(\infty)} M^{(\infty)}.
\end{aligned}
\tag{44}
$$

In order to define a continuous RH problem, we define sectionally analytic function $\Phi(\lambda)$ as follows:

$$
\begin{aligned}
&Y_{n(j)}^{(\infty)} = \Phi_j^{(\infty)}\, e^{Q(\lambda)} \lambda^{D_n^{(\infty)}}, \qquad Y_{n(j)}^{(0)} = \Phi_j^{(0)}\, e^{Q(\lambda)} \left(\frac{1}{\lambda}\right)^{D_n^{(0)}}, \quad j = 1, \dots, 4\\
&Y_n^{(1)} = \Phi^{(1)}\, e^{Q(\lambda)} (\lambda - 1)^{D^{(1)}}, \quad Y_n^{(-1)} = \Phi^{(-1)}\, e^{Q(\lambda)} (\lambda + 1)^{D^{(-1)}},
\end{aligned}
\tag{45}
$$

where

$$
Q(\lambda) = \frac{c_3}{2}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3. \tag{46}
$$

The orientation as indicated in figure 2 allows the splitting of the complex $\lambda$-plane in + and − regions. Then (41) imply certain jumps for the sectionally analytic function $\Phi$ which is represented by $\Phi^{(-1)}$, $\Phi^{(0)}$, $\Phi^{(1)}$, $\Phi_j^{(0)}$ and $\Phi_j^{(\infty)}$, $j = 1, \dots, 4$, in the regions indicated in figure 2, and we obtain the following RH-problem:

$$
\Phi^+(\hat\lambda) = \Phi^-(\hat\lambda)[e^{Q(\hat\lambda)} V\, e^{-Q(\hat\lambda)}] \text{ on } C, \qquad \Phi = I + O\left(\frac{1}{\lambda}\right) \text{ as } \lambda \to \infty, \tag{47}
$$

where $C$ is the sum of the all contours, and the jump matrices $V$ are given by

$$
\begin{aligned}
V_{CB_1} &= \lambda^{D_n^{(\infty)}}(E^{(0)})^{-1}\left(\frac{1}{\lambda}\right)^{-D_n^{(0)}},\\
V_{C\infty} &= \lambda^{D_n^{(\infty)}}G_1^{(\infty)}\lambda^{-D_n^{(\infty)}},\\
V_{OC} &= \left(\frac{1}{\lambda}\right)^{D_n^{(0)}}\left(G_1^{(0)}\right)^{-1}\left(\frac{1}{\lambda}\right)^{-D_n^{(0)}},\\
V_{DC} &= \left(\frac{1}{\lambda}\right)^{D_n^{(0)}}\left(G_1^{(0)}\right)^{-1}E^{(0)}G_1^{(\infty)}\lambda^{-D_n^{(\infty)}},\\
V_{D\infty} &= \lambda^{D_n^{(\infty)}}\left(G_2^{(\infty)}\right)^{-1}\lambda^{-D_n^{(\infty)}},\\
V_{OD} &= \left(\frac{1}{\lambda}\right)^{D_n^{(0)}}G_2^{(0)}\left(\frac{1}{\lambda}\right)^{-D_n^{(0)}},\\
V_{\widehat{E_2E_3}} &= \lambda^{D_n^{(\infty)}}[E^{(-1)}]^{-1}(\lambda+1)^{-D^{(1)}},\\
V_{\underset{\smile}{E_2E_3}} &= (\lambda+1)_+^{D^{(1)}}E^{(-1)}\lambda^{-D_n^{(\infty)}},\\
V_{E_1E_2} &= \lambda^{D_n^{(\infty)}}J^{(-1)}\lambda^{-D_n^{(\infty)}},\\
V_{DE_1} &= \lambda^{D_n^{(\infty)}}\left(E^{(0)}G_1^{(\infty)}G_2^{(\infty)}\right)^{-1}G_1^{(0)}G_2^{(0)}\left(\frac{1}{\lambda}\right)^{-D_n^{(0)}},\\
V_{E_1F} &= \left(\frac{1}{\lambda}\right)^{D_n^{(0)}}\left(G_1^{(0)}G_2^{(0)}\right)^{-1}E^{(0)}G_1^{(\infty)}G_2^{(\infty)}J^{(-1)}\lambda^{-D_n^{(\infty)}},\\
V_{FO} &= \left(\frac{1}{\lambda}\right)^{D_n^{(0)}}G_3^{(0}\left(\frac{1}{\lambda}\right)^{-D_n^{(0)}},\\
V_{F\infty} &= \lambda^{D_n^{(\infty)}}\left(G_3^{(\infty)}\right)^{-1}\lambda^{-D_n^{(\infty)}},\\
V_{FA} &= \lambda^{D_n^{(\infty)}}\left(E^{(0)}G_1^{(\infty)}G_2^{(\infty)}J^{(-1)}G_3^{(\infty)}\right)^{-1}G_1^{(0)}G_2^{(0)}G_3^{(0)}\left(\frac{1}{\lambda}\right)^{-D_n^{(0)}},\\
V_{OA} &= \left(\frac{1}{\lambda}\right)_+^{D_n^{(0)}}(G_4^{(0)})^{-1}\left(\frac{1}{\lambda}\right)^{-D_n^{(0)}},\\
V_{A\infty} &= \lambda^{D_n^{(\infty)}}G_4^{(\infty)}(\lambda)_+^{-D_n^{(\infty)}},\\
V_{B_1B_2} &= \lambda^{D_n^{(\infty)}}(J^{(1)})^{-1}\lambda^{-D_n^{(\infty)}},\\
V_{\widehat{B_2B_3}} &= \lambda^{D_n^{(\infty)}}(E^{(1)})^{-1}(\lambda-1)_+^{-D^{(1)}},\\
V_{\underset{\smile}{B_2B_3}} &= (\lambda-1)^{D^{(1)}}E^{(1)}\lambda^{-D_n^{(\infty)}},\\
V_{B_1A} &= \left(\frac{1}{\lambda}\right)_+^{D_n^{(0)}}\left(\prod_{j=1}^{4}G_j^{(0)}\right)^{-1}E^{(0)}G_1^{(\infty)}G_2^{(\infty)}J^{(-1)}G_3^{(\infty)}G_3^{(\infty)}(\lambda)_+^{-D_n^{(\infty)}},
\end{aligned}
\tag{48}
$$

and $V_{E_3\infty} = V_{\overline{E_2E_3}} = V_{B\infty} = V_{\overline{B_2B_3}} = I$. Since we have associated the branch cuts $A\infty$, $OA$, $E_1E_2$ and $B_1B_2$ with $\lambda^{D_n^{(\infty)}}$, $\left(\frac{1}{\lambda}\right)^{D_n^{(0)}}$, $(\lambda+1)^{D^{(-1)}}$, and $(\lambda-1)^{D^{(1)}}$, respectively. The subscript

+ appearing in the definition of $V_{\underset{\smile}{E_2E_3}}$, $V_{OA}$, $V_{A\infty}$, $V_{\underset{\smile}{B_2B_3}}$ and $V_{B_1A}$ indicate that we consider the relevant boundary values from + region, i.e. $(\lambda)_+ = |\lambda|e^{2i\pi}$.

By construction $\Phi_n(z)$ satisfies the continuous RH problem and this can be checked by the product of the jump matrices $V$ at the intersection points. The product conditions give

$$\begin{aligned}
&C : (V_{C\infty})^{-1} V_{CB_1} (V_{OC})^{-1} V_{DC} = I, && D : (V_{OD})^{-1} V_{DC} (V_{D\infty})^{-1} V_{DE_1} = I,\\
&E_1 : V_{DE_1} V_{E_1F} \left(V_{E_1E_2}\right)^{-1} = I, && E_2 : \left(V_{\overset{\frown}{E_2E_3}}\right)_+ V_{\underset{\smile}{E_2E_3}} \left(V_{E_1E_2}\right)^{-1} = I,\\
&E_3 : V_{\underset{\smile}{E_2E_3}} V_{\overset{\frown}{E_2E_3}} = I, && F : (V_{FO})^{-1} V_{E_1F} (V_{F\infty})^{-1} V_{FA} = I,\\
&A : (V_{A\infty})^{-1} V_{FA} (V_{OA})^{-1} V_{B_1A} = I, && B_1 : V_{B_1A} \left(V_{B_1B_2}\right)^{-1} V_{CB_1} = I,\\
&B_2 : V_{\underset{\smile}{B_2B_3}} \left(V_{B_1B_2}\right)^{-1} \left(V_{\overset{\frown}{B_2B_3}}\right)_+ = I, && B_3 : V_{\overset{\frown}{B_2B_3}} V_{\underset{\smile}{B_2B_3}} = I.
\end{aligned} \tag{49}$$

The product conditions at the intersection points $A, B_2, B_3, C, D, E_1, E_2, E_3, F$ are satisfied identically and the product condition at point $B_1$ is satisfied because of the consistency condition (39) of the monodromy data. In equation (49), $\left(V_{\overset{\frown}{E_2E_3}}\right)_+$ indicates that $(\lambda + 1)$ term in $V_{\overset{\frown}{E_2E_3}}$ must be evaluated as $(\lambda + 1)_+$, and $\left(V_{\underset{\smile}{B_2B_3}}\right)_+$ indicates that $(\lambda - 1)$ term must be evaluated as $(\lambda - 1)_+$.

The RH problem (47) is equivalent to following Fredholm integral equation:

$$\Phi^-(\lambda) = I + \frac{1}{2i\pi} \int_C \frac{\Phi^-(\hat{\lambda})[V(\hat{\lambda})V^{-1}(\lambda) - I]}{\hat{\lambda} - \lambda} \, d\hat{\lambda}, \tag{50}$$

where $C$ is the sum of all contours. Hence, the solution of the discrete second Painlevé equation can be obtained by solving the associated RH-problem (47). The jump matrices of the associated RH-problem are given in terms of the monodromy data, which are such that only two of them are arbitrary. Once the solution $\Phi$ of the associated RH-problem is obtained, the solution $x_n$ of $dP_{II}$ can be written as

$$x_n = -(\Phi_{-1})_{12}, \tag{51}$$

where

$$\Phi = I + \Phi_{-1}\lambda^{-1} + \Phi_{-2}\lambda^{-2} + \cdots, \qquad \text{as} \quad \lambda \to \infty, \tag{52}$$

and $(\Phi_{-1})_{12}$ is $(1, 2)$ entry of $\Phi_{-1}$.

## 5. Derivation of the linear problem

In this section, we show that once the sectionally analytic function $\Phi(\lambda)$ satisfying the RH-problem (47) is known, then the coefficients $M_n$ and $L_n$ of the linear differential equation (2) can be obtained and hence the solution of $dP_{II}$.

*Derivation of $M_n$.* We define $M_n$ by $M_n(\lambda) = \frac{\partial Y_n}{\partial \lambda}[Y_n(\lambda)]^{-1}$. Since both $\frac{\partial Y_n}{\partial \lambda}$, and $Y_n(\lambda)$ admit the same jumps, it follows that $M_n(\lambda)$ is holomorphic in complex $\lambda$-plane except at $\lambda = 0$, where it has a pole of order three, and $\lambda = \pm 1$ where it has simple poles. Furthermore, $Y_n(\lambda) \sim \lambda^{D_n^{(\infty)}} e^{Q(\lambda)}$ as $\lambda \to \infty$, and thus

$$M_n(\lambda) = A_1\lambda + A_2 + A_3\frac{1}{\lambda} + A_4\frac{1}{\lambda^2} + A_5\frac{1}{\lambda^3} + A_6\frac{1}{\lambda - 1} + A_7\frac{1}{\lambda + 1}. \tag{53}$$

Since $Y_n(\lambda)$ and $\Phi(\lambda)$ are related by equation (45), and $\dfrac{\partial Y_n}{\partial \lambda} = M_n(\lambda)Y_n(\lambda)$, we have

$$\frac{\partial \Phi}{\partial \lambda} + \Phi\left[c_3\left(\lambda + \frac{1}{\lambda^3}\right)\sigma_3 + \frac{1}{\lambda}D_n^{(\infty)}\right] = M_n\Phi, \qquad \text{as} \quad \lambda \to \infty \tag{54a}$$

$$\frac{\partial\Phi}{\partial\lambda} + \Phi\left[c_3\left(\lambda + \frac{1}{\lambda^3}\right)\sigma_3 - \frac{1}{\lambda}D_n^{(0)}\right] = M_n\Phi, \qquad \text{as} \quad \lambda \to 0 \tag{54b}$$

$$\frac{\partial\Phi}{\partial\lambda} + \Phi\left[c_3\left(\lambda + \frac{1}{\lambda^3}\right)\sigma_3 + \frac{1}{\lambda - 1}D^{(1)}\right] = M_n\Phi, \qquad \text{as} \quad \lambda \to 1 \tag{54c}$$

$$\frac{\partial\Phi}{\partial\lambda} + \Phi\left[c_3\left(\lambda + \frac{1}{\lambda^3}\right)\sigma_3 + \frac{1}{\lambda + 1}D^{(-1)}\right] = M_n\Phi, \qquad \text{as} \quad \lambda \to -1. \tag{54d}$$

For large $\lambda$, $\Phi$ has the expansion

$$\Phi = I + \frac{1}{\lambda}\Phi_{-1} + \frac{1}{\lambda^2}\Phi_{-2} + O\left(\frac{1}{\lambda^3}\right). \tag{55}$$

Substituting (55) into (54*a*) gives

$$\begin{aligned} &O(\lambda) : A_1 = c_3\sigma_3, \\ &O(1) : A_2 = [\Phi_{-1}, A_1], \\ &O(\lambda^{-1}) : A_3 = D_n^{(\infty)} + [\Phi_{-2}, A_1] - A_2\Phi_{-1}. \end{aligned} \tag{56}$$

Therefore, $A_1 = M_1$, $A_2$ can be written as $A_2 = M_2$, where $(\Phi_{-1})_{12} = -x_n$, $(\Phi_{-1})_{21} = x_{n-1}$, and $(A_3)_{11} = -(A_3)_{22} = c_2 + n - 2c_3x_nx_{n-1}$. Thus, $A_3$ can be taken as $A_3 = M_3$.

For small $\lambda$, $\Phi$ has the expansion

$$\Phi = I + \lambda\Phi_1 + \lambda^2\Phi_2 + O(\lambda^3). \tag{57}$$

Substituting (57) into (54*b*) yields

$$\begin{aligned} &O(\lambda^{-3}) : \ A_5 = c_3\sigma_3, \\ &O(\lambda^{-2}) : \ A_4 = [\Phi_1, A_5], \\ &O(\lambda^{-1}) : \ A_3 = -D_n^{(0)} + [\Phi_2, A_5] - A_4\Phi_1. \end{aligned} \tag{58}$$

Therefore, $A_5 = M_5$, $A_4$ can be written as $A_4 = M_4$, where $(\Phi_1)_{21} = -x_n$, $(\Phi_1)_{12} = x_{n-1}$.

Since $\Phi(\lambda)$ is sectionally analytic and $\Phi(\lambda) = \Phi^{(j)}$, $j = \pm 1$ near $\lambda = \pm 1$, then (54*c*) and (54*d*) imply that

$$A_6 = \Phi^{(1)}(1)D^{(1)}[\Phi^{(1)}(1)]^{-1}, \qquad A_7 = \Phi^{(-1)}(-1)D^{(-1)}[\Phi^{(-1)}(-1)]^{-1}. \tag{59}$$

Thus, $\det A_j = -c_0^2/4$, and $\operatorname{tr} A_j = 0$, $j = 6, 7$. Moreover, the symmetry $Y^{(-1)}(\lambda) = \sigma_3 Y^{(1)}(-\lambda)\sigma_3$ implies that $A_7 = -A_6$. Therefore, we can take $A_6 = -A_7 = (c_0/2)\sigma_3$.

*Derivation of $L_n$.* Similar considerations imply that $L_n = L_1\lambda + L_2 + L_3\lambda^{-1}$, and

$$\Phi_{n+1}\lambda^{\sigma_3} = L_n\Phi_n \qquad \text{as} \qquad \lambda \to 0,\ \infty. \tag{60}$$

As $\lambda \to \infty$, substituting (55) into (60) yields

$$O(\lambda) : L_1 = \Omega_1, \qquad O(1) : L_2 = \Phi_{n+1,-1}\ \Omega_1 - \Omega_1\ \Phi_{n,-1}, \tag{61}$$

where

$$\Omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

As $\lambda \to 0$, substituting (57) into (60) gives

$$O(\lambda^{-1}) : L_3 = \Omega_2, \qquad O(1) : L_2 = \Phi_{n+1,1}\ \Omega_2 - \Omega_2\ \Phi_{n,1}, \tag{62}$$

where

$$\Omega_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Equations (61) and (62) show that $L_n$ is given as in (3).

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