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Elementary proofs of some identities of Ramanujan for the Rogers–Ramanujan functions

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ABSTRACT

In a handwritten manuscript published with his lost notebook, Ramanujan stated without proofs forty identities for the Rogers–Ramanujan functions. With one exception all of Ramanujan's identities were proved. In this paper, we provide a proof for the remaining identity together with new elementary proofs for two identities of Ramanujan which were previously proved using the theory of modular forms. Ramanujan stated that each of his formula was the simplest of a large class. Our proofs are constructive and permit us to obtain several analogous identities which could have been stated by Ramanujan and may very well belong to his large class of identities.

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1. Introduction

The Rogers–Ramanujan functions are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}, \tag{1.1}$$

where $(a;q)_0 := 1$ and, for $n \geqslant 1$,

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k).$$

These functions satisfy the famous Rogers–Ramanujan identities [7,5], [6, pp. 214–215]

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \quad \text{and} \quad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, \tag{1.2}$$

where

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n, \quad |q| < 1.$$

In a handwritten manuscript published with his lost notebook, Ramanujan stated without proofs forty identities for the Rogers–Ramanujan functions. The simplest yet the most elegant is the following identity which was proved by L.J. Rogers [8]

$$H(q)G\big(q^{11}\big) - q^2 G(q) H\big(q^{11}\big) = 1. \tag{1.3}$$

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Let $\chi(q) := (-q;q^2)_\infty$. The identities of Ramanujan that we prove in this paper are as follows.

Entry 1.1. *Define*

$$U := G(q^{17})H(q^2) - q^3 G(q^2)H(q^{17}) \quad and \quad V := G(q)G(q^{34}) + q^7 H(q)H(q^{34}).$$

Then

$$\frac{U}{V} = \frac{\chi(-q)}{\chi(-q^{17})} \tag{1.4}$$

and

$$U^4V^4 - qU^2V^2 = \frac{\chi^3(-q^{17})}{\chi^3(-q)}\left(1 + q^2\frac{\chi^3(-q)}{\chi^3(-q^{17})}\right)^2. \tag{1.5}$$

Entry 1.2.

$$\{G(q^2)G(q^{23}) + q^5 H(q^2)H(q^{23})\}\{G(q^{46})H(q) - q^9 G(q)H(q^{46})\} = \chi(-q)\chi(-q^{23}) + q + \frac{2q^2}{\chi(-q)\chi(-q^{23})}. \tag{1.6}$$

Entry 1.3.

$$\begin{aligned}&\{G(q)G(q^{94}) + q^{19}H(q)H(q^{94})\}\{G(q^{47})H(q^2) - q^9 G(q^2)H(q^{47})\}\\ &\quad = \chi(-q)\chi(-q^{47}) + 2q^2 + \frac{2q^4}{\chi(-q)\chi(-q^{47})} + q\sqrt{4\chi(-q)\chi(-q^{47}) + 9q^2 + \frac{8q^4}{\chi(-q)\chi(-q^{47})}}.\end{aligned} \tag{1.7}$$

D. Bressoud proved (1.4) in his thesis [4] and we will not provide another proof here. A.J.F. Biagioli claimed in [3] that he was going to prove (1.5), but a proof of (1.5) does not appear in his paper. With the exception of (1.5), Ramanujan's forty identities were proved by Rogers [8], G.N. Watson, [9], D. Bressoud [4], and A.J.F. Biagioli [3]. The methods of Rogers, Watson and Bressoud were elementary while Biagioli used the theory of modular forms. In [2], we extensively studied Ramanujan's forty identities and provided various elementary proofs except for five identities in the spirit of Ramanujan. Author in a recent work [10], gave a generalization of an identity of Rogers. Our generalization is actually based upon Bressoud's work who generalized and used Rogers identity to prove some of Ramanujan's identities. In [10], we also developed similar identities and provided new identities for the Rogers–Ramanujan functions and gave new elementary proofs for two of Ramanujan's identities. In this paper, we give elementary proofs for the remaining three identities above by employing some of the results obtained in [10]. We employ Ramanujan's modular equations of degree 23 and 47 and several identities of Ramanujan from his list of forty identities.

The rest of the paper is organized as follows. The preliminary results are given in Section 2. In the following sections, we give proofs of Entries 1.1–1.3 along the way we obtain various similar identities for the functions involved.

2. Definitions and preliminary results

We first recall Ramanujan's definition for a general theta function and some of its important special cases. Set

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.1}$$

For convenience, we also define

$$f_k(a,b) = \begin{cases} f(a,b) & \text{if } k \equiv 0 \pmod 2,\\ f(-a,-b) & \text{if } k \equiv 1 \pmod 2.\end{cases} \tag{2.2}$$

The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [1, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.3}$$

The three most important special cases of (2.1) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{2.4}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.5}$$

and

$$f(-q) := f\left(-q, -q^2\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty =: q^{-1/24}\eta(\tau), \tag{2.6}$$

where $q = \exp(2\pi i\tau)$, $\operatorname{Im}\tau > 0$, and η denotes the Dedekind eta-function. The product representations in (2.4)–(2.6) are special cases of (2.3). Also, after Ramanujan, define

$$\chi(q) := \left(-q; q^2\right)_\infty. \tag{2.7}$$

Using (2.3) and (2.6), we can rewrite the Rogers–Ramanujan identities (1.2) in the forms

$$G(q) = \frac{f(-q^2, -q^3)}{f(-q)} \quad \text{and} \quad H(q) = \frac{f(-q, -q^4)}{f(-q)}. \tag{2.8}$$

A useful consequence of (2.8) in conjunction with the Jacobi triple product identity (2.3) is

$$G(q)H(q) = \frac{f(-q^5)}{f(-q)}. \tag{2.9}$$

The function $f(a,b)$ also satisfies a useful addition formula. For each nonnegative integer n, let

$$U_n := a^{n(n+1)/2} b^{n(n-1)/2} \quad \text{and} \quad V_n := a^{n(n-1)/2} b^{n(n+1)/2}.$$

Then [1, p. 48, Entry 31]

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{2.10}$$

Two special cases of (2.10) which we frequently use are

$$\varphi(q) = \varphi\left(q^4\right) + 2q\psi\left(q^8\right) \tag{2.11}$$

and

$$\psi(q) = f\left(q^6, q^{10}\right) + qf\left(q^2, q^{14}\right). \tag{2.12}$$

Our proofs employ the following identities of Ramanujan from his list of forty identities.

Entry 2.1.

$$G(q)G\left(q^4\right) + qH(q)H\left(q^4\right) = \chi^2(q) = \frac{\varphi(q)}{f(-q^2)}. \tag{2.13}$$

Entry 2.2.

$$G(q)G\left(q^4\right) - qH(q)H\left(q^4\right) = \frac{\varphi(q^5)}{f(-q^2)}. \tag{2.14}$$

Entry 2.3.

$$G(q)H(-q) + G(-q)H(q) = \frac{2}{\chi^2(-q^2)} = \frac{2\psi(q^2)}{f(-q^2)}. \tag{2.15}$$

Entry 2.4.

$$G(q)H(-q) - G(-q)H(q) = \frac{2q\psi(q^{10})}{f(-q^2)}. \tag{2.16}$$

In the remainder of this section we collect several results from [10]. Let m be an integer and α, β, p and λ be positive integers such that

$$\alpha m^2 + \beta = p\lambda. \tag{2.17}$$

Let δ, ε be integers. Further let l and t be real and x and y be nonzero complex numbers. Recall that the general theta functions f, f_k are defined by (2.1) and (2.2). With the parameters defined this way, we set

$$R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y) := \sum_{\substack{k=0\\ n=2k+t}}^{p-1} (-1)^{\varepsilon k} y^k q^{\{\lambda n^2+p\alpha l^2+2\alpha nml\}/4} f_\delta\big(xq^{(1+l)p\alpha+\alpha nm}, x^{-1}q^{(1-l)p\alpha-\alpha nm}\big) \times f_{\varepsilon p+m\delta}\big(x^{-m}y^p q^{p\beta+\beta n}, x^m y^{-p} q^{p\beta-\beta n}\big). \tag{2.18}$$

We have

Lemma 2.5. *(See [10, Lemma 1].) Let l, t and z be integers with $z \in \{-1,1\}$. Define $\delta_1 := \varepsilon p + m\delta$ and assume that*

$$\varepsilon(p+t)+\delta(l+m) \equiv 1 \pmod 2. \tag{2.19}$$

Then,

$$R1(z,\varepsilon,\delta,l,t,\alpha,\beta,m,p) := R\left(\varepsilon,\delta,l-\frac{zm}{3},t+\frac{zp}{3},\alpha,\beta,m,p,\lambda,1,1\right) = (-1)^{\frac{(z+1)(1+\delta_1)}{2}} q^{\frac{1}{4}\{p\alpha l^2+p\beta/9\}} f\big(-q^{2p\beta/3}\big)\big\{S1+(-1)^{\varepsilon t/2}S2\big\}, \tag{2.20}$$

where

$$S1 = \sum_{\substack{n=1\\ n\equiv t \pmod 2}}^{p-1} (-1)^{\varepsilon(n-t)/2} q^{\frac{1}{4}\{\lambda n^2+2\alpha mnl-2n\beta/3\}} \frac{f(-q^{2\beta n/3}, -q^{2p\beta/3-2\beta n/3})}{f_{\delta_1}(q^{\beta n/3}, q^{2p\beta/3-\beta n/3})} \times f_\delta\big(q^{(1+l)p\alpha+\alpha mn}, q^{(1-l)p\alpha-\alpha mn}\big), \tag{2.21}$$

$$S2 = \begin{cases} f_\delta(q^{(1+l)p\alpha}, q^{(1-l)p\alpha}) & \text{if } t\equiv \delta_1+1\equiv 0 \pmod 2,\\ 0 & \text{otherwise.}\end{cases} \tag{2.22}$$

Lemma 2.6. *(See [10, Lemma 2].) Let l and t be integers. Define $\delta_1 := \varepsilon p + m\delta$ and assume that*

$$\varepsilon t+\delta(l+1) \equiv 1 \pmod 2. \tag{2.23}$$

Define

$$R2(\varepsilon,\delta,l,t,\alpha,\beta,m,p) := R\left(\varepsilon,\delta,l-\frac{1}{3},t,\alpha,\beta,m,p,\lambda,1,1\right).$$

If $\gcd(m,p)=1$, then

$$R2(\varepsilon,\delta,l,t,\alpha,\beta,m,p) = q^{\frac{p\alpha}{36}} f\big(-q^{2p\alpha/3}\big)\{S3+S4\}, \tag{2.24}$$

where

$$S3 = \sum_{\substack{n=1\\ n\equiv t \pmod 2}}^{p-1} (-1)^{\varepsilon(n-t)/2} q^{\frac{1}{4}\{\lambda n^2+2\alpha mn(l-1/3)+p\alpha l(l-2/3)\}} \frac{f(-q^{2\alpha(nm+lp)/3}, -q^{2p\alpha/3-2\alpha(nm+lp)/3})}{f_\delta(q^{\alpha(nm+lp)/3}, q^{2p\alpha/3-\alpha(nm+lp)/3})} \times f_{\delta_1}\big(q^{p\beta+\beta n}, q^{p\beta-\beta n}\big), \tag{2.25}$$

$$S4 = \begin{cases} (-1)^{(l+t\varepsilon)/2}\varphi_{\delta_1}(q^{p\beta}) & \text{if } t\equiv 0 \pmod 2,\\ 2(-1)^{\frac{m+l+\varepsilon(p-t)}{2}} q^{p\beta/4}\psi(q^{2p\beta}) & \text{if } p\equiv t\equiv \delta\equiv 1+m+l\equiv 1 \pmod 2,\\ 0 & \text{otherwise}.\end{cases} \tag{2.26}$$

Theorem 2.7. *(See [10, Theorem 2].) Let α, β, m, p, and λ be as before with $\alpha m^2+\beta = p\lambda$, and let ε, δ, l, t be integers with $(1+l)\delta+t\varepsilon\equiv 1 \pmod 2$. Assume further that $3\,|\,\alpha m$ and $\gcd(3,\lambda)=1$. Recall that $R1$ and $R2$ are defined by (2.20) and (2.24). Let α_1, β_1, m_1, and p_1 be another set parameters with $\alpha_1\beta_1=\alpha\beta$, $\alpha_1^2+m_1\beta_1^2=p_1\lambda$ and $\lambda|(\alpha m-\alpha_1 m_1)$. Set $a := (\alpha m-\alpha_1 m_1)/\lambda$. Then,*

$$R2(\varepsilon,\delta,l,t,\alpha,\beta,m,p) = R1(z,\delta,\varepsilon,l_1,t_1,1,\alpha\beta,\alpha m,\lambda), \tag{2.27}$$

where $l_1 := t+\alpha mz/3$, $t_1 := l-1/3-z\lambda/3$ and $z=\pm 1$ with $z\equiv -\lambda \pmod 3$. Moreover, if $3\,|\,\alpha_1 m_1$, then

$$R2(\varepsilon,\delta,l,t,\alpha,\beta,m,p) = R2(\varepsilon,\delta+a\varepsilon,l,t_2,\alpha_1,\beta_1,m_1,p_1), \tag{2.28}$$

where $t_2 = t+a(l-1/3)$.

If $3 \mid \beta_1$ *and* $\gcd(3, \alpha_1 m_1) = 1$, *then*

$$R2(\varepsilon, \delta, l, t, \alpha, \beta, m, p) = R1(y, \varepsilon, \delta + a\varepsilon, l_3, t_3, \alpha_1, \beta_1, m_1, p_1), \tag{2.29}$$

where $y = \pm 1$ *with* $y \equiv m_1 \pmod 3$, $l_3 = l - 1/3 + ym_1/3$, *and* $t_3 = t + a(l - 1/3) - yp_1/3$.

Lastly,

Theorem 2.8. *(See [10, Theorem 3].) Let* α, β, m, p, *and* λ *be as before with* $\alpha m^2 + \beta = p\lambda$, *and let* ε, δ, l, t *be integers with* $\varepsilon(p + t) + \delta(l + m) \equiv 1 \pmod 2$. *Assume that* $y = \pm 1$ *with* $y \equiv m \pmod 3$. *Assume further that* $3 \mid \beta$ *and* $\gcd(3, m\lambda) = 1$. *Recall that* $R1$ *and* $R2$ *are defined by* (2.24) *and* (2.20). *Let* α_1, β_1, m_1, *and* p_1 *be another set parameters as in Theorem* 2.7 *and set* $a := (\alpha m - \alpha_1 m_1)/\lambda$. *Then,*

$$R1(z, \varepsilon, \delta, l, t, \alpha, \beta, m, p) = R1(y, \delta, \varepsilon, l_1, t_1, 1, \alpha\beta, \alpha m, \lambda), \tag{2.30}$$

where $l_1 = t + (zp + \alpha my)/3$, $t_1 = l - (zm + y\lambda)/3$, $z = \pm 1$ *with* $z \equiv -\lambda \pmod 3$. *Moreover, if* $3 \mid \beta_1$ *and* $\gcd(3, \alpha_1 m_1) = 1$, *then*

$$R1(y, \varepsilon, \delta, l, t, \alpha, \beta, m, p) = R1(y_1, \varepsilon, \delta + a\varepsilon, l_2, t_2, \alpha_1, \beta_1, m_1, p_1), \tag{2.31}$$

where $l_2 = l - (ym - y_1 m_1)/3$, $t_2 = t + al + (yp - y_1 p_1 - aym)/3$, *and* $y_1 = \pm 1$ *with* $y_1 \equiv m_1 \pmod 3$. *If* $3 \mid \alpha_1 m_1$, *then*

$$R1(y, \varepsilon, \delta, l, t, \alpha, \beta, m, p) = R2(\varepsilon, \delta + a\varepsilon, l_3, t_3, \alpha_1, \beta_1, m_1, p_1), \tag{2.32}$$

where $l_3 = l + (1 - ym)/3$, $t_3 = t + al + y(p - am)/3$.

3. Proof of Entry 1.1

Let $S(q) := U(q)V(q)$, $Q := q^{17}$, and $T(q) := \chi^2(-q)\chi^2(-Q)$. The proof of (1.5) will follow from a series of identities given below. The last identity, (3.8), is clearly equivalent to (1.5). We have

$$\chi(-Q)U(q) = \frac{\chi(Q)}{\chi(-q^2)} - q^2 \frac{\chi(q)}{\chi(-Q^2)}, \tag{3.1}$$

$$2qV\big(q^2\big) = \chi^2\big(-Q^2\big)\left(\frac{\chi(q)}{\chi(Q)} - \frac{\chi(-q)}{\chi(-Q)}\right), \tag{3.2}$$

$$\chi\big(-Q^2\big)U(q)U(-q) = \frac{\chi(Q^2)}{\chi(-q^4)} + q^4 \frac{\chi(q^2)}{\chi(-Q^4)}, \tag{3.3}$$

$$2U\big(-q^2\big)V\big(q^4\big) = \chi^2\big(-Q^2\big)\left(\frac{\chi(q)}{\chi(Q)} + \frac{\chi(-q)}{\chi(-Q)}\right), \tag{3.4}$$

$$S(q)S(-q) - S\big(q^2\big) = \frac{4q^4}{T(q^2)}, \tag{3.5}$$

$$S(-q)S\big(q^2\big) - qS(q) = T(q), \tag{3.6}$$

$$S^3(q) - 5qS(q) = T(q) + \frac{4q^3}{T(q)}, \tag{3.7}$$

$$S^4(q) - qS^2(q) = \frac{\chi^3(-q^{17})}{\chi^3(-q)}\left(1 + q^2 \frac{\chi^3(-q)}{\chi^3(-q^{17})}\right)^2. \tag{3.8}$$

We start by proving (3.1). By (2.30) with the set of parameters $z = 1$, $\varepsilon = 1$, $\delta = 1$, $l = t = 0$, $\alpha = 17$, $\beta = 3$, $m = 1$ and $p = 4$ $(\lambda = 5)$, we find that

$$R1(1, 1, 1, 0, 0, 17, 3, 1, 4) = R1(1, 1, 1, 7, -2, 1, 51, 17, 5). \tag{3.9}$$

By Lemma 2.5, we also find that

$$R1(1, 1, 1, 0, 0, 17, 3, 1, 4) = q^{1/3} f\big(-q^8\big)\left(\varphi\big(-Q^4\big) - q^4 \frac{\varphi(-q^4)\psi(-Q^2)}{\psi(-q^2)}\right). \tag{3.10}$$

By several applications of (2.3) together with (2.8), we find that

$$\frac{f(-q^2, -q^3)}{f(q, q^4)} = \frac{f(-q)}{f(-q^5)} G\big(q^2\big), \quad \text{and} \quad \frac{f(-q, -q^4)}{f(q^2, q^3)} = \frac{f(-q)}{f(-q^5)} H\big(q^2\big). \tag{3.11}$$

Employing Lemma 2.5 again together with (3.11) with q replaced by Q^2 and by (2.8), we conclude that

$$\begin{aligned} &R1(1,1,1,7,-2,1,51,17,5) \\ &\quad = q^{1/3} f\left(-Q^{10}\right)\left(\frac{f(-q^4,-q^6)f(-Q^4,-Q^6)}{f(Q^2,Q^8)} + q^{14}\frac{f(-q^2,-q^8)f(-Q^2,-Q^8)}{f(Q^4,Q^6)}\right) \\ &\quad = q^{1/3} f\left(-Q^2\right) f\left(-q^2\right)\left(G\left(q^2\right)G\left(Q^4\right) + q^{14}H\left(q^2\right)H\left(Q^4\right)\right) \\ &\quad = q^{1/3} f\left(-q^2\right) f\left(-Q^2\right) V\left(q^2\right). \end{aligned} \tag{3.12}$$

Therefore, by (3.9)–(3.12), after replacing q^2 by q, and by (2.4)–(2.6), we arrive at

$$V(q) = \frac{f(-q^4)}{f(-q)f(-Q)}\left(\varphi\left(-Q^2\right) - q^2\frac{\varphi(-q^2)\psi(-Q)}{\psi(-q)}\right) = \frac{1}{\chi(-q)}\left(\frac{\chi(Q)}{\chi(-q^2)} - q^2\frac{\chi(q)}{\chi(-Q^2)}\right), \tag{3.13}$$

which, by (1.4), is equivalent to (3.1).

Next, we prove (3.2). Recall that

$$\begin{aligned} &G\left(Q^2\right)H\left(q^4\right) - q^6 H\left(Q^2\right)G\left(q^4\right) = U\left(q^2\right), \\ &G\left(Q^2\right)G(q) + q^7 H\left(Q^2\right)H(q) = V(q), \\ &G\left(Q^2\right)G(-q) - q^7 H\left(Q^2\right)H(-q) = V(-q). \end{aligned}$$

Regarding $G(Q^2)$, $q^6H(Q^2)$, and 1 as the "variables," we conclude from this triple of equations that

$$\begin{vmatrix} H(q^4) & -G(q^4) & U(q^2) \\ G(q) & qH(q) & V(q) \\ G(-q) & -qH(-q) & V(-q) \end{vmatrix} = 0. \tag{3.14}$$

Expanding this determinant (3.14) by the last column, using Entries 2.3 and 2.1, we deduce that

$$-2q\frac{U(q^2)}{\chi^2(-q^2)} - V(q)\chi^2(-q) + V(-q)\chi^2(q) = 0. \tag{3.15}$$

We should remark that by (1.4), the identity (3.15), is equivalent to

$$\chi(q)\chi(Q)U(-q) - \chi(-q)\chi(-Q)U(q) = 2q\frac{U(q^2)}{\chi^2(-q^2)}. \tag{3.16}$$

Therefore, by (3.15) and by two applications of (3.13) with q replaced by $-q$ in the first application, we find that

$$\begin{aligned} 2q\frac{U(q^2)}{\chi^2(-q^2)} &= \chi^2(q)\left(\frac{1}{\chi(q)}\left(\frac{\chi(-Q)}{\chi(-q^2)} - q^2\frac{\chi(-q)}{\chi(-Q^2)}\right)\right) - \chi^2(-q)\left(\frac{1}{\chi(-q)}\left(\frac{\chi(Q)}{\chi(-q^2)} - q^2\frac{\chi(q)}{\chi(-Q^2)}\right)\right) \\ &= \frac{\chi(-Q)}{\chi(-q)} - \frac{\chi(Q)}{\chi(q)} \\ &= \frac{\chi(-Q^2)}{\chi(-q^2)}\left(\frac{\chi(q)}{\chi(Q)} - \frac{\chi(-q)}{\chi(-Q)}\right), \end{aligned}$$

which, by (1.4), is equivalent to (3.2). We should remark that the proof of (3.2) similar to the proof of (3.1) can be given.

Next, we prove (3.3). By (2.30) with the set of parameters $z=1$, $\varepsilon=0$, $\delta=1$, $l=t=0$, $\alpha=17$, $\beta=3$, $m=1$ and $p=4$ ($\lambda=5$), we find that

$$R1(1,0,1,0,0,17,3,1,4) = R1(1,1,0,7,-2,1,51,17,5). \tag{3.17}$$

By Lemma 2.5, we find that

$$R1(1,0,1,0,0,17,3,1,4) = q^{1/3} f\left(-q^8\right)\left(\varphi\left(-Q^4\right) + q^4\frac{\varphi(-q^4)\psi(-Q^2)}{\psi(-q^2)}\right). \tag{3.18}$$

By using (1.2), (2.3), and some elementary product manipulations, we can show that

$$G(q)G(-q) = \frac{f(q^4,q^6)}{f(-q^2)} \quad \text{and} \quad H(q)H(-q) = \frac{f(q^2,q^8)}{f(-q^2)}. \tag{3.19}$$

By Lemma 2.5, (2.8), (3.19) and by Entry 2.4, we also find that

$$
\begin{aligned}
&R1(1,1,0,7,-2,1,51,17,5)\\
&= q^{1/3} f\left(-Q^{10}\right)\left(\frac{f(q^4,q^6)f(-Q^4,-Q^6)}{f(-Q^2,-Q^8)} - q^{14}\frac{f(q^2,q^8)f(-Q^2,-Q^8)}{f(-Q^4,-Q^6)} - 2q^8\psi\left(q^{10}\right)\right)\\
&= q^{1/3}\frac{f(-Q^{10})}{f(-Q^2,-Q^8)f(-Q^4,-Q^6)}\left(f\left(q^4,q^6\right)f\left(-Q^4,-Q^6\right)^2 - q^{14}f\left(q^2,q^8\right)f\left(-Q^2,-Q^8\right)^2\right.\\
&\quad \left.- 2q^8 f\left(-Q^2,-Q^8\right)f\left(-Q^4,-Q^6\right)\psi\left(q^{10}\right)\right)\\
&= q^{1/3}\frac{f(-Q^{10})f^2(-Q^2)f(-q^2)}{f(-Q^2,-Q^8)f(-Q^4,-Q^6)}\left(G(q)G(-q)G^2\left(Q^2\right) - q^{14}H(q)H(-q)H^2\left(Q^2\right)\right.\\
&\quad \left.- 2q^8 G\left(Q^2\right)H\left(Q^2\right)\frac{\psi(q^{10})}{f(-q^2)}\right)\\
&= q^{1/3} f\left(-q^2\right)f\left(-Q^2\right)\left(G(q)G(-q)G^2\left(Q^2\right) - q^{14}H(q)H(-q)H^2\left(Q^2\right)\right.\\
&\quad \left.- q^7 G\left(Q^2\right)H\left(Q^2\right)\left(G(q)H(-q) - G(-q)H(q)\right)\right)\\
&= q^{1/3} f\left(-q^2\right)f\left(-Q^2\right)\left(G(q)G\left(Q^2\right) + q^7H(q)H\left(Q^2\right)\right)\left(G(-q)G\left(Q^2\right) - q^7H(-q)H\left(Q^2\right)\right)\\
&= q^{1/3} f\left(-q^2\right)f\left(-Q^2\right)V(q)V(-q).
\end{aligned}
\tag{3.20}
$$

Therefore, by (3.17)–(3.20), we conclude that

$$
V(q)V(-q) = \frac{f(-q^8)}{f(-q^2)f(-Q^2)}\left(\varphi\left(-Q^4\right) + q^4\frac{\varphi(-q^4)\psi(-Q^2)}{\psi(-q^2)}\right). \tag{3.21}
$$

Now (3.3) follows by similar considerations as in (3.13) since the theta functions that appear are essentially the same.

Next, we prove (3.4). Observe by (2.4)–(2.6) and (2.11) that

$$
\chi^2(q) = \frac{\varphi(q)}{f(-q^2)} = \frac{\varphi(q^4) + 2q\psi(q^8)}{f(-q^2)} = \frac{\chi^2(q^4)}{\chi(-q^2)\chi(-q^4)} + 2q\frac{1}{\chi^2(-q^8)\chi(-q^2)\chi(-q^4)}. \tag{3.22}
$$

Therefore,

$$
2\chi^2\left(q^4\right) = \chi\left(-q^2\right)\chi\left(-q^4\right)\left(\chi^2(q) + \chi^2(-q)\right) \quad \text{and} \quad \frac{4q}{\chi^2(-q^8)} = \chi\left(-q^2\right)\chi\left(-q^4\right)\left(\chi^2(q) - \chi^2(-q)\right). \tag{3.23}
$$

By (3.1) with q replaced by q^4 and by (3.3) with q replaced by q^2, and by two applications of (3.23) with q replaced by q and Q, respectively, and by (3.2), we find that

$$
\begin{aligned}
\chi^2\left(-Q^4\right)U\left(-q^2\right)U\left(q^2\right)U\left(q^4\right) &= \frac{\chi^2(Q^4)}{\chi^2(-q^8)} - q^{16}\frac{\chi^2(q^4)}{\chi^2(-Q^8)}\\
&= \frac{\chi(-q^2)\chi(-q^4)\chi(-Q^2)\chi(-Q^4)}{8q}\left(\left(\chi^2(Q) + \chi^2(-Q)\right)\left(\chi^2(q) - \chi^2(-q)\right)\right.\\
&\quad \left.- \left(\chi^2(Q) - \chi^2(-Q)\right)\left(\chi^2(q) + \chi^2(-q)\right)\right)\\
&= \frac{\chi(-q^2)\chi(-q^4)\chi(-Q^2)\chi(-Q^4)}{4q}\left(\chi^2(q)\chi^2(-Q) - \chi^2(-q)\chi^2(Q)\right)\\
&= \frac{\chi(-q^2)\chi(-q^4)\chi^3(-Q^2)\chi(-Q^4)}{4q}\left(\frac{\chi(q)}{\chi(Q)} - \frac{\chi(-q)}{\chi(-Q)}\right)\left(\frac{\chi(q)}{\chi(Q)} + \frac{\chi(-q)}{\chi(-Q)}\right)\\
&= \frac{\chi(-q^2)\chi(-q^4)\chi(-Q^2)\chi(-Q^4)}{2}V\left(q^2\right)\left(\frac{\chi(q)}{\chi(Q)} + \frac{\chi(-q)}{\chi(-Q)}\right).
\end{aligned}
\tag{3.24}
$$

By two applications of (1.4), we observe that

$$
\chi^2\left(-Q^4\right)U\left(-q^2\right)U\left(q^2\right)U\left(q^4\right) = \chi^2\left(-Q^4\right)U\left(-q^2\right)V\left(q^2\right)\frac{\chi(-q^2)}{\chi(-Q^2)}V\left(q^4\right)\frac{\chi(-q^4)}{\chi(-Q^4)}. \tag{3.25}
$$

Now, we use (3.25) in the leftmost side of (3.24) and complete the proof of (3.4).

Next, we prove (3.5). By (3.3) and by (3.1) with q replaced by q^2, we find that

$$
\chi^2\left(-Q^2\right)U^2(q)U^2(-q) - \chi^2\left(-Q^2\right)U^2\left(q^2\right) = 4q^4\frac{1}{\chi(-q^2)\chi(-Q^2)}. \tag{3.26}
$$

From (3.26), by using (1.4), we obtain

$$\chi^2(-Q^2)S(q)S(-q)\frac{\chi(-q)}{\chi(-Q)}\frac{\chi(q)}{\chi(Q)}-\chi^2(-Q^2)S(q^2)\frac{\chi(-q^2)}{\chi(-Q^2)}=4q^4\frac{1}{\chi(-q^2)\chi(-Q^2)},$$

from which (3.5) readily follows.

Next, we prove (3.6). By adding (3.2) and (3.4), we find that

$$\chi^2(-Q^2)\frac{\chi(q)}{\chi(Q)}=U(-q^2)V(q^4)+qV(q^2). \tag{3.27}$$

In (3.27), we replace q by $-q$ and multiply the resulting identity with (3.27), we obtain that

$$\chi^4(-Q^2)\frac{\chi(q)}{\chi(Q)}\frac{\chi(-q)}{\chi(-Q)}=U^2(-q^2)V^2(q^4)-q^2V^2(q^2). \tag{3.28}$$

Now in (3.28) by replacing q^2 by q and employing (1.4) several times, we arrive at (3.6).

Now we prove (3.7). In (3.6), we replace q by $-q$ and multiply the resulting identity with (3.6), we find that

$$S(q)S(-q)\big(S^2(q^2)-q^2\big)-qS(q^2)\big(S^2(q)-S^2(-q)\big)=T(q)T(-q)=T(q^2). \tag{3.29}$$

By (3.5), and by (3.6) with q replaced by q^2, we also find that

$$\begin{aligned}S(q)S(-q)S(-q^2)S(q^4)&=\left(S(q^2)+\frac{4q^4}{T(q^2)}\right)\big(q^2S(q^2)+T(q^2)\big)\\&=q^2S^2(q^2)+S(q^2)\left(T(q^2)+\frac{4q^6}{T(q^2)}\right)+4q^4.\end{aligned} \tag{3.30}$$

Starting with the relations

$$\begin{aligned}&G(q^2)G(Q^4)+q^{14}H(q^2)H(Q^4)=V(q^2),\\&-q^3G(q^2)H(Q)+H(q^2)G(Q)=U(q),\\&q^3G(q^2)H(-Q)+H(q^2)G(-Q)=U(-q),\end{aligned}$$

and by arguing as in (3.14)–(3.16), we similarly find that

$$\chi(q)\chi(Q)V(-q)-\chi(-q)\chi(-Q)V(q)=2q^3\frac{V(q^2)}{\chi^2(-Q^2)}. \tag{3.31}$$

Next, we multiply (3.16) and (3.31) together, we find that

$$T(-q)S(-q)+T(q)S(q)-\chi(-q^2)\chi(-Q^2)\big(U(q)V(-q)+U(-q)V(q)\big)=4q^4\frac{S(q^2)}{T(q^2)}. \tag{3.32}$$

By (1.4) and by (3.4) we observe that

$$U(q)V(-q)+U(-q)V(q)=V(q)V(-q)\left(\frac{U(q)}{V(q)}+\frac{U(-q)}{V(-q)}\right)=2V(q)V(-q)\frac{U(-q^2)V(q^4)}{\chi^2(-Q^2)}. \tag{3.33}$$

In (3.32), we use (3.33) and the value of $T(q)$ (and $T(-q)$) given by (3.6), we arrive at

$$2S(q)S(-q)S(q^2)-q\big(S^2(q)-S^2(-q)\big)-2\frac{\chi(-q^2)}{\chi(-Q^2)}V(q)V(-q)U(-q^2)V(q^4)=4q^4\frac{S(q^2)}{T(q^2)}. \tag{3.34}$$

Observe that

$$\frac{\chi(-q^2)}{\chi(-Q^2)}V(q)V(-q)U(-q^2)V(q^4)=\sqrt{S(q)S(-q)S(-q^2)S(q^4)}. \tag{3.35}$$

Therefore (3.34) can be written as

$$2S(q)S(-q)S(q^2)-q\big(S^2(q)-S^2(-q)\big)-2\sqrt{S(q)S(-q)S(-q^2)S(q^4)}=4q^4\frac{S(q^2)}{T(q^2)}. \tag{3.36}$$

Now we multiply both sides of (3.36) by $S(q^2)$ and substitute the value of $S(q^2)(S^2(q)-S^2(-q))$ from (3.29), we deduce that

$$S(q)S(-q)S^2\left(q^2\right)+T\left(q^2\right)+q^2S(q)S(-q)-2S\left(q^2\right)\sqrt{S(q)S(-q)S\left(-q^2\right)S\left(q^4\right)}=4q^4\frac{S^2(q^2)}{T(q^2)}. \tag{3.37}$$

Now in (3.37), we substitute the value of $S(q)S(-q)$ from (3.5) to find that

$$S^3\left(q^2\right)+T\left(q^2\right)+\frac{4q^6}{T(q^2)}+q^2S\left(q^2\right)=2S\left(q^2\right)\sqrt{S(q)S(-q)S\left(-q^2\right)S\left(q^4\right)}. \tag{3.38}$$

Next, we return to (3.30) and use (3.38), we find that

$$S^4\left(q^2\right)-2S^2\left(q^2\right)\sqrt{S(q)S(-q)S\left(-q^2\right)S\left(q^4\right)}+S(q)S(-q)S\left(-q^2\right)S\left(q^4\right)=4q^4. \tag{3.39}$$

From (3.39), we conclude that

$$\sqrt{S(q)S(-q)S\left(-q^2\right)S\left(q^4\right)}=S^2\left(q^2\right)-2q^2. \tag{3.40}$$

Returning and using this in (3.30), we arrive at

$$S^3\left(q^2\right)-5q^2S\left(q^2\right)=T\left(q^2\right)+\frac{4q^6}{T(q^2)}, \tag{3.41}$$

which is (3.7) with q replaced by q^2.

Lastly, we prove (3.8). By (3.1), by (3.1) with q replaced by $-q$, and by (3.4), we find that

$$\begin{aligned}
\chi\left(-Q^2\right)U(q)U(-q)&=\left(\frac{\chi(Q)}{\chi(-q^2)}-q^2\frac{\chi(q)}{\chi(-Q^2)}\right)\left(\frac{\chi(-Q)}{\chi(-q^2)}-q^2\frac{\chi(-q)}{\chi(-Q^2)}\right)\\
&=\frac{\chi(-Q^2)}{\chi^2(-q^2)}-q^2\left(\frac{1}{\chi(q)\chi(-Q)}+\frac{1}{\chi(-q)\chi(Q)}\right)+q^4\frac{\chi(-q^2)}{\chi^2(-Q^2)}\\
&=\frac{\chi(-Q^2)}{\chi^2(-q^2)}\left(1+q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right)-q^2\frac{1}{\chi(-q^2)}\left(\frac{\chi(q)}{\chi(Q)}+\frac{\chi(-q)}{\chi(-Q)}\right)\\
&=\frac{\chi(-Q^2)}{\chi^2(-q^2)}\left(1+q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right)-2q^2\frac{1}{\chi(-q^2)\chi^2(-Q^2)}U\left(-q^2\right)V\left(q^4\right).
\end{aligned} \tag{3.42}$$

From (3.42), we have that

$$\begin{aligned}
\frac{\chi^2(-Q^2)}{\chi^4(-q^2)}\left(1+q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right)^2&=\left(\chi\left(-Q^2\right)U(q)U(-q)+2q^2\frac{1}{\chi(-q^2)\chi^2(-Q^2)}U\left(-q^2\right)V\left(q^4\right)\right)^2\\
&=\chi^2\left(-Q^2\right)U^2(q)U^2(-q)+4q^2\frac{1}{\chi(-q^2)\chi(-Q^2)}U(-q)U(q)U\left(-q^2\right)V\left(q^4\right)\\
&\quad+4q^4\frac{1}{\chi^2(-q^2)\chi^4(-Q^2)}U^2\left(-q^2\right)V^2\left(q^4\right).
\end{aligned} \tag{3.43}$$

We multiply both sides of (3.43) by $\chi(-q^2)\chi(-Q^2)$ and after several applications of (1.4), we arrive at

$$\frac{\chi^3(-Q^2)}{\chi^3(-q^2)}\left(1+q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right)^2=S(q)S(-q)T\left(q^2\right)+4q^2\sqrt{S(q)S(-q)S\left(-q^2\right)S\left(q^4\right)}+4q^4\frac{S(-q^2)S(q^4)}{T(q^2)}. \tag{3.44}$$

Next, we employ (3.5), (3.40), (3.6) and (3.7) both with q replaced by q^2, we find that

$$\begin{aligned}
\frac{\chi^3(-Q^2)}{\chi^3(-q^2)}\left(1+q^4\frac{\chi^3(-q^2)}{\chi^3(-Q^2)}\right)^2&=\left(S\left(q^2\right)+4q^4\frac{1}{T(q^2)}\right)T\left(q^2\right)+4q^2\left(S^2\left(q^2\right)-2q^2\right)+4q^4\left(\frac{q^2S(q^2)+T(q^2)}{T(q^2)}\right)\\
&=S\left(q^2\right)\left(T\left(q^2\right)+4q^6\frac{1}{T(q^2)}\right)+4q^2S^2\left(q^2\right)\\
&=S\left(q^2\right)\left(S^3\left(q^2\right)-5q^2S\left(q^2\right)\right)+4q^2S^2\left(q^2\right)\\
&=S^4\left(q^2\right)-q^2S^2\left(q^2\right),
\end{aligned}$$

which is (3.8) with q replaced by q^2.

4. Proof of Entry 1.2

Proof. Let $Q := q^{23}$, $A(q) := G(q^2)G(Q) + q^5 H(q^2)H(Q)$, and $B(q) := H(q)G(Q^2) - q^9 G(q)H(Q^2)$. From (2.28), we find that

$$R2(1,0,0,1,3,23,2,5) = R2(1,-9,0,4,69,1,1,10). \tag{4.1}$$

We employ Lemma 2.6 and argue similarly as in (3.12), we obtain that

$$\begin{aligned} &R2(1,0,0,1,3,23,2,5)\\ &\quad= q^{7/6} f\bigl(-q^{10}\bigr)\left(\frac{f(-q^4,-q^6)f(-Q^4,-Q^6)}{f(q^2,q^8)} + q^{10}\frac{f(-q^2,-q^8)f(-Q^2,-Q^8)}{f(q^4,q^6)}\right)\\ &\quad= q^{7/6} f\bigl(-q^2\bigr) f\bigl(-Q^2\bigr) A\bigl(q^2\bigr). \end{aligned} \tag{4.2}$$

Next, we employ Lemma 2.6 to find that

$$\begin{aligned} &R2(1,-9,0,4,69,1,1,10)\\ &\quad= q^{7/6} f\bigl(-Q^{20}\bigr)\left\{-q^2\frac{f(-Q^4,-Q^{16})f(-q^8,-q^{12})}{f(-Q^2,-Q^{18})} - q^{12}\frac{f(-Q^8,-Q^{12})f(-q^4,-q^{16})}{f(-Q^6,-Q^{14})}\right.\\ &\qquad\left. + \frac{f(-Q^8,-Q^{12})f(-q^6,-q^{14})}{f(-Q^4,-Q^{16})} + q^{38}\frac{f(-Q^4,-Q^{16})f(-q^2,-q^{18})}{f(-Q^8,-Q^{12})} + q^{18}\varphi\bigl(-q^{10}\bigr)\right\}. \end{aligned} \tag{4.3}$$

By using (1.2), (2.3), and some elementary product manipulations, we can show that

$$\frac{f(-q^2,-q^8)}{f(-q,-q^9)} = \frac{f(-q^2)}{f(-q^{10})}G(q), \qquad \frac{f(-q^4,-q^6)}{f(-q^3,-q^7)} = \frac{f(-q^2)}{f(-q^{10})}H(q). \tag{4.4}$$

By (4.4) with q replaced by Q^2 and by (1.2), we have

$$\begin{aligned} &-q^2\frac{f(-Q^4,-Q^{16})f(-q^8,-q^{12})}{f(-Q^2,-Q^{18})} - q^{12}\frac{f(-Q^8,-Q^{12})f(-q^4,-q^{16})}{f(-Q^6,-Q^{14})}\\ &\quad= -q^2\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\bigl(G\bigl(q^4\bigr)G\bigl(Q^2\bigr) + q^{10}H\bigl(q^4\bigr)H\bigl(Q^2\bigr)\bigr)\\ &\quad= -q^2\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}A\bigl(q^2\bigr). \end{aligned} \tag{4.5}$$

By (2.3), it is easy to verify that

$$f\bigl(-q^3,-q^7\bigr) = f\bigl(-q^2\bigr)H(-q)G\bigl(q^4\bigr) \quad\text{and}\quad f\bigl(-q,-q^9\bigr) = f\bigl(-q^2\bigr)G(-q)H\bigl(q^4\bigr). \tag{4.6}$$

Also from (2.8), we find

$$\frac{f(-q,-q^4)}{f(-q^2,-q^3)} = \frac{f(-q)}{f(-q^5)}H^2(q) \quad\text{and}\quad \frac{f(-q^2,-q^3)}{f(-q,-q^4)} = \frac{f(-q)}{f(-q^5)}G^2(q). \tag{4.7}$$

Using (4.6) with q replaced by q^2 and (4.7) with q replaced by Q^4, we find that

$$\begin{aligned} &\frac{f(-Q^8,-Q^{12})f(-q^6,-q^{14})}{f(-Q^4,-Q^{16})} + q^{38}\frac{f(-Q^4,-Q^{16})f(-q^2,-q^{18})}{f(-Q^8,-Q^{12})} + q^{18}\varphi\bigl(-q^{10}\bigr)\\ &\quad= \frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\left(G^2\bigl(Q^4\bigr)H\bigl(-q^2\bigr)G\bigl(q^8\bigr) + q^{38}H^2\bigl(Q^4\bigr)G\bigl(-q^2\bigr)H\bigl(q^8\bigr) + q^{18}\frac{\varphi(-q^{10})}{f(-q^4)}\frac{f(-Q^{20})}{f(-Q^4)}\right). \end{aligned} \tag{4.8}$$

Next, by Entry 2.2 with q replaced by $-q^2$ and by (2.9) with q replaced by Q^4, we deduce that

$$\frac{\varphi(-q^{10})}{f(-q^4)}\frac{f(-Q^{20})}{f(-Q^4)} = \bigl(G\bigl(-q^2\bigr)G\bigl(q^8\bigr) + q^2H\bigl(-q^2\bigr)H\bigl(q^8\bigr)\bigr)G\bigl(Q^4\bigr)H\bigl(Q^4\bigr). \tag{4.9}$$

Returning and using (4.9) in (4.8), we arrive at

$$\begin{aligned}&\frac{f(-Q^8,-Q^{12})f(-q^6,-q^{14})}{f(-Q^4,-Q^{16})}+q^{38}\frac{f(-Q^4,-Q^{16})f(-q^2,-q^{18})}{f(-Q^8,-Q^{12})}+q^{18}\varphi(-q^{10})\\&=\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}\big(G(q^8)G(Q^4)+q^{20}H(q^8)H(Q^4)\big)\big(H(-q^2)G(Q^4)+q^{18}G(-q^2)H(Q^4)\big)\\&=\frac{f(-q^4)f(-Q^4)}{f(-Q^{20})}A(q^4)B(-q^2).\end{aligned}\tag{4.10}$$

By (4.3), (4.5), and by (4.10), we conclude that

$$R2(1,-9,0,4,69,1,1,10)=q^{7/6}f(-q^4)f(-Q^4)\big(-q^2A(q^2)+A(q^4)B(-q^2)\big).\tag{4.11}$$

Therefore, by (4.1), (4.2) and by (4.11), we arrive at

$$f(-q^2)f(-Q^2)A(q^2)=f(-q^4)f(-Q^4)\big(-q^2A(q^2)+A(q^4)B(-q^2)\big).\tag{4.12}$$

Lastly, by replacing q^2 by q, we conclude that

$$\big(\chi(-q)\chi(-Q)+q\big)A(q)=B(-q)A(q^2).\tag{4.13}$$

Next, we prove the companion formula

$$\big(\chi(-q)\chi(-Q)+q\big)B(q)=A(-q)B(q^2).\tag{4.14}$$

By (2.28), we find that

$$R2(1,1,1,1,3,23,2,5)=R2(1,-8,1,-5,69,1,1,10)\tag{4.15}$$

and

$$R2(1,1,0,0,3,23,2,5)=R2(1,-8,0,3,69,1,1,10).\tag{4.16}$$

By Lemma 2.6, we find that

$$\begin{aligned}&R2(1,-8,1,-5,69,1,1,10)\\&=q^{5/12}f(-Q^{20})\bigg\{q^{11}\frac{f(-Q^2,-Q^{18})f(q,q^{19})}{f(Q,Q^{19})}-q\frac{f(-Q^6,-Q^{14})f(q^3,q^{17})}{f(Q^3,Q^{17})}+q^5\frac{\varphi(-Q^{10})\psi(q^5)}{\psi(Q^5)}\\&\quad-q^{23}\frac{f(-Q^6,-Q^{14})f(q^7,q^{13})}{f(Q^7,Q^{13})}+q^{55}\frac{f(-Q^2,-Q^{18})f(q^9,q^{11})}{f(Q^9,Q^{11})}\bigg\}.\end{aligned}\tag{4.17}$$

By employing Lemma 2.6, we also obtain that

$$\begin{aligned}&R2(1,-8,0,3,69,1,1,10)\\&=q^{5/12}f(-Q^{20})\bigg\{-q^9\frac{f(-Q^2,-Q^{18})f(q^9,q^{11})}{f(Q,Q^{19})}+\frac{f(-Q^6,-Q^{14})f(q^7,q^{13})}{f(Q^3,Q^{17})}-q^5\frac{\varphi(-Q^{10})\psi(q^5)}{\psi(Q^5)}\\&\quad+q^{24}\frac{f(-Q^6,-Q^{14})f(q^3,q^{17})}{f(Q^7,Q^{13})}-q^{57}\frac{f(-Q^2,-Q^{18})f(q,q^{19})}{f(Q^9,Q^{11})}\bigg\}.\end{aligned}\tag{4.18}$$

By employing (2.10) twice with $a=-q^2$, $b=-q^3$, $n=2$ and $a=-q$, $b=-q^4$, $n=2$ we easily find that

$$f(-q^2,-q^3)=f(q^9,q^{11})-q^2f(q,q^{19})\quad\text{and}\quad f(-q,-q^4)=f(q^7,q^{13})-qf(q^3,q^{17}).\tag{4.19}$$

Next, we add (4.17) with (4.18) then collect terms and use (4.19) to find that

$$\begin{aligned}&R2(1,-8,1,-5,69,1,1,10)+R2(1,-8,0,3,69,1,1,10)\\&=q^{5/12}f(-Q^{20})\bigg\{-q^9\frac{f(-Q^2,-Q^{18})f(-q^2,-q^3)}{f(Q,Q^{19})}+\frac{f(-Q^6,-Q^{14})f(-q,-q^4)}{f(Q^3,Q^{17})}\\&\quad-q^{23}\frac{f(-Q^6,-Q^{14})f(-q,-q^4)}{f(Q^7,Q^{13})}+q^{55}\frac{f(-Q^2,-Q^{18})f(-q^2,-q^3)}{f(Q^9,Q^{11})}\bigg\}.\end{aligned}\tag{4.20}$$

By using (4.19) again, this time with q replaced by Q, we find from (4.20) that

$$\begin{aligned}&R2(1,-8,1,-5,69,1,1,10)+R2(1,-8,0,3,69,1,1,10)\\&\quad=q^{5/12}f\left(-Q^{20}\right)\left\{-q^9\frac{f(-Q^2,-Q^{18})f(-q^2,-q^3)f(-Q^2,-Q^3)}{f(Q,Q^{19})f(Q^9,Q^{11})}\right.\\&\qquad\left.+\frac{f(-Q^6,-Q^{14})f(-q,-q^4)f(-Q,-Q^4)}{f(Q^3,Q^{17})f(Q^7,Q^{13})}\right\}.\end{aligned}\tag{4.21}$$

By several applications of (2.3), we can verify that

$$\frac{f(-q^2,-q^{18})f(-q^2,-q^3)}{f(q,q^{19})f(q^9,q^{11})}=\frac{f(-q)H(q^2)}{f(-q^{20})}\quad\text{and}\quad\frac{f(-q^6,-q^{14})f(-q,-q^4)}{f(q^3,q^{17})f(q^7,q^{13})}=\frac{f(-q)G(q^2)}{f(-q^{20})}.\tag{4.22}$$

By returning to (4.21) and using (4.22) with q replaced by Q, we conclude that

$$\begin{aligned}&R2(1,-8,1,-5,69,1,1,10)+R2(1,-8,0,3,69,1,1,10)\\&\quad=q^{5/12}f(-q)f(-Q)\left(H(q)G\left(Q^2\right)-q^9G(q)H\left(Q^2\right)\right)\\&\quad=q^{5/12}f(-q)f(-Q)B(q).\end{aligned}\tag{4.23}$$

By Lemma 2.6 and by (4.4), we find that

$$\begin{aligned}&R2(1,1,1,1,3,23,2,5)\\&\quad=-q^{17/12}f\left(-q^{10}\right)\left(\frac{f(-q^4,-q^6)f(-Q^4,-Q^6)}{f(-q^3,-q^7)}-q^9\frac{f(-q^2,-q^8)f(-Q^2,-Q^8)}{f(-q,-q^9)}\right)\\&\quad=-q^{17/12}f\left(-q^2\right)f\left(-Q^2\right)\left(H(q)G\left(Q^2\right)-q^9G(q)H\left(Q^2\right)\right)\\&\quad=-q^{17/12}f\left(-q^2\right)f\left(-Q^2\right)B(q).\end{aligned}\tag{4.24}$$

We employ Lemma 2.6 again and argue similarly as in (4.8)–(4.10) to deduce that

$$\begin{aligned}&R2(1,1,0,0,3,23,2,5)\\&\quad=q^{5/12}f\left(-q^{10}\right)\left(-q^5\frac{f(-q^2,-q^8)f(-Q^3,-Q^7)}{f(-q^4,-q^6)}-q^{18}\frac{f(-q^4,-q^6)f(-Q,-Q^9)}{f(-q^2,-q^8)}+\varphi\left(-Q^5\right)\right)\\&\quad=q^{5/12}f\left(-q^2\right)f\left(-Q^2\right)\left(H\left(q^2\right)G\left(Q^4\right)-q^{18}G\left(q^2\right)H\left(Q^4\right)\right)\left(G\left(q^2\right)G(-Q)-q^5H\left(q^2\right)H(-Q)\right)\\&\quad=q^{5/12}f\left(-q^2\right)f\left(-Q^2\right)B\left(q^2\right)A(-q).\end{aligned}\tag{4.25}$$

Therefore, by (4.15), (4.16), (4.23), (4.24), and by (4.25), we arrive at

$$-qf\left(-q^2\right)f\left(-Q^2\right)B(q)+f\left(-q^2\right)f\left(-Q^2\right)B\left(q^2\right)A(-q)=f(-q)f(-Q)B(q),\tag{4.26}$$

which is clearly equivalent to (4.14). Now in (4.14), we replace q by $-q$ and multiply the resulting identity with (4.13), we conclude that

$$A\left(q^2\right)B\left(q^2\right)=\left(\chi(q)\chi(Q)-q\right)\left(\chi(-q)\chi(-Q)+q\right).\tag{4.27}$$

Therefore, by (4.27) and by (1.6), it remains to prove that

$$\chi\left(-q^2\right)\chi\left(-Q^2\right)+q^2+\frac{2q^4}{\chi(-q^2)\chi(-Q^2)}=\left(\chi(q)\chi(Q)-q\right)\left(\chi(-q)\chi(-Q)+q\right).\tag{4.28}$$

It is straightforward to verify that (4.28) is equivalent to a modular equation of degree 23 of Schröter. This modular equation is also stated by Ramanujan and an elementary proof can be found in [1, p. 551, Entry 15(i)]. □

5. Proof of Entry 1.3

Proof. Let $Q:=q^{47}$, $A(q):=H(q^2)G(Q)-q^9G(q^2)H(Q)$, and $B(q):=G(q)G(Q^2)+q^{19}H(q)H(Q^2)$. The proof of Entry 1.3 is very similar to that of Entry 1.2 and so we skip most of the details.

From (2.27), we find that

$$R2(1,0,1,1,3,47,1,5)=R1(-1,0,1,0,4,1,141,3,10).\tag{5.1}$$

We employ Lemma 2.6 and argue similarly as in (3.12), we obtain that

$$\begin{aligned}
&R2(1,0,1,1,3,47,1,5)\\
&=q^{19/6}f\left(-q^{10}\right)\left(-\frac{f(-q^2,-q^8)f(-Q^4,-Q^6)}{f(q^4,q^6)}+q^{18}\frac{f(-q^4,-q^6)f(-Q^2,-Q^8)}{f(q^2,q^8)}\right)\\
&=-q^{19/6}f\left(-q^2\right)f\left(-Q^2\right)A\left(q^2\right). \qquad (5.2)
\end{aligned}$$

Next, we employ Lemma 2.5 to find that

$$\begin{aligned}
&R1(-1,0,1,0,4,1,141,3,10)\\
&=q^{19/6}f\left(-Q^{20}\right)\left\{q^4\frac{f(-Q^4,-Q^{16})f(-q^4,-q^{16})}{f(-Q^2,-Q^{18})}-q^{22}\frac{f(-Q^8,-Q^{12})f(-q^8,-q^{12})}{f(-Q^6,-Q^{14})}\right.\\
&\quad\left.-\frac{f(-Q^8,-Q^{12})f(-q^2,-q^{18})}{f(-Q^4,-Q^{16})}-q^{74}\frac{f(-Q^4,-Q^{16})f(-q^6,-q^{14})}{f(-Q^8,-Q^{12})}+q^{36}\varphi\left(-q^{10}\right)\right\}. \qquad (5.3)
\end{aligned}$$

By arguing as in (4.3)–(4.11), we conclude that

$$R1(-1,0,1,0,4,1,141,3,10)=q^{19/6}f\left(-q^4\right)f\left(-Q^4\right)\left(q^4A\left(q^2\right)-A\left(q^4\right)B\left(-q^2\right)\right). \qquad (5.4)$$

Therefore, by (5.1), (5.2) and by (5.4), we arrive at

$$-f\left(-q^2\right)f\left(-Q^2\right)A\left(q^2\right)=f\left(-q^4\right)f\left(-Q^4\right)\left(q^4A\left(q^2\right)-A\left(q^4\right)B\left(-q^2\right)\right). \qquad (5.5)$$

Lastly, by replacing q^2 by q, we conclude that

$$\left(\chi(-q)\chi(-Q)+q^2\right)A(q)=B(-q)A\left(q^2\right). \qquad (5.6)$$

Next, we prove the companion formula

$$\left(\chi(-q)\chi(-Q)+q^2\right)B(q)=A(-q)B\left(q^2\right). \qquad (5.7)$$

By (2.27), we find that

$$R2(0,1,0,1,3,47,1,5)=R1(-1,1,0,0,3,1,141,3,10) \qquad (5.8)$$

and

$$R2(0,1,0,0,3,47,1,5)=R1(-1,1,0,-1,3,1,141,3,10). \qquad (5.9)$$

By Lemma 2.5, we find that

$$\begin{aligned}
&R1(-1,1,0,0,3,1,141,3,10)\\
&=q^{5/12}f\left(-Q^{20}\right)\left\{-q^{19}\frac{f(-Q^2,-Q^{18})f(q^7,q^{13})}{f(Q,Q^{19})}+q^2\frac{f(-Q^6,-Q^{14})f(q,q^{19})}{f(Q^3,Q^{17})}-q^{10}\frac{\varphi(-Q^{10})\psi(q^5)}{\psi(Q^5)}\right.\\
&\quad\left.+q^{47}\frac{f(-Q^6,-Q^{14})f(q^9,q^{11})}{f(Q^7,Q^{13})}-q^{114}\frac{f(-Q^2,-Q^{18})f(q^3,q^{17})}{f(Q^9,Q^{11})}\right\}. \qquad (5.10)
\end{aligned}$$

By employing Lemma 2.5, we also obtain that

$$\begin{aligned}
&R1(-1,1,0,-1,3,1,141,3,10)\\
&=q^{5/12}f\left(-Q^{20}\right)\left\{-q^{20}\frac{f(-Q^2,-Q^{18})f(q^3,q^{17})}{f(Q,Q^{19})}+\frac{f(-Q^6,-Q^{14})f(q^9,q^{11})}{f(Q^3,Q^{17})}-q^{10}\frac{\varphi(-Q^{10})\psi(q^5)}{\psi(Q^5)}\right.\\
&\quad\left.+q^{49}\frac{f(-Q^6,-Q^{14})f(q,q^{19})}{f(Q^7,Q^{13})}-q^{113}\frac{f(-Q^2,-Q^{18})f(q^7,q^{13})}{f(Q^9,Q^{11})}\right\}. \qquad (5.11)
\end{aligned}$$

By arguing as in (4.19)–(4.23), we conclude that

$$\begin{aligned}
&R1(-1,1,0,-1,3,1,141,3,10)-R1(-1,1,0,0,3,1,141,3,10)\\
&=q^{5/12}f(-q)f(-Q)\left(G(q)G\left(Q^2\right)+q^{19}H(q)H\left(Q^2\right)\right)\\
&=q^{5/12}f(-q)f(-Q)B(q). \qquad (5.12)
\end{aligned}$$

By Lemma 2.6 and by (4.4), we find that

$$\begin{aligned}
&R2(0,1,0,1,3,47,1,5)\\
&=q^{29/12}f\left(-q^{10}\right)\left(\frac{f(-q^2,-q^8)f(-Q^4,-Q^6)}{f(-q,-q^9)}+q^{19}\frac{f(-q^4,-q^6)f(-Q^2,-Q^8)}{f(-q^3,-q^7)}\right)\\
&=q^{29/12}f\left(-q^2\right)f\left(-Q^2\right)\left(G(q)G\left(Q^2\right)+q^{19}H(q)H\left(Q^2\right)\right)\\
&=q^{29/12}f\left(-q^2\right)f\left(-Q^2\right)B(q).
\end{aligned}\tag{5.13}$$

We employ Lemma 2.6 again and argue similarly as in (4.8)–(4.10) to deduce that

$$\begin{aligned}
&R2(0,1,0,0,3,47,1,5)\\
&=q^{5/12}f\left(-q^{10}\right)\left(q^9\frac{f(-q^4,-q^6)f(-Q^3,-Q^7)}{f(-q^2,-q^8)}+q^{38}\frac{f(-q^2,-q^8)f(-Q,-Q^9)}{f(-q^4,-q^6)}+\varphi\left(-Q^5\right)\right)\\
&=q^{5/12}f\left(-q^2\right)f\left(-Q^2\right)\left(G\left(q^2\right)G\left(Q^4\right)+q^{38}H\left(q^2\right)H\left(Q^4\right)\right)\left(H\left(q^2\right)G(-Q)+q^9G\left(q^2\right)H(-Q)\right)\\
&=q^{5/12}f\left(-q^2\right)f\left(-Q^2\right)B\left(q^2\right)A(-q).
\end{aligned}\tag{5.14}$$

Therefore, by (5.8), (5.9), (5.12), (5.13), and by (5.14), we arrive at

$$-q^2f\left(-q^2\right)f\left(-Q^2\right)B(q)+f\left(-q^2\right)f\left(-Q^2\right)B\left(q^2\right)A(-q)=f(-q)f(-Q)B(q),\tag{5.15}$$

which is clearly equivalent to (5.7). Now in (5.7), we replace q by $-q$ and multiply the resulting identity with (5.6), we conclude that

$$A\left(q^2\right)B\left(q^2\right)=\left(q^2+\chi(q)\chi(Q)\right)\left(q^2+\chi(-q)\chi(-Q)\right).\tag{5.16}$$

By (1.7), and (5.16), it remains to prove that

$$\begin{aligned}
&\left(q+\chi\left(q^{1/2}\right)\chi\left(Q^{1/2}\right)\right)\left(q+\chi\left(-q^{1/2}\right)\chi\left(-Q^{1/2}\right)\right)\\
&=\chi(-q)\chi\left(-q^{47}\right)+2q^2+\frac{2q^4}{\chi(-q)\chi(-q^{47})}+q\sqrt{4\chi(-q)\chi\left(-q^{47}\right)+9q^2+\frac{8q^4}{\chi(-q)\chi(-q^{47})}}.
\end{aligned}\tag{5.17}$$

To prove (5.17), we employ Ramanujan's modular equation of degree 47 [1, pp. 446–447], namely,

$$\begin{aligned}
&\frac{1}{2}\left\{\varphi\left(q^{1/2}\right)\varphi\left(Q^{1/2}\right)+\varphi\left(-q^{1/2}\right)\varphi\left(-Q^{1/2}\right)\right\}-\frac{1}{2}\left\{\varphi(q)\varphi(Q)+\varphi(-q)\varphi(-Q)\right\}-2q^{12}\psi\left(q^2\right)\psi\left(Q^2\right)\\
&=q^2f(q)f(Q)+q^2f(-q)f(-Q)+2q^8f\left(-q^4\right)f\left(-Q^4\right).
\end{aligned}\tag{5.18}$$

We also make use of the well-known identity [1, p. 40, Entry 25(v), (vi)]

$$\varphi^2(q)=\varphi^2\left(q^2\right)+4q\psi^2\left(q^4\right).\tag{5.19}$$

Employing (5.19) twice with q replaced by q and Q, we find that

$$\begin{aligned}
&\left\{\varphi(q)\varphi(Q)+\varphi(-q)\varphi(-Q)\right\}^2\\
&=\varphi^2(q)\varphi^2(Q)+\varphi^2(-q)\varphi^2(-Q)+2\varphi^2\left(-q^2\right)\varphi^2(-Q)\\
&=\left(\varphi^2\left(q^2\right)+4q\psi^2\left(q^4\right)\right)\left(\varphi^2\left(Q^2\right)+4Q\psi^2\left(Q^4\right)\right)+\left(\varphi^2\left(q^2\right)-4q\psi^2\left(q^4\right)\right)\left(\varphi^2\left(Q^2\right)-4Q\psi^2\left(Q^4\right)\right)\\
&\quad+2\varphi^2\left(-q^2\right)\varphi^2\left(-Q^2\right)\\
&=2\left(\varphi^2\left(q^2\right)\varphi^2\left(Q^2\right)+16q^{48}\psi^2\left(q^4\right)\psi^2\left(Q^4\right)+\varphi^2\left(-q^2\right)\varphi^2\left(-Q^2\right)\right).
\end{aligned}$$

Now by replacing q by $q^{1/2}$ in the equation we have just obtained, we find that

$$\begin{aligned}
&\left\{\varphi\left(q^{1/2}\right)\varphi\left(Q^{1/2}\right)+\varphi\left(-q^{1/2}\right)\varphi\left(-Q^{1/2}\right)\right\}^2\\
&=2\left(\varphi^2(q)\varphi^2(Q)+16q^{24}\psi^2\left(q^2\right)\psi^2\left(Q^2\right)+\varphi^2(-q)\varphi^2(-Q)\right)\\
&=2f^2\left(-q^2\right)f^2\left(-Q^2\right)\left(\chi^4(q)\chi^4(Q)+\chi^4(-q)\chi^4(-Q)+16q^{24}\frac{1}{\chi^4(-q^2)\chi^4(-Q^2)}\right).
\end{aligned}\tag{5.20}$$

For simplicity, set $L:=\chi(q)\chi(Q)+\chi(-q)\chi(-Q)$ and $T:=\chi(-q^2)\chi(-Q^2)$. From (5.20), we conclude after some algebra that

$$\left\{\varphi\left(q^{1/2}\right)\varphi\left(Q^{1/2}\right)+\varphi\left(-q^{1/2}\right)\varphi\left(-Q^{1/2}\right)\right\}^2=2f^2\left(-q^2\right)f^2\left(-Q^2\right)\left(L^4-4TL^2+2T^2+\frac{16q^{24}}{T^4}\right).\tag{5.21}$$

Similarly,

$$\begin{aligned}&\varphi(q)\varphi(Q)+\varphi(-q)\varphi(-Q)\\&\quad=f\left(-q^2\right)f\left(-Q^2\right)\left(\chi^2(q)\chi^2(Q)+\chi^2(-q)\chi^2(-Q)\right)\\&\quad=f\left(-q^2\right)f\left(-Q^2\right)\left(\left(\chi(q)\chi(Q)+\chi(-q)\chi(-Q)\right)^2-2\chi\left(-q^2\right)\chi\left(-Q^4\right)\right)\\&\quad=f\left(-q^2\right)f\left(-Q^2\right)\left(L^2-2T\right).\end{aligned}\tag{5.22}$$

In (5.18), we divide each term by $f(-q^2)f(-Q^2)$ and use (5.21) and (5.22), we conclude that

$$\frac{\sqrt{2}}{2}\sqrt{L^4-4TL^2+2T^2+\frac{16q^{24}}{T^4}}-\frac{1}{2}\left(L^2-2T\right)-\frac{2q^{12}}{T^2}=q^2L+\frac{2q^8}{T}.\tag{5.23}$$

In (5.23), we solve for the expression with the square root, then square both sides and obtain

$$\left(-TL-2q^6+2q^2T\right)\left(TL+2q^6\right)\left(4T^3-T^2L^2+2q^2T^2L+8T^2q^4+4TLq^6+4q^8T-4q^{12}\right)=0.$$

It is easy to see that the first two factors do not vanish identically. Therefore, we conclude that

$$4T^3-T^2L^2+2q^2T^2L+8T^2q^4+4TLq^6+4q^8T-4q^{12}=0.\tag{5.24}$$

In (5.24), we divide each term by $-T^2$ and then complete it to squares to find that

$$\left(L-q^2-\frac{2q^6}{T}\right)^2=4T+9q^4+\frac{8q^8}{T}.\tag{5.25}$$

Lastly, using (5.25), we arrive at

$$\begin{aligned}2q^4+\frac{2q^8}{T}+T+q^2\sqrt{4T+9q^4+\frac{8q^8}{T}}&=2q^4+\frac{2q^8}{T}+T+q^2\left(L-q^2-\frac{2q^6}{T}\right)\\&=q^4+T+q^2L\\&=q^4+\chi\left(-q^2\right)\chi\left(-Q^2\right)+q^2\left(\chi(q)\chi(Q)+\chi(-q)\chi(-Q)\right)\\&=\left(\chi(q)\chi(Q)+q^2\right)\left(\chi(-q)\chi(-Q)+q^2\right),\end{aligned}$$

which is (5.17) with q replaced by q^2. Hence, the proof of Entry 1.3 is complete. □

We can also prove by similar arguments that

$$A(q)A(-q)=\left(1+\frac{2q^2}{\chi(-q^2)\chi(-Q^2)}\right)A\left(q^2\right)$$

and

$$B(q)B(-q)=\left(1+\frac{2q^2}{\chi(-q^2)\chi(-Q^2)}\right)B\left(q^2\right).$$

These two identities in turn provides the following alternative to (5.16)

$$A(q)B(q)=\left(q^2+\chi(q)\chi(Q)\right)\left(1+\frac{2q^2}{\chi(-q^2)\chi(-Q^2)}\right).$$

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