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A non-cooperative support for equal division in estate division problems[☆]

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ABSTRACT

We consider estate division problems and show that for any claim game based on a (estate division) rule satisfying *efficiency*, *equal treatment of equals*, and *order preservation of awards*, *all* (pure strategy) Nash equilibria induce *equal division*. Next, we consider (estate division) rules satisfying *efficiency*, *equal treatment of equals*, and *claims monotonicity*. Then, for claim games with at most three agents, again *all* Nash equilibria induce *equal division*. Surprisingly, this result does not extend to claim games with more than three agents. However, if *nonbossiness* is added, then *equal division* is restored.

1. Introduction

We consider estate division problems, a generalization of bankruptcy problems, in which a positive-valued estate has to be divided among a set of agents. Clearly, if the agents' claims add up to less than the estate, no conflict occurs and each agent can receive his claimed amount. However, if the sum of the agents' claims exceeds the estate, then bankruptcy occurs. The class of bankruptcy problems has been extensively studied using various approaches such as the normative (axiomatic) or the game-theoretical approach (cooperative or noncooperative). For extensive surveys of the literature, we refer to Moulin (2002) and Thomson (2003).

In bankruptcy problems the agents' claims are normally considered as fixed inputs to the problem. However, in many real life situations it is impossible or difficult to check the validity of claims, e.g., if the profit of a joint project should be split among the project participants, but inputs are not perfectly observable or difficult to compare. Other examples are claims based on moral property rights, entitlements (see Gächter and Riedl, 2005) or subjective needs (see Pulido et al., 2002). If the authority in charge of the estate lacks the ability to verify claims or verification is too costly, agents are likely to behave strategically to ensure larger shares of the estate for themselves.

We model this type of situation with a simple noncooperative game that resembles Nash's classical demand game (Nash, 1953). Given the estate to divide and based on a (estate division) rule, agents simply submit claims which are restricted to not exceed a common upper bound. We analyze the (pure strategy) Nash equilibria of the resulting *claim game*. We do not fix any specific rule, but only require the rule to satisfy basic and appealing properties.

First, we require the rule to satisfy *efficiency*, *equal treatment of equals*, and *order preservation of awards*.[1] Then, all agents claiming the largest possible amount is a Nash equilibrium and *all* Nash equilibria lead to *equal division* (Theorem 1). Second, we replace *order preservation of awards* with *claims monotonicity*.[2] Again, all agents claiming the largest possible amount is a Nash equilibrium. However, in contrast to the previous result, we show that *equal division* is guaranteed for *all* Nash equilibria only for claim games with at most three agents (Theorem 2). Surprisingly, this result does not extend to claim games with more than three agents (Example 1). Nevertheless, if *nonbossiness* is added, then *equal division* in *all* Nash equilibria is restored (Theorem 3).[3]

All our results point towards the same intuitive message: if it is impossible or difficult to test the legitimacy of claims, the conflict will escalate to the highest possible level at which claims are no longer informative. As a result, *equal division* is the "non-discriminating" outcome in Nash equilibrium. In other words, *equal division* is not only a normatively appealing division method,

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[1] *Efficiency*: the estate is allocated if the sum of claims is larger than (or equal to) the estate. *Equal treatment of equals*: any two agents with identical claims receive the same awards. *Order preservation of awards*: if an agent has a higher claim than another agent, then he does not receive less than that agent.

[2] *Claims monotonicity*: other things equal, an agent does not receive less after an increase in his claim.

[3] *Nonbossiness*: no agent can change other agents' awards by changing his claim unless his award changes as well.

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but it is also the result of a natural noncooperative game.[4] These findings might explain why in certain instances *equal division* is applied right away even without asking for agents' claims. For instance, pre-1975 US Admiralty law divides liabilities equally among parties if they are both found negligent (see Feldman and Kim, 2005). British Shipping Law, until the act of 1911, applied equal division of costs in case of a collision between two ships, however much the degree of their faults or negligence may differ. This practice has originated from a medieval rule, which was originally intended to be applied only in cases where negligence cannot be perfectly proven (see Porges and Thomas, 1963). Finally, in Arizona, California, Idaho, Louisiana, Nevada, New Mexico, Texas, Washington, and Wisconsin, "community property law" implements equal split of assets and wealth accumulated during marriage in case of a divorce.

A number of articles also consider strategic aspects of claim problems (see Thomson, 2003, Section 7). The articles closely related to ours are Chun (1989), Moreno-Ternero (2002), Herrero (2003), and Bochet et al. (2010) in that the games they consider do not focus on a specific rule, but a class of rules that is determined by basic properties. Chun (1989) considers a noncooperative game where agents propose rules and a sequential revision procedure then converges to *equal division*. Moreno-Ternero (2002) constructs a noncooperative game, the equilibrium of which converges to the proportional rule. A noncooperative game similar (in a sense dual) to the one in Chun (1989) is constructed by Herrero (2003) who shows convergence to the constrained equal losses rule. Bochet et al. (2010) consider the problem of allocating an estate when agents have single-peaked preferences and study direct revelation games associated with (allocation) rules. They prove that *uniform division* is the only Nash equilibrium outcome for rules satisfying certain properties (Bochet et al., 2010, Theorem 2).[5] Note that *uniform division* (for single-peaked preferences) is similar in spirit to equal division (for monotone preferences) in that a uniform allocation divides the estate as equally as possible by either taking agents's peaks as upper bounds (in case of bankruptcy) or as lower bounds (in case of an excess supply). Hence, in this paper as well as in Bochet et al. (2010), based on agents preferences, we implement the appropriate notion of equality in allocation.[6]

The paper is organized as follows. In Section 2, we introduce claim games as well as various properties for underlying rules and three well-known rules (the proportional, the constrained equal awards, and the constrained equal losses rule). In Section 3, we establish our *equal division* Nash equilibria results (Theorems 1–3, and Example 1) and discuss the independence of assumptions needed to establish our results (Remarks 1–3), the relation between claim games and divide-the-dollar games (Remark 4), and the Nash equilibria obtained for the proportional, the constrained equal awards, and the constrained equal losses rule (Remark 5). We conclude in Section 4.

2. The claim game

In a claim game, an *estate* $E > 0$ has to be divided among a *set of agents* $N = \{1, \ldots, n\}$. We assume that agents' preferences are strictly monotone over the amounts of the estate they receive. Then, the estate E and the set of agents N determine an *estate division problem*.

A *strategy* for an agent $i \in N$ is a *claim* $c_i \geq 0$ belonging to her non-empty *strategy set* $C_i = [0, k]$ $(k > 0)$. For example, we could assume for all $i \in N$, $C_i = [0, E]$. The *set of strategy profiles (claims vectors)* is denoted by $C = C_1 \times \cdots \times C_n$. Hence, for all $i \in N$, the *maximal claim* $\bar{c}_i \equiv \max C_i = k$ and the *maximal claims vector* $\bar{c} \equiv (\bar{c}_i)_{i \in N}$. We assume that the estate E, the set of agents N, and the set of strategy profiles C are fixed.

We use the following notations in the sequel. For each $c \in C$ and each $S \subseteq N, S \neq \emptyset$, let $c_S = \sum_{i \in S} c_i$. For each $c \in C$, each $i \in N$, and each $c'_i \in C_i$, $(c'_i, c_{-i}) \in C$ denotes the claims vector obtained from c by replacing c_i with c'_i.

A claim game's *outcome function* is a *(estate division) rule* $R : C \to \mathbb{R}^N_+$ that associates with each strategy profile $c \in C$ an *awards vector* $R(c) \in \mathbb{R}^N_+$ such that $\sum R_i(c) \leq E$ and $R(c) \leq c$;[7] $R_i(c)$ denotes the amount of estate E that agent i obtains under strategy profile c. We do not fix the rule that determines the outcomes of a claim game and therefore denote a *claim game* by $\Gamma(R)$.

Note that the rules that determine the outcomes of our claim games can be thought of as extended bankruptcy rules (see Thomson, 2003, for a comprehensive survey on the axiomatic and game-theoretic analysis of bankruptcy problems). We now introduce some properties of rules. All properties are stated for a generic rule R.

Efficiency: the largest possible amount of E is assigned taking claims as upper bounds, i.e., for all $c \in C$, [if $c_N \geq E$, then $\sum R(c)_i = E$] and [if $c_N \leq E$, then $R(c) = c$].

Note that we do not require that E has to be completely allocated among the agents if no bankruptcy occurs ($E < c_N$).[8]

The following property requires that the awards to agents whose claims are equal should be equal.

Equal treatment of equals: for all $c \in C$ and all $i, j \in N$ such that $c_i = c_j$, $R_i(c) = R_j(c)$.

By the next requirement, the ordering of awards should conform to the ordering of claims, i.e., if agent i's claim is larger than agent j's claim, i should receive at least as much as agent j does.

Order preservation of awards: for all $c \in C$ and all $i, j \in N$ such that $c_i > c_j$, $R_i(c) \geq R_j(c)$.

Order preservation of awards is a weakening of the standard *order preservation* property introduced by Aumann and Maschler (1985) (which additionally also requires *order preservation of losses*).[9]

The following monotonicity property requires that, other things equal, if an agent's claim increases, he should receive at least as much as he did initially.

Claims monotonicity: for all $c \in C$, all $i \in N$, and all $c'_i \in C_i$ such that $c_i < c'_i$, $R_i(c) \leq R_i(c'_i, c_{-i})$.

Most well-known bankruptcy rules satisfy all the properties mentioned above; e.g., the constrained equal awards, the constrained equal losses, and the proportional rule. We introduce efficient extensions of these well-known bankruptcy rules to the

4 A game-theoretical interpretation of our result is that if a rule satisfies certain natural and appealing properties (see our results above), it can be used to implement *equal division*.

5 The properties they consider – *peak-only*, *efficiency*, *symmetry*, *others-oriented peak monotonicity*, *peak continuity*, and *strict own-peak monotonicity* – are similar in spirit to the ones we consider for estate division problems.

6 Intuitively speaking, Bochet et al. (2010) require more properties to obtain their result to accommodate the role agents' peaks play as upper or lower bounds when allocating the estate as equally as possible.

7 Note that $R(c) \leq c$ if and only if for all $i \in N$, $R_i(c) \leq c_i$.

8 In Appendix B we describe what happens if we require that the estate E is always completely allocated among the agents. Then, *efficiency* is already incorporated in the definition of a rule and results essentially do not change.

9 *Order preservation*. A rule R satisfies *order preservation* if for all $c \in C$ and all $i, j \in N$ such that $c_i \geq c_j$, $R_i(c) \geq R_j(c)$ (*order preservation of awards*) and $c_i - R_i(c) \geq c_j - R_j(c)$ (*order preservation of losses*).

(estate division) rules that serve as outcome functions for our claim games.

The constrained equal awards rule allocates the estate as equally as possible taking claims as upper bounds.

Constrained equal awards rule, CEA. For each $c \in C$,

(i) if $c_N \leq E$, then $CEA(c) = c$ and
(ii) if $c_N \geq E$, then for all $j \in N$, $CEA_j(c) = \min\{c_j, \lambda_{\text{cea}}\}$, where λ_{cea} is such that $\sum \min\{c_i, \lambda_{\text{cea}}\} = E$.

The constrained equal losses rule allocates the shortage of the estate in an equal way, keeping awards bounded below by zero.

Constrained equal losses rule, CEL. For each $c \in C$,

(i) if $c_N \leq E$, then $CEL(c) = c$ and
(ii) if $c_N \geq E$, then for all $j \in N$, $CEL_j(c) = \max\{0, c_j - \lambda_{\text{cel}}\}$,where λ_{cel} is such that $\sum \max\{0, c_i - \lambda_{\text{cel}}\} = E$.

The proportional rule allocates the estate proportionally with respect to claims.

Proportional rule, P. For each $c \in C$,

(i) if $c_N \leq E$, then $P(c) = c$ and
(ii) if $c_N \geq E$, then $P(c) = \lambda_p c$, where $\lambda_p = \frac{E}{c_N}$.

3. Nash equilibria and equal division

First, we are interested in (pure strategy) Nash equilibria of the claim game. A claims vector $c \in C$ is a *Nash equilibrium (in pure strategies) of claim game* $\Gamma(R)$ if for all $i \in N$ and all $c'_i \in C_i$, $R_i(c) \geq R_i(c'_i, c_{-i})$; we call $R(c)$ the *Nash equilibrium outcome*. Since agents' preferences are strictly monotone over the amounts of the estate they receive, any Nash equilibrium of a claim game that is based on an *efficient* rule has to distribute the whole estate if that is possible given upper bounds $\bar{c}$ on reported claims vectors. This implies that at any Nash equilibrium c at which agents do not claim their maximal possible amounts ($c \neq \bar{c}$), the sum of reported claims must add up to at least the estate ($c_N \geq E$).

Lemma 1. *If rule R is* efficient, *then for any Nash equilibrium c of the claim game* $\Gamma(R)$, $c \neq \bar{c}$ *implies* $c_N \geq E$.

Proof. Let rule R be *efficient* and assume that $c \neq \bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R)$ such that $c_N < E$. Let $\alpha \equiv E - c_N > 0$ and define for some $j \in N$ such that $c_j < \bar{c}_j$, $c'_j \equiv \min\{\bar{c}_j, c_j + \alpha\} > c_j$. Then, by *efficiency*, $R(c) = c$ and $R(c'_j, c_{-j}) = (c'_j, c_{-j})$. Hence, $R_j(c) = c_j < c'_j = R_j(c'_i, c_{-i})$; contradicting that c is a Nash equilibrium of $\Gamma(R)$. □

We denote by $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^N_{++}$ the *one-vector*.

Equal division. Given an estate $E > 0$, $\frac{E}{n}\mathbf{1} \in \mathbb{R}^N_{++}$ denotes the corresponding *equal division vector*.

Next, we show that for claim games where agents have equal maximal strategies and the underlying rule satisfies *efficiency*, *equal treatment of equals*, and *order preservation of awards*, (a) claiming the maximal amount is always a Nash equilibrium and (b) all Nash equilibria induce equal division.

Theorem 1. *Let rule R satisfy* efficiency, equal treatment of equals, *and* order preservation of awards. *Then,*

(a) *the maximal claims vector* $\bar{c}$ *is a Nash equilibrium of the claim game* $\Gamma(R)$ *and*
(b) *the outcome in all Nash equilibria of the claim game* $\Gamma(R)$ *is* $\min\{k, \frac{E}{n}\}\mathbf{1}$.

Proof.

(a) We prove that $\bar{c} = k\mathbf{1}$ is a Nash equilibrium of the claim game $\Gamma(R)$. By *efficiency* and *equal treatment of equals*, $R(\bar{c}) = \min\{k, \frac{E}{n}\}\mathbf{1}$. If $R(\bar{c}) = k\mathbf{1}$, then each agent already gets the largest possible amount and $\bar{c}$ is a Nash equilibrium.

Thus, assume that $R(\bar{c}) = \frac{E}{n}\mathbf{1} < k\mathbf{1}$. Let $i \in N$ and $c'_i \neq \bar{c}_i$. Thus, for all $j \neq i$, $c'_i < k = \bar{c}_j$. Hence, by *order preservation of awards*, for all $j \neq i$, $R_j(c'_i, \bar{c}_{-i}) \geq R_i(c'_i, \bar{c}_{-i})$. Suppose that $R_i(c'_i, \bar{c}_{-i}) > \frac{E}{n}$. Then, for all $l \in N$, $R_l(c'_i, \bar{c}_{-i}) > \frac{E}{n}$ and $\sum R_l(c'_i, \bar{c}_{-i}) > E$; a contradiction. Thus, $R_i(c'_i, \bar{c}_{-i}) \leq \frac{E}{n} = R_i(\bar{c})$ and $\bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R)$.

(b) Suppose that c is a Nash equilibrium of the claim game $\Gamma(R)$ and $R(c) \neq \min\{k, \frac{E}{n}\}\mathbf{1}$. Then, for some $i \in N$, $R_i(c) < \min\{k, \frac{E}{n}\}$. Let $c'_i = k$ (possibly $c'_i = c_i$). Since c is a Nash equilibrium, $R_i(c'_i, c_{-i}) \leq R_i(c) < \min\{k, \frac{E}{n}\}$. In particular, (i) $R_i(c'_i, c_{-i}) < \frac{E}{n}$.

Since by (a), $\bar{c}$ is a Nash equilibrium such that $R(\bar{c}) = \min\{k, \frac{E}{n}\}\mathbf{1}$, we know that $c \neq \bar{c}$. Thus, by Lemma 1, $c_N \geq E$. Recall that $c'_i = k \geq c_i$. Hence, $c'_i + \sum_{l \neq i} c_l \geq E$ and by *efficiency*, (ii) $\sum R_l(c'_i, c_{-i}) = E$. For all $j \neq i$ such that $c_j < k = c'_i$, by *order preservation of awards* and (i), $R_j(c'_i, c_{-i}) \leq R_i(c'_i, c_{-i}) < \frac{E}{n}$. For all $j \neq i$ such that $c_j = k = c'_i$, by *equal treatment of equals* and (i), $R_j(c'_i, c_{-i}) = R_i(c'_i, c_{-i}) < \frac{E}{n}$. Hence, $\sum R_l(c'_i, c_{-i}) < E$; a contradiction to (ii). □

Remark 1 (*Independence of Assumptions in* Theorem 1).

(i) Suppose that not all agents have equal maximal claims, i.e., there exist $i, j \in N$ such that $\bar{c}_i \neq \bar{c}_j$. Then, an "unequal" Nash equilibrium is possible even if the rule satisfies *efficiency*, *equal treatment of equals*, and *order preservation of awards*; e.g., for the proportional rule $\bar{c}$ is a Nash equilibrium, but for all i, j such that $\bar{c}_i \neq \bar{c}_j$, $P_i(\bar{c}) \neq P_j(\bar{c})$.
(ii) The following rule R satisfies *equal treatment of equals*, *order preservation of awards*, but not *efficiency*. If $c \neq \bar{c}$, then $R(c) = P(c)$ and $R(\bar{c}) = 0\mathbf{1}$. Clearly, $\bar{c}$ is not a Nash equilibrium of the claim game $\Gamma(R)$ and the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ is never an equilibrium outcome.
(iii) A serial dictatorship rule that first serves agents with the highest claims lexicographically (i.e., if several agents have the highest claim, then first serve the agent with the lowest index and so on) satisfies *efficiency* and *order preservation of awards*, but not *equal treatment of equals*. There are Nash equilibria, e.g., $c = k\mathbf{1}$ when $nk > E$, at which agent 1 receives more than agent n.
(iv) The following rule R satisfies *efficiency* and *equal treatment of equals*, but not *order preservation of awards*. Rule R first assigns the estate E proportionally (and *efficiently*) among all agents who have a claim different from that of agent 1. Then, if some part of the estate is left, R allocates it equally (and *efficiently*) among the remaining agents. For $\bar{c} \neq E\mathbf{1}$, $\bar{c}$ is not a Nash equilibrium of the claim game $\Gamma(R)$ and the equal division vector $\min\{k, \frac{E}{n}\}\mathbf{1}$ is not an equilibrium outcome. □

In Ashlagi et al. (2008, Corollaries 1 and 2) two corresponding results are obtained using *order preservation* (see Footnote 9) (instead of *equal treatment of equals* and *order preservation of awards*) or *others oriented claims monotonicity*[10](instead of *order preservation of awards*).

Next, we show that for claim games where agents have equal maximal strategies and the underlying rule satisfies *efficiency*, *equal treatment of equals*, and *claims monotonicity*, (a) claiming the maximal amount is always a Nash equilibrium and (b) for $n \leq 3$, all Nash equilibria induce equal division.

Theorem 2. *Let rule R satisfy* efficiency, equal treatment of equals, *and* claims monotonicity. *Then,*

[10] *Others oriented claims monotonicity*: for all $c \in C$ and all $i \in N$ such that $c_i < c'_i$, $R_j(c) \geq R_j(c'_i, c_{-i})$ for all $j \neq i$.

(a) *the maximal claims vector $\bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R)$ and*
(b) *for $n \leq 3$, the outcome in all Nash equilibria of the claim game $\Gamma(R)$ is* $\min\{k, \frac{E}{n}\}\mathbf{1}$.

Proof.

(a) By *claims monotonicity*, for each agent i it is a weakly dominant strategy to claim $\bar{c}_i$. Hence, $\bar{c}$ is a Nash equilibrium of $\Gamma(R)$.

(b) For $n = 1$, the proof is obvious and therefore omitted.

For $n = 2$, Ashlagi et al. (2008, Lemma 3) prove that *efficiency*, *equal treatment of equals*, and *claims monotonicity* imply *order preservation of awards*. By Theorem 1 (b), *efficiency*, *equal treatment of equals*, and *order preservation of awards* imply the result.

Let $n = 3$. Suppose that c is a Nash equilibrium of the claim game $\Gamma(R)$ and $R(c) \neq \min\{k, \frac{E}{3}\}\mathbf{1}$. Without loss of generality, we assume that $c_1 \leq c_2 \leq c_3$.

Case 1. $i \in \{1, 2\} \equiv \{i, j\}$ and $R_i(c) < \min\{k, \frac{E}{3}\}$.

Let $c'_i = c_3$ (possibly $c'_i = c_1$). Since c is a Nash equilibrium, $R_i(c'_i, c_{-i}) \leq R_i(c) < \min\{k, \frac{E}{3}\}$. In particular, (i) $R_i(c'_i, c_{-i}) < \frac{E}{3}$.

Since by (a), $\bar{c}$ is a Nash equilibrium such that $R(\bar{c}) = \min\{k, \frac{E}{n}\}\mathbf{1}$, we know that $c \neq \bar{c}$. Thus, by Lemma 1, $c_N \geq E$. Hence, $c'_i + \sum_{l \neq i} c_l \geq E$ and by *efficiency*, (ii) $\sum R_l(c'_i, c_{-i}) = E$. By *equal treatment of equals* and (i), $R_3(c'_i, c_{-i}) = R_i(c'_i, c_{-i}) < \frac{E}{3}$. Hence, (ii) implies (iii) $R_j(c'_i, c_{-i}) > \frac{E}{3}$.

Recall that $j \in \{1, 2\}$ and therefore, $c_j \leq c_3$. Let $c'_j = c_3$ and consider $(c'_i, c'_j, c_3) = (c_3\mathbf{1})$. By *equal treatment of equals*, (i) and (iii) imply $c_j < c_3$. Hence, $c'_i + c'_j + c_3 > E$ and by *efficiency*, (iv) $\sum R_l(c'_i, c'_j, c_3) = E$. By *claims monotonicity*, $R_j(c'_i, c'_j, c_3) \geq R_j(c'_i, c_{-i}) > \frac{E}{3}$ and by *equal treatment of equals*, $R_j(c'_i, c'_j, c_3) = R_i(c'_i, c'_j, c_3) = R_3(c'_i, c'_j, c_3) > \frac{E}{3}$. Hence, $\sum R_l(c'_i, c'_j, c_3) > E$; a contradiction to (iv).

Case 2. $R_3(c) < \min\{k, \frac{E}{3}\}$.

First, $R_3(c) < \min\{k, \frac{E}{3}\}$ implies (v) $R_3(c) < \frac{E}{3}$. Furthermore, if $c_2 = c_3$, then by *equal treatment of equals*, $R_2(c) = R_3(\hat{c}) < \min\{k, \frac{E}{3}\}$, and we are done by Case 1. Hence, assume that $c_2 < c_3$. Let $c'_3 = c_2$ and consider (c'_3, c_{-3}).

Since by (a), $\bar{c}$ is a Nash equilibrium such that $R(\bar{c}) = \min\{k, \frac{E}{n}\}\mathbf{1}$, we know that $c \neq \bar{c}$. Thus, by Lemma 1, $c_N \geq E$. Hence, by *efficiency*, (vi) $\sum R_l(c) = E$. Then, (v) and (vi) imply $R_1(c) > \frac{E}{3}$ or $R_2(c) > \frac{E}{3}$. Note that $R_1(c) \leq c_1 \leq c_2$ and $R_2(c) \leq c_2$. Thus, $c'_3 > R_3(c)$ and $c_1 + c_2 + c'_3 \geq \sum R_l(c) = E$, where the last equality follows from (vi). By *efficiency*, (vii) $\sum R_l(c'_3, c_{-3}) = E$. By *claims monotonicity*, (viii) $R_3(c'_3, c_{-3}) \leq R_3(c) < \frac{E}{3}$. By *equal treatment of equals*, $R_2(c'_3, c_{-3}) = R_3(c'_3, c_{-3}) < \frac{E}{3}$. Hence, (vii) implies (ix) $R_1(c'_3, c_{-3}) > \frac{E}{3}$.

Let $c'_1 = c_2$ and consider $(c'_1, c'_3, c_2) = (c_2\mathbf{1})$. By *equal treatment of equals*, (viii) and (ix) imply $c_1 < c'_3 = c_2$. Hence, $c'_1 + c'_3 + c_2 > E$ and by *efficiency*, (x) $\sum R_l(c'_1, c'_3, c_2) = E$. By *claims monotonicity*, $R_1(c'_1, c'_3, c_2) \geq R_1(c'_3, c_{-3}) > \frac{E}{3}$ and by *equal treatment of equals*, $R_1(c'_1, c'_3, c_2) = R_2(c'_1, c'_3, c_2) = R_3(c'_1, c'_3, c_2) > \frac{E}{3}$. Hence, $\sum R_l(c'_1, c'_3, c_2) > E$; a contradiction to (x). □

Remark 2 (*Independence of Assumptions in* Theorem 2)**.** Note that claims monotonicity alone implies Theorem 2 (a). Hence, we show independence only for Theorem 2 (b).

(i) The proof that it is essential that all agents have equal maximal claims is the same as in Remark 1 (i).

(ii) The following rule R satisfies *claims monotonicity* and *equal treatment of equals*, but not *efficiency*. If at c exactly one agent i claims $c_i = \bar{c}_i$, then he receives $R_i(c) = \min\{k, \frac{E}{n}\}$ and for all $j \neq i$, $R_i(c) = 0$. Furthermore, $R(\bar{c}) = \min\{k, \frac{E}{n}\}\mathbf{1}$ and for all other claims vectors c, $R(c) = 0\mathbf{1}$. Then, Nash equilibria which do not induce equal division exist, e.g., for $N = \{1, 2, 3\}$ and $E = k = 1$, $\tilde{c} = (1, 0, 0)$ resulting in the equilibrium outcome $R(c) = (1/3, 0, 0)$.

(iii) To prove that *equal treatment of equals* is needed one can use the serial dictatorship rule as described in Remark 1 (iii) – it satisfies *efficiency* and *claims monotonicity*, but not *equal treatment of equals*.

(iv) To prove that *claims monotonicity* is needed one can use the rule as described in Remark 1 (iv) – it satisfies *efficiency* and *equal treatment of equals*, but not *claims monotonicity*. □

In the following example, we show that when $n > 3$, *efficiency*, *equal treatment of equals*, and *claims monotonicity* are not sufficient to guarantee equal division in all Nash equilibria of the claim game $\Gamma(R)$. This example represents a claim game with an asymmetric equilibrium despite the fact that players enter the game in a symmetric way. Note that all players have the same maximal claims, the rule used satisfies *equal treatment of equals*, and yet there is an equilibrium at which players receive unequal payoffs.

Example 1. Let $N = \{1, 2, 3, 4\}$, $E = 1$, and for all $i \in N$, $C_i = [0, 1]$. Before we define rule R of claim game $\Gamma(R)$, we introduce some notation.

Let $H = 1/2$ and $L = 1/3$ be two points, which we will use to partition the set of claim profiles. For all profiles $c \in C$, let $LH(c) = \{i \in N : L \leq c_i \leq H\}$, $L(c) = \{i \in N : c_i < L\}$, and $H(c) = \{i \in N : c_i > H\}$. For all $j = 0, 1, \ldots, 4$, denote by C^j the set of claim profiles in which j agents claim between L and H and $n - j$ agents claim more than H. That is, $C^j = \{c \in C : |LH(c)| = j \text{ and } |H(c)| = n - j\}$. Let $P = C \setminus \cup_{j=0}^4 C^j$. Note that for all claim profiles $c \in P$, there exists some agent $i \in N$ for which $c_i < L$. Furthermore, note that the collection of sets C^j $(j = 1, \ldots, 4)$ and P partition the set of claim profiles C.

For all $c \in P$, define $B(c)$ to be the maximal set of agents such that

(i) $c_{B(c)} \leq 1$, i.e., the sum of claims of agents in $B(c)$ does not exceed the estate, and
(ii) for all $i, l \in N$, if $c_i \geq c_l$ and $l \in B(c)$, then $i \in B(c)$, i.e., if agent l is a member of $B(c)$, then all agents with claims larger than or equal to c_l are also members of $B(c)$.

Let $D(c) = \{i \in N \setminus B(c) : \text{for all } l \in N \setminus B(c),\ c_i \geq c_l\}$, i.e., if $B(c) \neq N$, then $D(c) \neq \emptyset$ contains the set of agents that have the highest claim among the agents in $N \setminus B(c)$. Note that $D(c) = \emptyset$ if and only if $c_N \leq 1$. Finally, let $A(c) = B(c) \cup D(c)$.

Roughly speaking, rule R works as follows. For claim profiles in $\cup_{j=0}^4 C^j$, we specify awards to agents according to their claims being larger than H or not. For claim profiles in P, rule R does the following: it first ranks agents from highest claim to lowest claim. Then, to all agents in the set $B(c)$, R gives their full claim, and allocates the residual amount equally to agents in $D(c)$. All other agents receive 0.

$$R_i(c, 1) = \begin{cases} 1/4, & c \in C^4 \cup C^0, \\ 1/6, & c_i \leq H \text{ and } c \in C^3, \\ 1/2, & c_i > H \text{ and } c \in C^3, \\ 1/3, & c_i \leq H \text{ and } c \in C^2, \\ 1/6, & c_i > H \text{ and } c \in C^2, \\ 0, & c_i \leq H \text{ and } c \in C^1, \\ 1/3, & c_i > H \text{ and } c \in C^1, \\ c_i, & c \in P \text{ and } i \in B(c), \\ \dfrac{1 - c_{B(c)}}{|D(c)|}, & c \in P \text{ and } i \in D(c), \\ 0, & c \in P \text{ and } i \notin A(c). \end{cases}$$

We prove in Appendix A (Claim 1) that rule R satisfies *efficiency*, *equal treatment of equals*, and *claims monotonicity*. Next, we show that the profile of claims $c = (1, 1/3, 1/3, 1/3)$ is an equilibrium for $\Gamma(R)$ and since $R(c) = (1/2, 1/6, 1/6, 1/6)$, we have a violation of equal division in equilibrium.

Note that $c \in C^3$. Then, a unilateral deviation by agent 1 can only result in a claim profile that belongs to one of the sets C^3, C^4, or P, which induces the amounts (for agent 1) 1/2, 1/4, or 0, respectively. Since at c agent 1 obtains 1/2, no unilateral deviation from c is beneficial for agent 1. Next, a unilateral deviation by agent $k \in \{2, 3, 4\}$ can only result in a claim profile that belongs to one of the sets C^2, C^3, or P, which induces the amounts (for agent k) 1/6, 1/6, or 0, respectively. Since at c agent k obtains 1/6, no unilateral deviation from c is beneficial for agent k. □

Note that the rule described in Example 1 violates *order preservation of awards* (see Footnote 9) and *others oriented claims monotonicity* (see Footnote 10). Furthermore, this rule is *discontinuous* and it is an open problem if a continuous example can be constructed.

Finally, we show that equal division is restored in the equilibrium result of Theorem 2 for more than three agents by adding a non-manipulation (or robustness) property: nonbossiness (Satterthwaite and Sonnenschein, 1981) requires that no agent can change other agents' awards by changing his claim without changing his own award.

Nonbossiness: for all $c \in C$ and all $i \in N$ such that $R_i(c) = R_i(c'_i, c_{-i}), R_j(c) = R_j(c'_i, c_{-i})$ for all $j \neq i$.

Theorem 3. *Let rule R satisfy* efficiency, equal treatment of equals, claims monotonicity, *and* nonbossiness. *Then,*

(a) *the maximal claims vector $\bar{c}$ is a Nash equilibrium of the claim game $\Gamma(R)$ and*
(b) *the outcome in all Nash equilibria of the claim game $\Gamma(R)$ is* $\min\{k, \frac{E}{n}\}\mathbf{1}$.

Proof.
(a) By *claims monotonicity*, for each agent i it is a weakly dominant strategy to claim $\bar{c}_i$. Hence, $\bar{c}$ is a Nash equilibrium of $\Gamma(R)$.

(b) We first prove that *equal treatment of equals*, *claims monotonicity*, and *nonbossiness* imply equal division for all Nash equilibria. Suppose that c is a Nash equilibrium of the claim game $\Gamma(R)$ and for some $i, j \in N$, (i) $R_i(c) \neq R_j(c)$. Hence, by *equal treatment of equals* $c_i \neq c_j$. Without loss of generality assume that $c_i < c_j$. Let $c'_i = c_j$ and consider (c'_i, c_{-i}). Since c is a Nash equilibrium, $R_i(c'_i, c_{-i}) \leq R_i(c)$. By *claims monotonicity*, $R_i(c'_i, c_{-i}) \geq R_i(c)$. Hence, (ii) $R_i(c'_i, c_{-i}) = R_i(c)$. Thus, by *nonbossiness*, (iii) $R_j(c'_i, c_{-i}) = R_j(c)$. Then, (i)–(iii) imply $R_i(c'_i, c_{-i}) \neq R_j(c'_i, c_{-i})$; a contradiction to *equal treatment of equals*. Therefore, for all $i, j \in N$, $R_i(c) = R_j(c)$, which proves an equal division vector is induced by all Nash equilibria. By *efficiency*, this equal division vector equals $\min\{k, \frac{E}{n}\}\mathbf{1}$. □

From the proof of Theorem 3 it becomes clear that even without *efficiency* Nash equilibria outcomes respect equal division. However, without *efficiency*, some part of the estate might be wasted.

Note that the rule described in Example 1 violates *nonbossiness*.

Remark 3 (*Independence of Assumptions in* Theorem 3)**.** Note that *claims monotonicity* alone implies Theorem 2 (a). Hence, we show independence only for Theorem 2 (b).

(i) The proof that it is essential that all agents have equal maximal claims is the same as in Remark 1 (i).
(ii) As explained after the proof of Theorem 3, *efficiency* is only needed to obtain the *efficient* equal division vector as equilibrium outcome. The following inefficient proportional rule satisfies *equal treatment of equals*, *claims monotonicity*, and *nonbossiness*, but not *efficiency*: for all $c \in C, P'(c) = P(c; E/2)$, where $P(c; E/2)$ denotes the outcome of the proportional rule where only half the estate $\frac{E}{2}$ is allocated.
(iii) To prove that *equal treatment of equals* is needed one can use the serial dictatorship rule as described in Remark 1 (iii)—it satisfies *efficiency*, *claims monotonicity*, and *nonbossiness* but not *equal treatment of equals.*
(iv) To prove that *claims monotonicity* is needed one can use rule R as described in Remark 1 (iv)—it satisfies *efficiency*, *equal treatment of equals*, and *nonbossiness*, but not *claims monotonicity*.
(v) To prove that *nonbossiness* is needed one can use the same rule as in Example 1—it satisfies *equal treatment of equals*, and *claims monotonicity*, but not *nonbossiness*. □

Remark 4 (*Divide-the-Dollar Games versus Claim Games*)**.** In the *divide-the-dollar game* (a simple version of Nash's, 1953, demand game), two agents simultaneously submit their claims over a dollar. If the sum of claims does not exceed a dollar, each agent receives his claim. Otherwise, both agents receive nothing. This simple version of the game has infinitely many (pure strategy) Nash equilibrium outcomes, namely any division of the dollar. Brams and Taylor (1994) and Anbarcı (2001) consider adaptations of the divide-the-dollar game that exhibit equal division in Nash equilibrium.

Claim games can be considered as modified versions of the divide-the-dollar game (or Nash's, 1953, demand game), the main difference being that agents are not punished as severely as they are in divide-the-dollar games (i.e., receiving nothing) whenever the sum of claims exceeds the estate. In a claim game, if the sum of claims does not exceed the value of the estate, all agents receive their claims (as in a divide-the-dollar game). If, on the other hand, the sum of claims exceeds the value of the estate, instead of not allocating the estate at all, a division rule with certain properties solves the dispute over the estate and (in contrast to the divide-the-dollar game) an efficient outcome is obtained. We consider entire classes of rules (determined by normative properties they share), whereas most divide-the-dollar games use a fixed reference allocation (e.g., the zero share vector). Similar to Brams and Taylor (1994) and Anbarcı (2001), our modifications induce equal division in Nash equilibrium.

It is worthwhile mentioning that the rules we use for claim games satisfy most properties that Brams and Taylor (1994) require for *reasonable* payoff schemes: (i) equal claims are treated equally, (ii) no agent receives more than what he claimed, (iii) if the sum of claims does not exceed the estate, then every agent receives his claim, (iv) if the sum of claims exceed the estate, nevertheless, the whole estate is allocated, and (v) if all claims are higher than $\frac{E}{n}$, then the highest claimant does no better than the lowest claimant (our rules only might fail to satisfy property (v)). Hence, our results provide an alternative answer to the question ''Can one alter the payoff structure of the divide-the-dollar game in a *reasonable* way so that the egalitarian outcome is a noncooperative solution of the corresponding game?'' raised by Brams and Taylor (1994). □

Remark 5 (*(Strong) Nash Equilibria for $\Gamma(P)$, $\Gamma(CEA)$ and $\Gamma(CEL)$*)**.** The proportional rule, the constrained equal awards rule, and the constrained equal losses rule satisfy all properties introduced in this article. Hence, for these rules, claiming the largest possible amount is always an equal-division Nash equilibrium. For the proportional rule and the constrained equal losses rule, this is the unique Nash equilibrium of the associated claim game. However, if agents are allowed to claim more than an equal share of the estate, the constrained equal awards rule admits multiple (in fact infinitely many) equal-division Nash equilibria. This difference stems from the fact that under the proportional rule and the constrained equal losses rule, claiming the whole estate is a strictly dominant strategy for all agents whereas under the constrained equal awards rule, it is a weakly dominant strategy.

Note that all Nash equilibria for the proportional rule, the constrained equal awards rule, and the constrained equal losses rule are also strong (i.e., there exist no coalition that can make each of its members strictly better off using a joint deviation). However, this is not a general result: there exist rules satisfying *equal treatment of equals*, *efficiency*, *order preservation of awards*, and *claims monotonicity* such that the corresponding claim game has Nash equilibria that are not strong (an example is available from the authors upon request). □

4. Concluding remarks

We analyze situations where an estate should be distributed among a set of agents, but claims to the estate are impossible or difficult to verify. We model a simple and intuitive claim game where, given the estate and a (estate division) rule satisfying some basic properties, agents simply announce their claims. Our game can be thought of as a modified Nash demand game (or divide-the-dollar game) where players are punished less severely than in the standard version of the game. Our results show that first of all, claiming the largest possible amount is *always* a Nash equilibrium. Of course, this is an intuitive and not very surprising result. However, in addition, we show that even though we do not focus on any specific outcome function to be used in our claim game, *equal division* is the unique Nash equilibrium outcome. Since most well-known rules satisfy all the properties we require (e.g., the proportional rule, the constrained equal awards rule, and the constrained equal losses rule), our results can be interpreted as a non-cooperative support for equal division in estate division conflicts. The advantage of this implementation result is that *equal division* is the outcome that is based on the (strategic) choices of the agents rather than a fixed outcome that is externally imposed; given the desirable properties of the underlying rule this process might therefore be considered fair and its outcome thus acceptable. Finally, future research on this topic might analyze situations in which partial verification is possible and agents spend resources to support their claims (e.g., hiring a lawyer in a court case).

Appendix A

Claim 1. *Rule R as defined in Example 1 satisfies* equal treatment of equals, efficiency, *and* claims monotonicity.

Proof. *Equal treatment of equals* follows immediately from the definition of rule R.

Efficiency: Note that for all $c \in \cup_{j=0}^{4} C^j$, $c_N \geq 1$ and $\sum R_l(c) = 1$. Assume that $c \in P$. If $c_N \leq 1$, then $R(c) = c$. Finally, if $c_N > 1$, then $\sum R_l(c) = \sum_{i\in B(c)} R_i(c) + \sum_{i\in D(c)} R_i(c) + \sum_{i\in N\setminus A(c)} R_i(c) = c_{B(c)} + \sum_{i\in D(c)} \frac{1-c_{B(c)}}{|D(c)|} + 0 = 1$.

Claims Monotonicity: Let $i \in N$, $c = (c_i, c_{-i})$, and $c' = (c'_i, c_{-i})$ such that $c_i < c'_i$. We show that $R_i(c) \leq R_i(c')$ for the following (exhaustive) cases.

Case 1: $c, c' \in P$.

If $i \notin A(c)$, then $R_i(c) = 0 \leq R_i(c')$. If $i \in A(c)$, then $i \in A(c')$, i.e., $i \in B(c')$ or $i \in D(c')$. If $i \in B(c')$, then $R_i(c) \leq c_i \leq c'_i = R_i(c')$ and we are done. Assume that $i \in D(c')$. Since $c_i < c'_i$, for all $j \in N \setminus B(c)$, $c_j < c'_i$. Hence, $A(c') \setminus \{i\} \subseteq B(c)$. Therefore, for all $j \in B(c')$, $R_j(c') = R_j(c)$, and for all $j \in D(c') \setminus \{i\}$, $R_j(c') \leq R_j(c)$. Thus, we showed that for all $j \in A(c') \setminus \{i\}$, $R_j(c') \leq R_j(c)$. Since for all $j \in N \setminus A(c')$, $R_j(c') = 0$ and R is *efficient* it follows that $R_i(c) \leq R_i(c')$.

Case 2: $c \in P$ and for some $j \in \{0, 1, 2, 3, 4\}$, $c' \in C^j$.

Note that $L(c) = \{i\}$ and $LH(c) \cup H(c) = N \setminus \{i\}$. Since $L \geq 1/3$, $c_{N\setminus\{i\}} \geq 1$ and $\sum_{l\neq i} R_l(c) = 1$. Thus, $R_i(c) = 1 - \sum_{l\neq i} R_l(c) = 0 \leq R_i(c')$.

Case 3: for some $j \in \{0, 1, 2, 3, 4\}$, $c \in C^j$ and $c' \in C^j$.

Note that either $[i \in LH(c)$ and $i \in LH(c')]$ or $[i \in H(c)$ and $i \in H(c')]$. Thus, $R_i(c) = R_i(c')$.

Case 4: for some $j \in \{1, 2, 3, 4\}$, $c \in C^j$ and $c' \in C^{j-1}$.

Note that $i \in LH(c)$ and $i \in H(c')$. Then, by the definition of R, for $j = 1, 4$, $R_i(c) < R_i(c')$ and for $j = 2, 3$, $R_i(c) = R_i(c')$. □

Appendix B

In this appendix we describe what happens if we require that the estate E is always completely allocated among the agents. Formally, a *(full division) rule* is a function $R : C \to \mathbb{R}^N_+$ that associates with each claims vector $c \in C$ an *awards vector* $x \in \mathbb{R}^N_+$ such that $\sum x_i = E$.

Note that Lemma 1 does not hold anymore, i.e., it is not always the case that in every Nash equilibrium c of the claim game $\Gamma(R)$, $c_N \geq E$; e.g., for the constant rule that always assigns E/n to each agent, every claims vector is a Nash equilibrium. Although this lemma is used in some of our proofs, it is used only in order to show that in equilibrium the entire estate is allocated. Hence, the fact that the whole estate is always allocated can be used instead of Lemma 1. Furthermore, all results that state that the division vector in a Nash equilibrium is $\min\{k, \frac{E}{n}\}\mathbf{1}$ are changed to have the $\frac{E}{n}\mathbf{1}$ division vector. To summarize, Theorems 1–3 hold with minimal changes in the statements and proofs. Finally, the only adjustment of Example 1 needed to fit the model described here is to change rule R in Example 1 to rule $\tilde{R}$ as follows: for every $c \notin P$ let $\tilde{R}(c) = R(c)$ and for every $c \in P$ let $\tilde{R}(c) = \frac{E}{4}$.

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