Discretization of hyperbolic type Darboux integrable equations preserving integrability

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I. INTRODUCTION

The problem of integrable discretization of the integrable partial differential equation (PDE) is very complicated and not enough studied. The same is true for evaluating the continuum limit for discrete models.1 In the present paper we undertake an attempt to clarify the connection between the Liouville type partial differential equations and their discrete analogues. One unexpected observation is that there are pairs of equations, one is continuous and the other one is semi-discrete, having a common integral. Inspired by these examples, we introduced a method of discretization of PDE having a nontrivial integral. Similar ideas are used in Ref. 2 where a method of construction of difference scheme for ordinary differential equations preserving the classical Lie group is suggested. Let us begin with the necessary definitions.

We consider discrete equations of the form

\[ v(n + 1, m + 1) = f(v(n, m), v(n + 1, m), v(n, m + 1)), \]  
and semi-discrete chains

\[ t(n + 1, x) = f(x, t(n, x), t(n + 1, x), t_x(n, x)). \]

Equations (1) and (2) are discrete and semi-discrete analogues of hyperbolic equations

\[ u_{xy} = f(x, y, u, u_x, u_y). \]

Functions \( v = v(n, m) \), \( t = t(n, x) \), and \( u = u(x, y) \) depend on discrete variables \( n \) and \( m \) and continuous variables \( x \) and \( y \). Throughout the paper we use the following notations:

\[ v_{i,j} = v(n + i, m + j); \quad v_i = v_{i,0}; \quad t_j = v_{0,j}; \quad t_i = t(n + i, x). \]

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For Eq. (3), function $W(x, y, u, u_x, u_y, \ldots, \partial^k u / \partial y^k)$ is called an $x$-integral of order $k$, if $D_x W = 0$ and $W_{\partial^j u / \partial y^j} \neq 0$, and function $\tilde{W}(x, y, u, u_x, u_y, \ldots, \partial^m u / \partial x^m)$ is called a $y$-integral of order $m$, if $D_y \tilde{W} = 0$ and $\tilde{W}_{\partial^j u / \partial x^j} \neq 0$. Here, $D_x$ and $D_y$ denote the total derivatives with respect to $x$ and $y$. Equation (3) is called the Darboux integrable if it possesses nontrivial $x$- and $y$-integrals.

For Eq. (2), function $F(x, n, t_m, t_{m+1}, \ldots)$ is called an $x$-integral of order $m' - m + 1$, if $D_x F = 0$ and $F_{t_m} \neq 0$. Function $l(x, n, t, u_x, u_y, \ldots)$ is called an $n$-integral of order $n$, if $DI = I$ and $l_{x} \neq 0$. Here, $D$ is the forward shift operator in $n$, i.e., $Dh(n, x) = h(n + 1, x)$. Equation (2) is called the Darboux integrable if it possesses nontrivial $x$- and $n$-integrals.

For Eq. (1), function $I(n, m, \bar{v}_k, \bar{v}_k+1, \ldots)$ is called an $n$-integral of order $k' - k + 1$, if $DI = I$ and $I_{\bar{v}_k} \neq 0$. Function $\bar{I}(n, m, v_r, v_{r+1}, v_{r+2}, \ldots)$ is called an $m$-integral of order $r' - r + 1$, if $D\bar{I} = \bar{I}$ and $I_{\bar{v}_r} \neq 0$. Here, $D$ and $D$ are the forward shift operators in $n$ and $m$, respectively. Equation (1) is called the Darboux integrable, if it possesses nontrivial $n$- and $m$-integrals (see also Ref. 3).

Continuous equations (3) are very well studied. In particular, the question of describing all the Darboux integrable equations (3) is completely solved (see Refs. 4–7). All Eqs. (3) possessing $x$- and $y$-integrals of order 2 are described by the following theorem.

**Theorem 1.1:** (see Ref. 7) Any Eq. (3), for which there exist second order $x$- and $y$-integrals, under the change of variables $x \to X(x)$, $y \to Y(y)$, and $u \to U(x, y, u)$, can be reduced to one of the kind:

1. $u_{xx} = e^u$, $\tilde{W} = u_{xx} - 0.5u_x^2$, $W = u_{yy} - 0.5u_y^2$;
2. $u_{xy} = e^u u_y$, $\tilde{W} = u_x - e^u$, $W = \frac{u_{xx}}{u_x} - u_y$;
3. $u_{xy} = e^u \sqrt{u_x^2 - 4}$, $\tilde{W} = u_{xx} - 0.5u_x^2 - 0.5e^{2u}$, $W = \frac{u_{yy} - u_x^2 + 4}{\sqrt{u_x^2 - 4}}$;
4. $u_{xy} = u_x u_y \left( \frac{1}{x - u_x} - \frac{1}{x - u_x} \right)$, $\tilde{W} = \frac{u_{xx}}{u_x} - \frac{2u_x}{x - u_x} + \frac{1}{u_x - x}$, $W = \frac{u_{yy}}{u_y} - \frac{2u_y}{u_y - y} + \frac{1}{u_y - y}$;
5. $u_{xy} = \psi(u) \beta(u_x) \bar{\beta}(u_y)$, $\left( l\psi \right)' = \psi$, $\beta' = -u_x$, $\bar{\beta}' = -u_y$, $\tilde{W} = \frac{u_{xx}}{\beta(u_x)} - \psi(u)\beta(u_x)$, $W = \frac{u_{yy}}{\bar{\beta}(u_y)} - \psi(u)\bar{\beta}(u_y)$;
6. $u_{xy} = \frac{\beta(u_x)\bar{\beta}(u_y)}{u_x}$, $\beta' + c \beta = -u_x$, $\bar{\beta}' + c \bar{\beta} = -u_y$, $\tilde{W} = \frac{u_x}{\beta(u_x)} - \frac{\beta}{\bar{\beta}}$, $W = \frac{u_y}{\bar{\beta}(u_y)} - \frac{\bar{\beta}}{\beta}$;
7. $u_{xy} = -2 \frac{u_x}{\sqrt{x-y}}$, $\tilde{W} = \frac{u_{xx}}{\sqrt{x-y}} + 2 \frac{\sqrt{x-y}}{x-y}$, $W = \frac{u_{yy}}{\sqrt{y-x}} + 2 \frac{\sqrt{y-x}}{x-y}$; and
8. $u_{xy} = \frac{1}{(x+y)\beta(u_x)\beta(u_y)}$, $\beta' = \beta^3 + \beta^2$, $\bar{\beta}' = \bar{\beta}^3 + \bar{\beta}^2$, $\tilde{W} = u_{xx} \beta(u_x) - \frac{1}{(x+y)\beta(u_x)}$, $W = u_{yy} \bar{\beta}(u_y) - \frac{1}{(x+y)\beta(u_y)}$.

On the contrary, the problem of describing all Eqs. (1) or (2) possessing both integrals (so-called the Darboux integrable equations) is very far from being solved (the problem of classification is solved only for a very special kind of semi-discrete equations), it would be beneficial for further classification to obtain new Darboux-integrable equations (1) and semi-discrete chains (2). It was observed that many chains (2) and their continuum limit equations (3) possess the same $n$- and $y$-integrals.

The main aim of the present paper is the discretization of Eqs. (3) preserving the structure of $y$-integrals of order 2: we take $y$-integral for each of eight classes of Theorem 1.1 and find the semi-discrete chain (2) possessing the given $n$-integral ($y$-integral). The next theorem introduces a list of semi-discrete models of the Darboux integrable equations (3) from Theorem 1.1 with integrals of order 2.
**Theorem 1.2:** Below is the list of Eqs. (2) possessing the given $n$-integral $I$:

<table>
<thead>
<tr>
<th>Given $n$ - integral</th>
<th>The corresponding chain $t_{1x}$</th>
<th>$n$ - integral of the chain $I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I = t_{1x} - 0.5t_{1x}^2$</td>
<td>$t_{1x} = t_x + C e^{0.5t_{1x}(t_{1x} - t_{1x})}$</td>
<td>$I = t_{1x} - 0.5t_{1x}^2$ (1*)</td>
</tr>
<tr>
<td>$I = t_x - e^t$</td>
<td>$t_{1x} = t_x - e^t + e^t$</td>
<td>$I = t_x - e^t$ (2*)</td>
</tr>
<tr>
<td>$I = \frac{t_{1x}}{t_x} - t_x$</td>
<td>$t_{1x} = K (t_{1x} t_x)$, where $K t_{1x} K t_x = K - 1$</td>
<td>$I = \frac{t_{1x}}{t_x} - t_x$ (2* b)</td>
</tr>
<tr>
<td>$I = t_{1x} - 0.5t_{1x} - 0.5e^{2t}$</td>
<td>$t_{1x} = t_x + \sqrt{e^{2t} + R e^{t_{1x} t_{1x}} + e^{2t}}$</td>
<td>$I = t_{1x} - 0.5t_{1x} - 0.5e^{2t}$ (3* a)</td>
</tr>
<tr>
<td>$I = t_{1x} - \frac{t_{1x} t_{1x} + 4}{\sqrt{t_x^2 - 4}}$</td>
<td>$t_{1x} = (1 + R e^{t_{1x}}) t_x$</td>
<td>$I = t_{1x} - \frac{t_{1x} t_{1x} + 4}{\sqrt{t_x^2 - 4}}$ (3* b)</td>
</tr>
<tr>
<td>$I = \frac{t_x}{t_x} - \frac{t_{1x}}{t_x} + \frac{1}{t_x}$</td>
<td>$t_{1x} = \left( \frac{t_{1x} t_x}{t_{1x} t_x - 2 t_x} \right) t_x$</td>
<td>$I = \frac{t_x}{t_x} - \frac{t_{1x}}{t_x} + \frac{1}{t_x}$ (4*)</td>
</tr>
<tr>
<td>$I = \frac{t_{1x}}{\beta(t_{1x})} - \psi(t) \beta(t_{1x})$</td>
<td>$t_{1x} = K (t_{1x}) t_x$, where $K t_{1x} K t_x = K - 1$</td>
<td>$I = \frac{t_{1x}}{\beta(t_{1x})} - \psi(t) \beta(t_{1x})$ (5*)</td>
</tr>
<tr>
<td>$I = t_{1x} - \frac{t_{1x} t_{1x}}{t_x}$</td>
<td>$t_{1x} = K (t_{1x}) t_x$, where $K t_{1x} K t_x = K - 1$</td>
<td>$I = \frac{t_{1x}}{R} - \frac{K t_x}{K t_{1x}}$ (6*)</td>
</tr>
<tr>
<td>$I = \frac{t_{1x}}{\beta(t_{1x})} - \frac{2 t_{1x}}{x + R}$</td>
<td>$t_{1x} = \left( \frac{t_{1x} t_x}{t_{1x} t_x + x + R} \right)$</td>
<td>$I = \frac{t_{1x}}{\beta(t_{1x})} - \frac{2 t_{1x}}{x + R}$ (7*)</td>
</tr>
</tbody>
</table>

where $C$, $R$, and $L$ are constants.

It is seen from Theorem 1.2 that in the cases (5*), (6*), and (8*), the form of the given $n$-integral has been narrowed down by a special choice of function $\beta(t_x)$.

Note that Eq. (1*) was found in Ref. 9. Equation (3*a) for $R = 2$ was found in Ref. 3. Eqs. (2*a) and (3*a) were found in Ref. 8. To our knowledge, the other equations from Theorem 1.2 are new.

It is remarkable that each equation in Theorem 1.2 also admits a nontrivial $x$-integral. It means that discretization preserving the structure of $y$-integrals sends Darboux integrable equations (3) into Darboux integrable chains (2).

The next theorem lists $x$-integrals for chains from Theorem 1.2.

**Theorem 1.3:** (I) Equations (2*b), (5*), and (6*) from Theorem 1.2 having the form $t_{1x} = K (t_{1x}) t_x$, admit $x$-integral $F(t_{1x})$, where function $F$ is a solution of $F_t + K(t_{1x}) F_t = 0$ with a given function $K(t_{1x})$.

(II) $x$-integrals of Eqs. (1*), (2*a), (3*a), (3*b), (4*), (7*), and (8*) are $F = e^{0.5t_{1x}} + e^{0.5t_{1x}}$, $F = e^{0.5t_{1x}} e^{0.5t_{1x}} (e^{0.5t_{1x}} - e^{0.5t_{1x}})^{-1}$, $F = e^{0.5t_{1x}} + e^{0.5t_{1x}} + b = e^{0.5t_{1x}} + e^{0.5t_{1x}} + b$ with $a = 2(4 - R^2)^{-1/2}$, $b = R(4 - R^2)^{-1/2}$, $F = \sqrt{R e^{2t_{1x}} + 2 e^{0.5t_{1x}} + \sqrt{R e^{2t_{1x}} + 2 e^{0.5t_{1x}}}}$, $F = (t_1 - t)(t_2 - t)^{-1}(t_1 + L)^{-1}$, $F = (2 t_1 - t - t_2)/(2C^2) - 1/(x + R)$, and $F = (t_1 - t + C)/(x + y)$ correspondingly.

In the case of partial differential equations having integrals in both directions allows one to deduce an explicit formula for general solution. Also, we can find explicit formulas for general solutions for the Darboux integrable semi-discrete equations. Let us illustrate with an example.

**Example:** Find general solution to the chain

$$t_{1x} - t_x = t_1^2 - t_2^2.$$  

Equation (4) has an $n$-integral $I = t_x - t_2^2$. Since $DI = I$, any solution of the chain satisfies the following Riccati equation:

$$t_x - t_2^2 = C(x),$$  

where $C(x)$ is an arbitrary function. Suppose that $t = \phi$ is a particular solution of (5), i.e., $\phi_x - \phi^2 = C(x)$. Then general solution is given by

$$t = \phi + z,$$  

(6)
where \( z = t - \phi \) solves the following Bernoulli equation:

\[
 z_e - z^2 - 2z\phi = 0. \tag{7}
\]

By the substitution \( \frac{1}{z} = \nu \), we get inhomogeneous linear differential equation

\[
 \nu' + 2\phi\nu + 1 = 0, \tag{8}
\]

with the solution

\[
 \nu = (-\int_{x_0}^{x} e^{\int_{\nu}^{\nu_0} 2\phi(s)ds} dx' + k(n))e^{-\int_{x_0}^{x} 2\phi(s)ds}. \tag{9}
\]

Denote \( \varphi(x) = \int_{x_0}^{x} e^{\int_{\nu}^{\nu_0} 2\phi(s)ds} dx' \), then \( C(x) = -\varphi(x) + k(n) \). Then differentiate \( \varphi(x) \) and get

\[
 \varphi' = e^{\int_{x_0}^{x} 2\phi(s)ds}. \tag{10}
\]

Take the logarithmic derivative of (10) with respect to \( x \), we get \( \frac{\varphi''}{\varphi'} = 2\phi(x) \). For (9), we have

\[
 v = (-\varphi(x) + k(n))e^{-\int_{x_0}^{x} 2\phi(s)ds} = \frac{-\varphi(x) + k(n)}{\varphi'} \quad \text{and} \quad z = \frac{-\varphi'}{-\varphi(x) + k(n)}. \]

Thus, general solution for (5) takes the form

\[
 \tau(x, n) = z + \phi = \frac{\varphi''}{2\varphi'} + \frac{\varphi'}{-\varphi(x) + k(n)}. \tag{11}
\]

However, some Darboux integrable semi-discrete equations do not admit explicit formula of such a kind for general solution (see Ref. 18).

One can also apply the discretization method preserving the structure of integrals for semi-discrete chains (2): take \( x \)-integral for a semi-discrete chain and find discrete equation (1) with the given \( m \)-integral (\( x \)-integral).

In spite of the absence of the complete classification for Darboux-integrable semi-discrete chains (2) there is a large variety of such chains in the literature (see, for instance, Refs. 3, 8, and 10). The procedure of obtaining fully discrete equations for a given integral is a difficult task and requires further investigation. As a rule it is reduced to a very complicated functional equation. We illustrate the application of the discretization method on chains (1*), (4*), and (7*) from Theorem 1.2. The discrete analogues of the chains are introduced in the next remark.

**Remark 1.4:** Below is the list of Eqs. (1) possessing the given \( m \)-integral \( \bar{I} \):

<table>
<thead>
<tr>
<th>Given ( m )-integral</th>
<th>The corresponding equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I = e^{(\nu(t) - t)/2} + e^{(\nu(t) - t)/2} )</td>
<td>( e^{\nu_1 + \nu} = \frac{1}{e^{\nu_1 + \nu}} )</td>
</tr>
<tr>
<td>( I = (\nu_1 - \nu)(\overline{\nu_2} + \nu)^{-1}(\nu_1 + L)^{-1} )</td>
<td>( \nu_1, 1 = \frac{L(\overline{\nu_2} + \nu)^{-1} + \nu_1}{L + \nu_2} )</td>
</tr>
<tr>
<td>( I = 2\nu_1 - \nu - \nu_2 )</td>
<td>( \nu_1, 1 = \nu_1 + h(\overline{\nu_1} - \nu), ; z = h(2\nu - h(z)) )</td>
</tr>
</tbody>
</table>

Equations (1*), (4*), and (7*) have, respectively, the following \( n \)-integrals \( I = e^{(\nu(t) - t)/2} + e^{(\nu(t) - t)/2}, \; I = (\overline{\nu_1} - \nu_1)(\nu_1 - \nu)^{-1}(\overline{\nu_2} + L)^{-1}, \) and \( I = \nu_1 - \nu - h^{-1}(\overline{\nu_1} - \nu) \) with \( h^{-1} \) being the inverse function of function \( h \) that satisfies the functional equation \( z = h(2\nu - h(z)) \).

Equation (1*) from Remark 1.4 appeared in Ref. 11, Eq. (4*) belongs to ABS list. Unfortunately, we failed to answer the question whether equation \( z = h(2\nu - h(z)) \) has any solution different from linear one \( h(z) = z + C \).

The article is organized as follows. Theorem 1.2 is proved in Sec. II. The proof of Theorem 1.3 is omitted. Chains (1*), (2*a), and (3*a) are of the form \( t_{1k} = t_k + d(t, t_1) \), and their \( x \)-integrals can be seen in Ref. 8. One can find \( x \)-integrals for chains (3*b), (4*), (7*), and (8*) by direct calculations. In Sec. III, the discretization of chains (1*), (4*), and (7*) from Remark 1.4 is presented and for each obtained discrete equation the second integral is found. In Sec. IV, the Conclusion is drawn.
II. PROOF OF THEOREM 1.2

A. Case (1*)

Consider all chains (2) with \( n \)-integral of the form \( I = t_{xx} - \frac{1}{2}t_x^2 \). Equality \( DI = I \) implies

\[
f_x + f_t t_x + f_t f + f_{tt} t_{xx} - \frac{1}{2} f^2 = t_{xx} - \frac{1}{2} t_x^2. \tag{12}\]

By comparing the coefficients before \( t_{xx} \) in (12), we have \( f_{tt} = 1 \). Therefore,

\[
f(x, t, t_1, t_2) = t_x + d(x, t, t_1). \tag{13}\]

We substitute (13) into (12) and we get \( d_x + d_t t_x + d_{tt} t_x + d_{tt} t_{xx} - \frac{1}{2} d^2 = -\frac{1}{2} t_x^2 \), or equivalently, \( d_x + d_{tt} - d = 0 \) and \( d_x + d_{tt} d - \frac{1}{2} d^2 = 0 \). We solve the last two equations simultaneously and find that \( d = e^{\phi} K(x, t_1 - t) \), where \( K = Ce^{-\frac{1}{2}(t_1 - t)} \) and \( C \) is an arbitrary constant. Therefore, chain (2) with \( n \)-integral \( I = t_{xx} - \frac{1}{2} t_x^2 \) becomes \( t_{tx} = t_x + C e^{(t_1 + t)/2} \).

B. Case (2*)

Consider all chains (2) with \( n \)-integral \( I = t_x - e^t \). Equality \( DI = I \) implies \( f - e^{\phi} = t_x - e^t \), which gives the equation \( t_{tx} = f = t_x - e^t + e^t \).

C. Case (2*)

Consider all chains (2) with \( n \)-integral \( I = \frac{t_x}{t_x} - t_x \). Equality \( DI = I \) implies

\[
f_x + f_t t_x + f_t f + f_{tt} t_{xx} - f = \frac{t_{xx}}{t_x} - t_x. \tag{14}\]

By comparing the coefficients before \( t_{xx} \) in (14), we have \( f_{tt}/f = 1/t_x \), that is, \( f = K(x, t, t_1)/t_x \). Substitute \( f = K(x, t, t_1)t_x \) into (14) and we have \( \frac{k_x}{k} + \frac{k_{tt}}{k} t_x + K_t t_x - K t_x = -\frac{1}{k} t_x \), or equivalently (by comparing the coefficients before \( t_x \) and \( t_x^0 \)), we get \( \frac{k_x}{k} + K_t = K - 1 \) and \( K_x = 0 \). Therefore, equations \( t_{tx} = K(t, t_1)t_x \), where \( K \) satisfies \( \frac{k_x}{k} + K_t = K - 1 \), are the only chains (2) that admit \( n \)-integral \( I \) of the form \( I = \frac{t_x}{t_x} - t_x \).

D. Case (3*)

Consider all chains (2) with \( n \)-integral \( I = t_{xx} - \frac{1}{2} t_x^2 - \frac{1}{2} t^2 \). Equality \( DI = I \) implies

\[
f_x + f_t t_x + f_t f + f_{tt} t_{xx} - \frac{1}{2} f^2 - \frac{1}{2} e^{2\phi} t_{xx} = t_{xx} - \frac{1}{2} t_x^2 - \frac{1}{2} e^{2\phi}. \tag{15}\]

By comparing the coefficients before \( t_{xx} \) in (15), we have \( f_{tt} = 1 \), that is, \( f(x, t, t_1, t_2) = t_x + d(x, t, t_1) \). Substitute \( f(x, t, t_1, t_2) = t_x + d(x, t, t_1) \) into (15) and we have

\[
d_x + d_t t_x + d_{tt} (t_x + d) - \frac{1}{2} (t_x + d)^2 - \frac{1}{2} e^{2\phi} = -\frac{1}{2} t_x^2 - \frac{1}{2} e^{2\phi}. \tag{16}\]

Compare the coefficients before \( t_x \) and \( t_x^0 \) in (16) and we get

\[
d_t + d_{tt} - d = 0, \quad d_t + d_{tt} d - \frac{1}{2} d^2 - \frac{1}{2} e^{2\phi} = -\frac{1}{2} t_x^2 - \frac{1}{2} e^{2\phi}. \tag{17}\]

The first equation in (17) has a solution \( d = e^t K(x, t_1 - t) \). Substitution of this expression into the second equation of (17) gives \( e^{-\phi} K_x + K_{tt} - K^2 - \frac{1}{2} e^{-2\phi} = 0 \). Since \( K \) depends on \( U = t_1 - t \) and \( x \), then \( K_x = 0 \) and the last equation becomes \( 2K'K + K^2 = 1 - e^{-2\phi} \), and hence, \( d = e^t K = \sqrt{e^{2\phi} + e^{2\phi} + R e^{2\phi}} \), where \( R \) is an arbitrary constant. Therefore, chain (2) with \( n \)-integral \( I = t_{xx} - \frac{1}{2} t_x^2 - \frac{1}{2} e^{2\phi} \) becomes

\[
t_{tx} = t_x + \sqrt{e^{2\phi} + e^{2\phi} + R e^{2\phi}}, \quad R = \text{const.}
\]
E. Case (3*b)

Consider all chains (2) with $n$-integral $I = \frac{\omega - t_x^2 + 4}{\sqrt{t_x^2 - 4}}$. Equality $DI = I$ implies

$$f_x + f_i t_x + f_i f + f_i t_{xx} - f^2 + 4 \sqrt{f^2 - 4} = t_{xx} - t_x^2 + 4 \sqrt{t_x^2 - 4},$$

(18)

By comparing the coefficients before $t_{xx}$ in (18), we get

$$f(t) = \frac{1}{\sqrt{t_x^2 - 4}},$$

that is

$$\text{arccosh} \frac{t_x}{2} = \text{arccosh} \frac{t}{2} + K(x, t, t_1).$$

Thus,

$$f(x, t, t_1, t_2) = A t_x + B \sqrt{t_x^2 - 4},$$

(19)

where $A(x, t, t_1) = \cosh K$, $B(x, t, t_1) = \sinh K$, $A^2 - B^2 = 1$. Note that $f = 2\cosh((\text{arccosh} \frac{t}{2}) + K)$, i.e., $\sqrt{f^2 - 4} = 2\sinh((\text{arccosh} \frac{t}{2}) + K) = 2(\sqrt{t_x^2 - 1}\cosh K + \frac{t}{2}\sinh K)$, or $\sqrt{f^2 - 4} = B t_x + A \sqrt{t_x^2 - 4}$. Substitute (19) into (18) and we have

$$t_x A_x + B_x \sqrt{t_x^2 - 4} + t_x^2 A + t_x B_x \sqrt{t_x^2 - 4} + (t_x A_{t_1} + B_{t_1} \sqrt{t_x^2 - 4})(A t_x + B \sqrt{t_x^2 - 4}) - (A t_x + B \sqrt{t_x^2 - 4})^2 + 4 = -(B t_x + A \sqrt{t_x^2 - 4})^2,$$

that can be written shortly as

$$(t_x^2 - 4)(\alpha_1 + \alpha_2 t_x^2) = (\alpha_3 + \alpha_4 t_x + \alpha_5 t_x^2)^2,$$

(20)

where $\alpha_1 = B_x$, $\alpha_2 = B_x + A_{t_1} + B_{t_1} A - 2AB + B$, $\alpha_3 = -4B_{t_1} B + 4B^2 + 4 - 4A$, $\alpha_4 = A_{t_1}$, and $\alpha_5 = A_{t_1} + A_{t_1} A + B_{t_1} B - A^2 - B^2 + A$. We compare the coefficients before $t_x^2$, $t_{x}^3$, $t_{xx}$, $t_{xxx}$, and $t_{xx}^2$ in (20) and we have

$$\alpha_2^2 = \alpha_5^2, \quad 2\alpha_3 \alpha_2 = 2\alpha_{2} \alpha_{2}, \quad \alpha_1^2 - 4\alpha_2^2 = \alpha_3^2 + 2\alpha_3 \alpha_2, \quad -8\alpha_1 \alpha_2 = 2\alpha_3 \alpha_2,$$

that implies $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, which is possible only if $A = 1 + \Re e^{i t_1}$ and $B = \sqrt{R^2 e^{2i(t_1 + t_2)}} + 2\Re e^{i(t_1 + t_2)}$, where $R = \text{const}$. Therefore, by (19), the chain (2) with $n$-integral $I = \frac{t_x^2 - t_x^2 + 4}{\sqrt{t_x^2 - 4}}$ becomes $t_{xx} = (1 + \Re e^{i t_1}) t_x + \sqrt{R^2 e^{2i(t_1 + t_2)}} + 2\Re e^{i(t_1 + t_2)} \sqrt{t_x^2 - 4}$.

F. Case (4*)

Consider chains (2) with $n$-integral $I = \frac{\omega - t_x^2}{t_x - x}$. Equality $DI = I$ implies

$$f_x + f_i t_x + f_i f + f_i t_{xx} = \frac{2 f}{t_x - x} + \frac{1}{t_x - x} = \frac{t_{xx}}{t_x - x} - \frac{2 t_x}{t_x - x} + \frac{1}{t_x - x}.$$  

(21)

We compare the coefficients before $t_{xx}$ and we have $f_i / f = 1 / t_x$, that is, $f = t_x K(x, t, t_1)$. Substitute $f = t_x K$ into (21) and we have

$$K_x t_x + K_t t_x^2 + K_t K t_x^2 \frac{2 K t_x}{t_x - x} + \frac{1}{t_x - x} = -2 t_x \frac{1}{t_x - x} + \frac{1}{t_x - x}.$$  

(22)

By comparing the coefficients before $t_x$ and $t_{xx}$ in (22), we get

$$K_x = \frac{K}{t_x} = \frac{2 K}{t_x - x} - \frac{2}{t_x - x}, \quad K_{t_x} = -\frac{1}{t_x - x} + \frac{1}{t_x - x}.  

(23)

We solve two equations of (23) simultaneously and we have $K = \frac{n + 1}{t_x + L} \frac{n - 1}{t_x - L}$, where $L$ is an arbitrary constant. Therefore, any chain (2) with $n$-integral $I = \frac{\omega - t_x^2}{t_x - x}$ becomes $t_{xx} = \frac{n + 1}{t_x + L} \frac{n - 1}{t_x - L} t_x$.

G. Case (5*)

Consider all chains (2) with $n$-integral $I = \frac{\omega - t_x^2}{t_x - x}$. where $\beta = \beta(t_x)$, $\psi = \psi(t)$, and $\beta \beta' = -t_x$. We have, $2 \beta \beta' = -2 t_x$, i.e., $\beta^2 = -t_x^2 + M^2$, or $\beta = \sqrt{M^2 - t_x^2}$, where $M$ is an arbitrary
constant. Equality \( DI = I \) implies
\[
\frac{f_x + f_{it} + f_{ii} f + f_{i} t_{xx}}{\beta(f)} - \psi(t_1)\beta(f) = \frac{t_{xx}}{\beta(t_x)} - \psi(t)\beta(t_x).
\] (24)

We compare the coefficients before \( t_{xx} \) and we have \( f_i / \beta(f) = 1/\beta(t_x) \), which implies that either \((5^a)\): \( M = 0, \beta(t_x) = it_x, \) and \( t_{1x} = K(x, t, t_1)t_x, \)

or \((5^b)\): \( M \neq 0 \) and then \( \arcsin \frac{\gamma}{M} = \arcsin \frac{\gamma}{M} + L(x, t, t_1) \), that is,
\[
f = t_x A(x, t, t_1) + \sqrt{M^2 - t_x^2} B(x, t, t_1), \quad A^2 + B^2 = 1.
\] (25)

In case \((5^a)\), we substitute \( t_{1x} = f = K(x, t, t_1)t_x \) into (24), use that \( \beta(t_x) = it_x \), and obtain
\[
K_x = 0, \quad \frac{K_i}{K} + K_t + \psi(t_1)K = \psi(t).
\] (26)

Therefore, the chains (2) with \( n \)-integral \( I = \frac{t_x}{\beta(t_x)} - i \psi(t)t_x \) are equations \( t_{1x} = K(t, t_1)t_x \), where function \( K \) satisfies (26).

Let us consider case \((5^b)\). Note that
\[
M^2 - f^2 = M^2 - A^2 t_x^2 - 2ABt_x \sqrt{M^2 - t_x^2} - B^2 M^2 + B^2 t_x^2 = (Bt_x - A \sqrt{M^2 - t_x^2})^2
\]
and \( \beta(f) = \pm(Bt_x - A \sqrt{M^2 - t_x^2}) \), \( \beta(t_x) = \sqrt{M^2 - t_x^2} \). Substitute (25) into (24) and we get
\[
A_{it} + B_{\sqrt{M^2 - t_x^2}} + A_t t_x^2 + B_{\sqrt{M^2 - t_x^2}}(A_{it} + B_{\sqrt{M^2 - t_x^2}}) = \pm(Bt_x - A \sqrt{M^2 - t_x^2}) \psi(t_1) - \sqrt{M^2 - t_x^2} \psi(t),
\]
or the same,
\[
(M^2 - t_x^2)(\alpha_1 + \alpha_2 t_x^2) = (\alpha_3 + \alpha_4 t_x^2), \quad (27)
\]
where \( \alpha_1 = B_t, \quad \alpha_2 = B^2t_x - A^2 M^2 \psi(t_1) - A^2 \psi(t)M^2, \quad \alpha_3 = B B_t M^2 - A^2 M^2 \psi(t_1) - A^2 \psi(t)M^2, \quad \alpha_4 = A_{it} + B_{\sqrt{M^2 - t_x^2}}(A_{it} + B_{\sqrt{M^2 - t_x^2}}) \). We compare the coefficients before \( t_x^k, \quad k = 0, 1, 2, 3, 4, \) in (27) and find that \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0 \), which is possible only if \( \psi = R \) is a constant function, that contradicts to the equation \( (ln \psi)'' = \psi^2 \). Therefore, case \((5^b)\) is not realized.

\textbf{H. Case \((6^a)\)}

Consider chains (2) with \( n \)-integrals \( I = \frac{t_x}{\beta(t_x)} - \frac{\beta(t_1)}{t} \), where \( \beta = \beta(t_x) \) and \( \beta' + c \beta = -t_x \).

The equality \( DI = I \) implies
\[
\frac{f_x}{\beta(f)} = \frac{1}{\beta(t_x)}, \quad (28)
\]

and
\[
\frac{f_x + f_{xt} + f_{it} f}{\beta(f)} = \frac{\beta(f)}{t_1} = -\frac{\beta(t_x)}{t}.
\] (29)

Differentiation of (28) with respect to \( x, t, \) and \( t_1 \) gives
\[
f_{xt} = \frac{\beta'(f)}{\beta(t_x)} f_x, \quad f_{xt} = \frac{\beta'(f)}{\beta(t_x)} f_x, \quad f_{i} t_{t1} = \frac{\beta'(f)}{\beta(t_x)} f_{t1}.
\] (30)

First we differentiate (29) with respect to \( t_x \), use (30), and we get
\[
\frac{1}{\beta(f)} f_t + \frac{1}{\beta(t_x)} f_{t1} = -\frac{(c \beta(f) + f)}{t_1 \beta(t_x)} + \frac{c}{t} + \frac{t_x}{t \beta(t_x)}.
\] (31)
Next we differentiate (31) with respect to \( t_t \), use (30), and arrive to the equality
\[
\left\{ \frac{t_x}{\beta(t_x)} - \frac{f}{\beta(f)} \right\} f_t = - \frac{(c\beta(f) + f)t_x}{t_t \beta(t_x)} - \frac{\beta(f)}{t_t} + \frac{\beta(t_x)}{t} + \frac{c t_x}{t} + \frac{t^2}{1} \frac{\beta(t_x)}{t}.
\]

There are two possibilities: either (6.4), when
\[
A := \frac{t_x}{\beta(t_x)} - \frac{f}{\beta(f)} = 0,
\]
or (6.6), when
\[
f_t = \frac{\beta(f)\beta(t_x)}{t_t \beta(t_x) - f \beta(t_x)} \left\{ \frac{c\beta(f) + f}{t_t \beta(t_x)} - \frac{\beta(f)}{t} + \frac{\beta(t_x)}{t} + \frac{c t_x}{t} + \frac{t^2}{1} \frac{\beta(t_x)}{t} \right\}.
\]

Let us first consider the case (6.4). It follows from (32) and (28) that \( f_t/f = 1/t_t \), that is, \( f = K(x, t, t_t)t_t \). We substitute \( f = K(x, t, t_t)t_t \) into (29), use \( \beta(f)/t_t = (\beta(t_x)/f)/(t_t/t_x) = b(t_x)K/t_t \), and we obtain
\[
K_x + t_x \left\{ \frac{K_x}{K} + K_{t_t} \right\} = \frac{\beta^2(t_x)}{t_x} \left\{ \frac{K}{t_t} - \frac{1}{t} \right\},
\]
that is, \( K_x = 0, \beta(t_x) = \sqrt{R^2 t_x^2 + C t_t}, \) \( R = \text{const}, B = \text{const} \), and
\[
\frac{K_x}{K} + K_{t_t} = R^2 \left\{ \frac{K}{t_t} - \frac{1}{t} \right\}.
\]

Substitution of \( \beta(t_x) = \sqrt{R^2 t_x^2 + C t_t} \) into (32) shows that \( \beta(t_x) = R t_t \). Therefore, in the case (6.4), the \( n \)-integral is \( I = \int \frac{t_x}{R t_t - R t_t} \) and the corresponding chain (2) is of the form \( t_t = K(t, t_t)t_t \), where \( K \) satisfies (34).

Let us now study the case (6.6). It follows from (33) and (31) that
\[
f_t = \frac{f \beta(t_x)\beta(f)}{\beta(f) t_t - f \beta(t_x)} \left\{ \frac{c\beta(f) + f}{t_t \beta(t_x)} - \frac{\beta(t_x)}{t} + \frac{c t_x}{t} + \frac{t^2}{1} \frac{\beta(t_x)}{t} \right\}.
\]

First we differentiate (33) with respect to \( t_t \) and find \( f_{t_t} \), use the expression for \( f_t \) from (35) and \( \beta'(f) = - (f + c\beta(f))/\beta(f) \) to express \( f_{t_t} \) in terms of \( \beta(f), \beta(t_t), f, t_t, t \), and \( t_x \). Then we differentiate (35) with respect to \( t_t \) and find \( f_{t_{t_t}} \), use the expression for \( f_t \) from (33) and \( \beta'(f) = -(f + c\beta(f))/\beta(f) \) to express \( f_{t_{t_t}} \) in terms of \( \beta(f), \beta(t_t), f, t_t, t \), and \( t_x \).

Direct calculations show that
\[
f_{t_{t_t}} - f_{t_t} = \frac{2\beta(f)\beta(t_x)(c\beta(f) + f^2)}{tt_t^2(\beta(t_x)f - \beta(f)t_x)^2}.
\]

Equality \( f_{t_{t_t}} = f_{t_t} \) yields (i) \( \beta^2(f) + c\beta(f) + f^2 = 0 \), i.e., \( \beta(f) = Af, \beta(t_x) = At_x \), where \( A = \frac{-c - \sqrt{c^2 - 4}}{2} \), or (ii) \( f = t_t^{-1}(c\beta(t_x) + t_x) \).

Let us consider the case (i). It follows from (28) that \( f = K(x, t, t_t)t_t \). The same considerations as in part (6.4) show that the chain (2) in this case is \( t_{t_t} = K(t, t_t)t_t \), where function \( K(t, t_t) \) satisfies (34).

Let us consider the case (ii). It follows from (28) that \( \beta(f) = t_t^{-1}((1 - c^2)\beta(t_x) - ct_x) \). We substitute this expression for \( \beta(f) \) into (29) and get \( c^2(2 - c^2)\beta^2(t_x) + 2ct(1 - c^2)t_x\beta(t_x) - c^2 t_x^2 = 0 \), that implies that (i) \( c = 0 \), (II) \( c^2 = 2 \), (III) \( \beta(t_x) = \frac{\sqrt{2}t_x}{t_t}, \) or (IV) \( \beta(t_x) = -\frac{1}{t_t} \). Cases (II) and (IV) are not realized, each of them is incompatible with \( \beta\beta' + c\beta = -t_x \). Case (III) is realized only for \( c = 2 \) (with \( \beta(t_x) = t_x \)) and \( c = -2 \) (with \( \beta(t_x) = t_x \)). Therefore, using \( f = t_t^{-1}(c\beta(t_x) + t_x) \) and the fact that \( c = 0 \) (with \( \beta(t_x) = \pm it_x \)) or \( c = \pm 2(\beta(t_x) = \pm t_x) \) we arrive to a chain (2) of the form \( t_{t_t} = \pm \frac{t_t}{2} \). Note that chains \( t_{t_t} = \pm t_{t_t}^{-1}t_t \) with \( \beta(t_x) = \pm t_x \) or \( \beta(t_x) = \pm it_x \) is of the form \( t_{t_t} = K(t, t_t)t_t \), where \( K \) satisfies (34) with \( R^2 = 1 \) for \( t_{t_t} = -t_{t_t}^{-1}t_t \) or \( R^2 = -1 \) for \( t_{t_t} = t_{t_t}^{-1}t_t \).
I. Case (7*)

Consider chains (2) with n-integral \( I = \frac{t}{\sqrt{t}} + 2 \sqrt{t} \), \( y = \text{const} \). Equality \( DI = I \) implies

\[
\frac{f_x + f_1 t_x + f_1 f + f_{1,x} t_{xx}}{\sqrt{t}} + 2 \sqrt{t} = \frac{t_{xx}}{\sqrt{t}} + 2 \sqrt{t, x + y}.
\]

(36)

By comparing the coefficients before \( t_{xx} \), we have \( f_1, / \sqrt{t} = 1/ \sqrt{t, x} \), or

\[
f = (\sqrt{t} + K(x, t, t_1))^2.
\]

(37)

Substitute (37) into (36) and get \( K_x + K_t x + K_t x + 2 K_t \sqrt{t} K + K_t K^2 + \frac{K}{\sqrt{t}} = 0 \). We compare the coefficients before \( \sqrt{t} \), \( t_x \), and \( t_x^0 \) and have \( 2 K_t K = 0 \), i.e., \( K = L(x, t) \); \( K_t + K_t = 0 \), i.e., \( K = \frac{C}{x + y} \), \( C = \text{const} \). Therefore, chain (2) with n-integral \( I = \frac{t}{\sqrt{t}} + 2 \sqrt{t} \) becomes \( t_{xx} = (\sqrt{t} + \frac{C}{x + y})^2 \), where \( C \) and \( y \) are arbitrary constants.

J. Case (8*)

Consider chains (2) with n-integral \( I = \beta(t_x) t_{xx} - \frac{1}{(x + y) \beta(t)} \), where \( y \) is an arbitrary constant and \( \beta(t) = \beta_3(t_x) + \beta_2(t_x) \). The equality \( DI = I \) gives

\[
\beta(f_x + f_1 t_x + f_1 f + f_{1,x} t_{xx}) - \frac{1}{(x + y) \beta(f)} = \beta(t_x) t_{xx} - \frac{1}{(x + y) \beta(t_x)}.
\]

that implies

\[
\beta(f) f_x = \beta(t_x).
\]

(38)

and

\[
\beta(f_x + f_1 t_x + f_1 f) = \frac{1}{(x + y) \beta(f)} - \frac{1}{(x + y) \beta(t_x)}.
\]

(39)

Differentiate (38) with respect to \( x, t \), and \( t_1 \) and get

\[
f_{tx} = -(\beta(f) + 1) \beta(t_x) f_x, \quad f_{tx} = -(\beta(f) + 1) \beta(t_x) f_x, \quad f_{tx} = -(\beta(f) + 1) \beta(t_x) f_x.
\]

(40)

Now differentiate (39) with respect to \( t_x \), we have

\[
\beta(f) f_x + \beta(t_x) f_{tx} = \frac{1}{x + y} - \frac{\beta(t_x)}{(x + y) \beta(f)}.
\]

(41)

Differentiate (41) with respect to \( t_x \) and get \( f_{tx} = -\frac{1}{(x + y) \beta(f)} \). The last equation together with (41), (38), and (39) gives

\[
f_{tx} = \frac{1}{(x + y) \beta(f)}, \quad f_t = \frac{1}{(x + y) \beta(f)}, \quad f_x = \frac{\beta(t_x)}{\beta(f)}
\]

(42)

and

\[
f_x = \frac{1}{x + y} \left\{ \frac{1}{\beta(f)} - \frac{1}{(x + y) \beta(f)} - \frac{t_x}{\beta(f)} + \frac{f}{\beta(f)} \right\}.
\]

(43)

Since, by (42) and (43), \( f_{tx} = f_{tx} = \frac{1}{\beta(f) \beta(t_x)} (\beta(f) + 1) \), then \( \beta(f) = -1 \), and, therefore, by (42), we have \( f_{tx} = (x + y)^{-1} \), \( f_t = -(x + y)^{-1} \), and \( f_{tx} = 1 \). Hence, \( f(x, t, t_1, t_x) = t_x + \frac{t}{x + y} + C(x) \).

We substitute this expression for \( f \) into (43) and obtain \( C(x) = C(x + y)^{-1} \), where \( C \) is an arbitrary constant. Therefore, with the n-integral \( I = \frac{t}{\sqrt{t}} + 2 \sqrt{t} \), the chain (2) becomes \( t_{xx} = t_x + \frac{t}{x + y} + C(x + y)^{-1} \), where \( y \) is an arbitrary constant.
III. PROOF OF REMARK 1.4

A. Case 1**

Consider all Eqs. (1) with m-integral \( \tilde{I} = e^{v_1-v} + e^{v_2-v} \). Denote by \( e^{-v_j} = w_j, j = 0, 1, 2 \), and \( e^{-v_1} = \bar{w}_1 \). In new variables \( \tilde{I} = \frac{v_2 + w}{w_1} \) is an m-integral of equation \( w_{1,1} = g(w, w_1, \bar{w}_1) \). \( \tilde{D} \tilde{I} = \tilde{I} \) implies

\[
\frac{w_2 + w}{w_1} = g_1 + \bar{w}_1.
\]

We differentiate both sides of (44) with respect to \( w_2 \) and apply the shift operator \( D^{-1} \), we have

\[
\frac{1}{w_1} = \frac{g_1 w_2}{g} \implies D^{-1}\left(\frac{1}{w_1}\right) = D^{-1}\left(\frac{g_1 w_2}{g}\right) \implies g_{w_1} = \frac{\bar{w}_1}{w}.
\]

Therefore,

\[
g = \frac{\bar{w}_1 w_1}{w} + c(w, \bar{w}_1), \quad g_1 = \frac{g w_2}{w_1} + c(w_1, g).
\]

We substitute (45) into (44) and get,

\[
g = \frac{w}{w_1} = c(w_1, g) + \bar{w}_1.
\]

Substitution of (45) into (46) implies that \( c(w, \bar{w}_1)w = c(w_1, g)w_1 \), or the same, \( c(w, \bar{w}_1)w = D(c(w, \bar{w}_1)w) \). Suppose that equation \( w_{1,1} = g(w, w_1, \bar{w}_1) \) does not admit an m-integral of the first order, then \( c(w, \bar{w}_1)w = D(c(w, \bar{w}_1)w) = C = \text{const} \). Thus, \( c(w, \bar{w}_1) = C/w \). Finally, \( g(w, w_1, \bar{w}_1) = \frac{\bar{w}_1 w_1}{w} + C w^{-1} \). Therefore, the Eqs. (1) with m-integral \( \tilde{I} = e^{v_1-v} + e^{v_2-v} \) become \( e^{v_1+v} = (C + e^{-v_1+v})^{-1} \), where \( C \) is an arbitrary constant. Note that this equation is symmetric with respect to variables \( v_1 \) and \( \bar{v}_1 \). Therefore, m-integral for the equation can be obtained by simply changing in m-integral variables \( v_j \) into variables \( \bar{v}_j, j = 1, 2 \).

B. Case 4**

Consider Eqs. (1) with m-integral \( \tilde{I} = \frac{(v_1 - v)(v_2 + L)}{(v_1 - v + L)} \). Equation \( v_{1,1} = f(v, v_1, \bar{v}_1) \) can be rewritten as \( v_{1,1} = r(v, v_1, \bar{v}_1) \). Equality \( \tilde{D} \tilde{I} = \tilde{I} \) implies

\[
\frac{(f - \bar{v}_1)(f_1 + L)}{(f_1 - \bar{v}_1)(f + L)} = \frac{(v_1 - v)(v_2 + L)}{(v_2 - v)(v_1 + L)}
\]

(47)

Take the logarithmic derivative of (47) with respect to \( v_2 \) and then apply the shift operator \( D^{-1} \), we get

\[
\frac{f v_2}{f_1 + L} - \frac{f v_2}{f_1 - \bar{v}_1} = \frac{1}{v_2 + L} - \frac{1}{v_2 - v} \implies \frac{f v_1}{(f + L)(f - r)} = \frac{v_1 + L}{(v_1 + L)(v_1 - v_1)}. \]

(48)

We conclude from the second equation of (48) that

\[
\frac{f + L}{f - r} = \frac{v_1 + L}{v_1 - v_1} K(v, \bar{v}_1).
\]

(49)

Take the logarithmic derivative of (49) with respect to \( v_{1,1} \) and get \( f - r = r_{v_{1,1}}(v_1 - v_{1,1}) \). Differentiation of the last equality with respect to \( v_1 \) yields \( f_{v_1} = r_{v_{1,1}} \). We differentiate (48) with respect to \( v_{1,1} \) and use the fact that \( f_{v_1} = r_{v_{1,1}} \), we obtain \( f_{v_1} = \pm \frac{f - r}{v_1 - v_{1,1}} \).

First assume that \( f_{v_1} = -\frac{f - r}{v_1 - v_{1,1}} \). We have, \( f - r = D(v, v_1, \bar{v}_1)(v_1 - v_{1,1})^{-1} \). It follows from \( r_{v_{1,1}} = -\frac{f - r}{v_1 - v_{1,1}} \) that \( f - r = C(v, v_1, \bar{v}_1)(v_1 - v_{1,1})^{-1} \), and, therefore, \( f - r = C(v, \bar{v}_1)(v_1 - v_{1,1})^{-1} \). We substitute this expression for \( f - r \) into (49) and see that \( f + L = C(v, \bar{v}_1)K(v, \bar{v}_1)(v_1 + L)(v_1 - v_{1,1})^{-2} \) which is impossible since \( f \) does not depend on \( v_{1,1} \).

Now consider the case when \( f_{v_1} = \frac{f - r}{v_1 - v_{1,1}} \). We have, \( f - r = (v_1 - v_{1,1})C(v, v_1, \bar{v}_1) \). Also, \( r_{v_{1,1}} = \frac{f - r}{v_1 - v_{1,1}} \) implies that \( f - r = (v_1 - v_{1,1})C(v, v_1, \bar{v}_1) \). One can see that \( D(v, v_{1,1}, \bar{v}_1) \)
\[= C(v, v_1, \bar{v}_1) = C(v, \tilde{v}_1). \text{ Therefore, } f - r = C(v, \bar{v}_1)(v_1 - v_{-1}). \text{ It follows from (49) that}
\]
\[f = A(v, \bar{v}_1)v_1 + A(v, \tilde{v}_1)L - L,
\]
where \( A = CK. \) Note that \( A = A(v, \bar{v}_1) \) and \( A_1 = A(v_1, f(v, v_1, \bar{v}_1)) \). Substitute (50) into (47), we get
\[(A v_1 + \alpha L - \bar{v}_1)(A_1 v_2 + A_1 L)(v_2 - v)(v_1 + L)
\]
\[= (A_1 v_2 + A_1 L - \bar{v}_1)(A v_1 + \alpha L)(v_1 - v)(v_2 + L),
\]
and compare the coefficients before \( v_2^2 \), we have
\[A_1(A v_1 + \alpha L - \bar{v}_1)(v_1 + L) = A_1(A v_1 + \alpha L)(v_1 - v).
\]
It follows from (51) that \( A_1 = 0 \) or, by comparing the coefficients before \( v_1 \), one gets \( A = \frac{L + \bar{v}_1}{L + \bar{v}} \).

Therefore, by (50), we have the equation \( v_{1,1} = f = \frac{L(\bar{v}_1 + v_1 + v_1 \bar{v})}{L + \bar{v}} \). Note that the equation is symmetric with respect to variables \( v_1 \) and \( \bar{v}_1 \). This observation allows one to write down an \( n \)-integral \( I \) by a given \( m \)-integral \( \bar{I} \) by changing in \( I \) variables \( v_j \) into variables \( \bar{v}_j, j = 1, 2. \)

C. Case 7**

Consider all Eqs. (1) with \( m \)-integral \( F = 2v - v_1 - v_{-1} = D^{-1} \bar{I} \), where \( \bar{I} = 2v_1 - v - v_2 \). Equation \( v_{1,1} = f(v, v_1, \bar{v}_1) \) can be rewritten as \( v_{1,1} = r(v, v_{-1}, \bar{v}_1) \). Equality \( \bar{D}F = F \) implies
\[2\bar{v}_1 - f - r = 2v - v_1 - v_{-1}.
\]
We apply \( \frac{\partial}{\partial \bar{v}_1} \) and \( \frac{\partial}{\partial v} \) to (52) and find that \( f_{\bar{v}_1} = 1 \) and \( r_{v_{-1}} = 1 \). Therefore, \( f = v_1 + h(v, \bar{v}_1) \) and \( r = v_{-1} + q(v, \bar{v}_1) \). Substitute these expressions for \( f \) and \( r \) into (52) and get
\[q = 2\bar{v}_1 - 2v - h.
\]
Equation \( v_{1,1} = f = v_1 + h(v, v_1) \) can be rewritten as
\[\bar{v}_1 = v + h(v_{-1}, v_{-1,1}) = v + h(v_{-1}, v_{-1} + q(v, \bar{v}_1)).
\]
First differentiate (54) with respect to \( v_{-1} \) and then apply the shift operator \( D^{-1} \), we get \( D^{-1}h_v + D^{-1}h_{\bar{v}} = 0 \), that is, \( h = h(\bar{v}_1 - v) \). Equations (52)–(54) give \( \bar{v}_1 - v = h(2\bar{v}_1 - 2v - h) \), or by taking \( \epsilon = \bar{v}_1 - v \), one gets \( \epsilon = h(2\epsilon - h(\epsilon)) \). Therefore, the equation with \( m \)-integral \( \bar{I} = 2v_1 - v - v_2 \) becomes \( v_{1,1} = v_1 + h(\bar{v}_1 - v) \), where \( h \) solves a functional equation \( \epsilon = h(2\epsilon - h(\epsilon)) \). This equation \( v_{1,1} = v_1 + h(\bar{v}_1 - v) \) admits also an \( n \)-integral. Since the equation is of the form \( Dz = h(z) \) with \( z = \bar{v}_1 - v_1 \) then we have \( D(z - h^{-1}(z)) = z - h^{-1}(z) \). Actually, \( D(z - h^{-1}(z)) = D(z) - z = h(z) - z = z - h^{-1}(z) = z - h^{-1}(z) \). Here, we use the identity \( h(z) - z = z - h^{-1}(z) \) which is equivalent to the functional equation \( z = h(2z - h(z)) \).

IV. CONCLUSIONS

The problem of discretization of the Liouville type partial differential equations is discussed. Besides purely theoretical interest as a bridge between two parallel realizations of the integrability theory, this subject has an important practical significance. There are two-dimensional Toda field equations corresponding to each semi-simple or of Kac-Moody type Lie algebra (see Refs. 12 and 13). The question is open whether there exist integrable discrete versions of these. Different particular cases are studied in Refs. 14–16. In this article a step is done towards the solution of the problem. An effective method of discretization is suggested based on integrals. It is known that the Bäcklund transform is a kind of discretization (see Refs. 3 and 17). We would like to stress that our method of discretization essentially differs from that one. Even though for some exceptional cases the semi-discrete equation obtained realizes the Bäcklund transformation of the original PDE for the other examples, it is not the case.
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