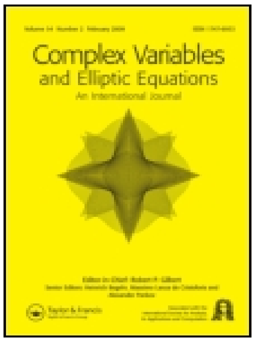


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# Quaternionic Hilbert spaces and a von Neumann inequality

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# Quaternionic Hilbert spaces and a von Neumann inequality

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We show that Drury's proof of the generalisation of the von Neumann inequality to the case of contractive rows of $N$-tuples of commuting operators still holds in the quaternionic case. The arguments require a seemingly new result on tensor products of quaternionic Hilbert spaces.

**Keywords:** von Neumann inequality; Drury–Arveson space; quaternionic Hilbert spaces; reproducing kernel Hilbert spaces; tensor products

**AMS Subject Classifications:** Primary 47A60; Secondary 46A32; 47B32; 47S10



## 1. Introduction

In 1978, Drury extended von Neumann inequality to the case of contractive rows of $N$-tuples of commuting operators [1]. Such an extension was done by Arveson as well in [2, Theorem 8.1]; see also [3]. To state their result, we first consider the reproducing kernel Hilbert space $\mathcal{A}$ with reproducing kernel $(1-\langle z,w\rangle)^{-1}$, where $z$ and $w$ belong to the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$ and $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{C}^N$. This space is often called the Drury–Arveson space. Letting $\omega_\alpha=|\alpha|!/\alpha!$, it can also be described as

$$\mathcal{A}=\left\{f(z)=\sum_{\alpha\in\mathbb{N}^N} z^\alpha f_\alpha : \|f\|_{\mathcal{A}}^2:=\sum_{\alpha\in\mathbb{N}^N}\frac{|f_\alpha|^2}{\omega_\alpha}<\infty\right\}.$$

We have used above the usual multi-index notation in which $\alpha!=\alpha_1!\cdots\alpha_N!$ and $|\alpha|=\alpha_1+\cdots+\alpha_N$ for $\alpha=(\alpha_1,\ldots,\alpha_N)\in\mathbb{N}^N$. Further, for two multi-indices $\alpha$ and $\beta$, we write $\alpha\geq\beta$ if $\alpha_\ell\geq\beta_\ell$ for all $\ell=1,\ldots,N$.

Next let $e^\ell$ denote the $N$-row vector with all entries 0 with the exception of the $\ell$-th which is 1, and let the backward shift operators $R_\ell$ be defined by

$$R_\ell f(z)=\sum_{\alpha\in\mathbb{N}^N} z^{\alpha-e^\ell}\frac{\alpha_\ell}{|\alpha|}f_\alpha \tag{1}$$

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for $\ell = 1, \ldots, N$; if $\alpha_\ell = 0$, we set $z^{\alpha - e^\ell} \frac{\alpha_\ell}{|\alpha|} = 0$ in the above sum. These operators are bounded on $\mathcal{A}$ and mutually commute. Moreover,

$$(R_\ell^* f)(z) = z_\ell f(z) =: (M_\ell f)(z),$$

which define the forward shift operators $M_\ell$, and

$$\sum_{\ell=1}^{N} R_\ell^* R_\ell = I_\mathcal{A} - CC^* \leq I_\mathcal{A}, \quad \text{where} \quad Cf = f(0),$$

$I$ is the identity operator, and for operators $A$, $B$ on a Hilbert space $\mathcal{H}$, the equivalent expressions $A \leq B$ and $B - A \geq 0$ denote that $B - A$ is positive in the sense that $\langle (B-A)h, h\rangle_\mathcal{H} \geq 0$ for all $h \in \mathcal{H}$ (see [2] for a proof of these facts). They can also be found in a number of later publications (see, e.g. [4]). We can now state the Drury–Arveson result.

Theorem 1.1 *Let $A_1, \ldots, A_N$ be bounded mutually commuting operators on a Hilbert space $\mathcal{H}$ such that $\sum_{\ell=1}^N A_\ell^* A_\ell \leq I_\mathcal{H}$. Then for every polynomial $Q(z)$ with complex coefficients, we have*

$$\|Q(A_1, \ldots, A_N)\|_\mathcal{H} \leq \|Q(M_1, \ldots, M_N)\|_\mathcal{A}.$$

The counterparts of the space $\mathcal{A}$ and of the operators $R_\ell$ have been recently introduced in the quaternionic setting [5,6], and the purpose of this article is to prove a von Neumann inequality in that setting. The lack of commutativity of the quaternions forces the choice of polynomials to be with real coefficients (Theorem 2.1).

## 2. The quaternionic Drury–Arveson space and the statement of the main theorem

We first briefly review the setting of hyperanalytic functions and the results of [5,6]. We denote by $\mathbb{H}$ the skew-field of real quaternions

$$\mathbb{H} = \left\{ x = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 : (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \right\},$$

where the units $\mathbf{e}_j$ satisfy the Cayley multiplication table [6, Section 2.1]. We also let $\overline{x} = x_0 - x_1\mathbf{e}_1 - x_2\mathbf{e}_2 - x_3\mathbf{e}_3$ and $|x|^2 = x\overline{x} = \overline{x}x$. A function $f$ defined on an open set $\Omega \subset \mathbb{R}^4$ is called left-hyperanalytic (we will simply say hyperanalytic) if

$$\frac{\partial}{\partial x_0} f + \mathbf{e}_1 \frac{\partial}{\partial x_1} f + \mathbf{e}_2 \frac{\partial}{\partial x_2} f + \mathbf{e}_3 \frac{\partial}{\partial x_3} f = 0$$

holds in $\Omega$. We use the terms hyperanalytic and hyperholomophic interchangeably.

The quaternionic variable $x$ is not hyperanalytic, nor is, in general, the product of two hyperanalytic functions. The functions $\zeta_\ell(x) = x_\ell - \mathbf{e}_\ell x_0$, $\ell = 1, 2, 3$, are hyperanalytic and they form the building blocks of the hyperanalytic polynomials; note that they do not commute and that they are not independent variables in the sense that they all depend on $x_0$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, let

$$\zeta^\alpha(x) = \zeta_1(x)^{\times\alpha_1} \times \zeta_2(x)^{\times\alpha_2} \times \zeta_3(x)^{\times\alpha_3},$$

where the symmetrized product of $a_1, \ldots, a_n \in \mathbb{H}$ is defined by

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}, \tag{2}$$

in which $S_n$ is the set of all permutations of the set $\{1, \ldots, n\}$ and where $\zeta_1(x)^{\times \alpha_1}$ means that the term $\zeta_1(x)$ appears $\alpha_1$ times among the $a_j$ in (2).

The Cauchy–Kovalevskaya product $\odot$ is an associative (but not commutative) product defined originally by Sommen in [7], which associates to two hyperanalytic functions another hyperanalytic function. It is different from the pointwise product which, in general, does not yield a hyperanalytic function, and is defined using the Cauchy–Kovalevskaya extension theorem. We will not review this aspect here, but mention that

$$(\zeta^\alpha p) \odot (\zeta^\beta q) = \zeta^{\alpha+\beta} pq, \quad \alpha, \beta \in \mathbb{N}^3, \quad p, q \in \mathbb{H}.$$

The quaternionic Drury–Arveson space is defined as the set

$$\mathcal{A}_{\mathbb{H}} = \left\{ f(x) = \sum_{\alpha \in \mathbb{N}^3} \zeta^\alpha(x) f_\alpha : f_\alpha \in \mathbb{H},\ \|f\|^2_{\mathcal{A}_{\mathbb{H}}} := \sum_{\alpha \in \mathbb{N}^3} \frac{|f_\alpha|^2}{\omega_\alpha} < \infty \right\}$$

[5, Definition 1.1]. It is a right quaternionic Hilbert space (definitions are recalled in Section 3) with the inner product

$$\langle f, g \rangle = \sum_{\alpha \in \mathbb{N}^3} \frac{1}{\omega_\alpha} \overline{g_\alpha} f_\alpha \quad \text{with} \quad g(x) = \sum_{\alpha \in \mathbb{N}^3} \zeta^\alpha(x) g_\alpha.$$

Its elements are hyperanalytic in the ellipsoid

$$\mathcal{E} = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1 \right\}.$$

It is the reproducing kernel (right quaternionic) Hilbert space with reproducing kernel

$$k(x, y) = (1 - \zeta_1(x)\overline{\zeta_1(y)} - \zeta_2(x)\overline{\zeta_2(y)} - \zeta_3(x)\overline{\zeta_3(y)})^{-\odot},$$

where $-\odot$ denotes the inverse with respect to the Cauchy–Kovalevskaya product; see [6, Proposition 2.10]. This means that

$$\langle f, k(\cdot, y) p \rangle_{\mathcal{A}_{\mathbb{H}}} = \overline{p} f(y), \quad p \in \mathbb{H}, \quad y \in \mathcal{E}.$$

We need to introduce the backward shift operators on $\mathcal{A}_{\mathbb{H}}$. We set

$$R_\ell f(x) = \sum_{\alpha \in \mathbb{N}^3} \zeta^{\alpha - e^\ell}(x) \frac{\alpha_\ell}{|\alpha|} f_\alpha,$$

where $e^1 = (1, 0, 0)$, $e^2 = (0, 1, 0)$, and $e^3 = (0, 0, 1)$. The operators $R_\ell$ are right linear and bounded on $\mathcal{A}_{\mathbb{H}}$ and satisfy

$$R_1^* R_1 + R_2^* R_2 + R_3^* R_3 = I_{\mathcal{A}_{\mathbb{H}}} - C^* C, \quad \text{where} \quad Cf = f(0).$$

Moreover, their adjoints are the operators of the Cauchy–Kovalevskaya product by the $\zeta_\ell$, that is,

$$R_\ell^* f = M_\ell f = \zeta_\ell \odot f = f \odot \zeta_\ell.$$

The adjoint of a right linear operator $A$ on a right quaternionic Hilbert space $\mathcal{H}$ is the right linear operator $A^*$ satisfying

$$\langle Ah_1, h_2\rangle_{\mathcal{H}} = \langle h_1, A^* h_2\rangle_{\mathcal{H}}, \quad h_1, h_2 \in \mathcal{H}.$$

If $A = A^*$, $A$ is called self-adjoint.

We now quote our main theorem.

THEOREM 2.1 *Let $\mathcal{H}$ be a right quaternionic Hilbert space and let $A_1, A_2, A_3$ be pairwise commuting right linear bounded operators on $\mathcal{H}$ such that*

$$A_1^* A_1 + A_2^* A_2 + A_3^* A_3 \le I_{\mathcal{H}}.$$

*Then for every hyperholomorphic polynomial $Q(x)$ with real coefficients, it holds that*

$$\|Q(A_1, A_2, A_3)\| \le \|Q(M_1, M_2, M_3)\|.$$

Let us make a number of comments on this result. The polynomials in the theorem have real coefficients because the Hilbert space $\mathcal{H}$ is a right Hilbert space. For $q \in \mathbb{H}$ and $T$ a right linear operator, the operator $qT$ makes sense only if $q$ is real, and the operator $Tq$ is right linear only if $q$ is real.

In the proof of Theorem 2.1, we need the following result on square roots, which is well known and has a number of proofs in the case of complex Hilbert spaces. As in this latter case, the proof for the quaternionic case is based on the power series expansion of $\sqrt{1-z}$ in the open unit disk, and we omit it.

LEMMA 2.2 *Let $T$ be a strictly contractive self-adjoint right linear operator on a right quaternionic Hilbert space $\mathcal{H}$. Then the operator*

$$D = I - \frac{1}{2}T + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}T^2 - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}T^3 + \cdots$$

*is a self-adjoint contractive right linear operator on $\mathcal{H}$ satisfying $D^2 = I - T$.*



## 3. Tensor products of quaternionic Hilbert spaces

Tensor products of quaternionic Hilbert spaces do not seem to have been much studied (see [8,9] for some results). The difficulty is the noncommutativity of the quaternions. To make our point, let us go back to the basic definitions. Let $R$ be a ring. Recall that if $\mathcal{G}$ is a right module over $R$ and $\mathcal{H}$ is a left module over $R$, the tensor product $\mathcal{G} \otimes_R \mathcal{H}$ is merely a group [10, p. 208]. To get more structure, we need, for example, $\mathcal{H}$ to be a two-sided $R$-module. Then the following result holds ([10, Theorem 5.5(iii)] or [11, Section 3]). If $\mathcal{H}$ is a two-sided $R$-module and $\mathcal{G}$ is a right $R$-module, then the tensor product $\mathcal{G} \otimes \mathcal{H}$ is a right $R$-module. Moreover,

$$(g \otimes h)r = (g \otimes hr) \quad \text{and} \quad (gp \otimes h) = (g \otimes ph), \quad h \in \mathcal{H},\ g \in \mathcal{G},\ r \in R.$$

We refer to [12] for information and references on quaternionic Hilbert spaces. We recall the following definition (see, e.g. [12, Definition 5.5] and the references therein). A right quaternionic pre-Hilbert space $\mathcal{G}$ is a right vector space on $\mathbb{H}$ endowed with an $\mathbb{H}$-valued form $\langle\cdot,\cdot\rangle$ that has the following properties:

(1) it is Hermitian: $\langle f,g\rangle = \langle \overline{g,f}\rangle, \quad f,g\in\mathcal{G}$;
(2) it is positive: $\langle f,f\rangle \geq 0, \quad f\in\mathcal{G}$;
(3) it is nondegenenate: $\langle f,f\rangle = 0 \Leftrightarrow f=0$;
(4) it is linear in the sense that $\langle fp,gq\rangle = \overline{q}\langle f,g\rangle p, \quad f,g\in\mathcal{G}, \quad p,q\in\mathbb{H}$.

The space $\mathcal{G}$ is a right quaternionic Hilbert space if it is complete with respect to the topology defined by the norm $\|f\| = \sqrt{\langle f,f\rangle}$.

Throughout this article, the notation $\mathcal{G}\otimes_{\mathbb{H}}\mathcal{H}$ is used for the topological tensor product of the quaternionic Hilbert spaces $\mathcal{G}$ and $\mathcal{H}$.

Theorem 3.1 *Let $\mathcal{H}$ be a separable two-sided quaternionic Hilbert space and let $\mathcal{G}$ be a separable right quaternionic Hilbert space. Then the tensor product $\mathcal{G}\otimes_{\mathbb{H}}\mathcal{H}$ endowed with the inner product*

$$\langle g_1\otimes h_1, g_2\otimes h_2\rangle_{\mathcal{G}\otimes_{\mathbb{H}}\mathcal{H}} = \langle(\langle g_1,g_2\rangle_{\mathcal{G}})h_1,h_2\rangle_{\mathcal{H}} \tag{3}$$

*is a right quaternionic Hilbert space.*

*Proof* Let $(h_i)$ (resp. $(g_i)$) be an orthonormal basis of $\mathcal{H}$ (resp. $\mathcal{G}$). Then (3) gives

$$\langle g_{i_1}\otimes h_{j_1}p, g_{i_2}\otimes h_{j_2}q\rangle_{\mathcal{G}\otimes_{\mathbb{H}}\mathcal{H}} = \langle\langle g_{j_1},g_{j_2}\rangle_{\mathcal{G}}h_{i_1}p,h_{i_2}q\rangle_{\mathcal{H}} = \begin{cases}\overline{q}\,p, & \text{if } (i_1,j_1)=(i_2,j_2);\\ 0, & \text{otherwise.}\end{cases}$$

It follows that the right span of the elements of the form $g_i\otimes h_j$ endowed with the inner product (3) is a right quaternionic pre-Hilbert space. Then the set of elements of the form $\sum_{i,j} g_i\otimes f_j c_{ij}$, where the $c_{ij}\in\mathbb{H}$ are such that

$$\sum_{i,j}|c_{ij}|^2 < \infty$$

is a right quaternionic Hilbert space. ■

## 4. Proof of the main theorem

We follow Drury's argument appropriately adapted to the quaternionic case. As in [1], it is convenient to work with the sequence space

$$\ell_2(\mathbb{N}^3,\omega,\mathbb{H}) = \left\{(f_\alpha)_{\alpha\in\mathbb{N}^3} : f_\alpha\in\mathbb{H},\ \sum_{\alpha\in\mathbb{N}^3}\omega_\alpha|f_\alpha|^2<\infty\right\}$$

rather than $\mathcal{A}_{\mathbb{H}}$, where $\omega$ denotes the sequence $(\omega_\alpha)$. We endow $\ell_2(\mathbb{N}^3,\omega,\mathbb{H})$ with the inner product

$$\langle(f_\alpha),(g_\alpha)\rangle_{\ell_2(\mathbb{N}^3,\omega,\mathbb{H})} = \sum_{\alpha\in\mathbb{N}^3}\omega_\alpha\overline{g_\alpha}f_\alpha.$$

The operators

$$(S_\ell f)_\alpha = f_{\alpha+e^\ell}, \quad \ell = 1, 2, 3,$$

are right linear and bounded on $\ell_2(\mathbb{N}^3, \omega, \mathbb{H})$, they commute pairwise, and their adjoints are given by

$$(S_\ell^* f)_\alpha = \frac{\alpha_\ell}{|\alpha|} f_{\alpha-e^\ell}, \quad \ell = 1, 2, 3.$$

Moreover,

$$S_1^* S_1 + S_2^* S_2 + S_3^* S_3 \le I_{\ell_2(\mathbb{N}^3, \omega, \mathbb{H})}.$$

These facts are proved as in the case of complex sequences and we omit their proofs here.

We first prove the theorem with $S_1, S_2, S_3$ in place of $M_1, M_2, M_3$. Let $A_1, A_2, A_3$ be as in the theorem. The operator

$$I_{\mathcal{H}} - A_1^* A_1 - A_2^* A_2 - A_3^* A_3 \tag{4}$$

is positive. We first assume that

$$r^2 I_{\mathcal{H}} - A_1^* A_1 - A_2^* A_2 - A_3^* A_3 \ge 0$$

for some $r \in (0, 1)$. The operator (4) is then strictly positive and we can apply Lemma 2.2 to define a positive operator $D$ such that

$$D^2 = I_{\mathcal{H}} - A_1^* A_1 - A_2^* A_2 - A_3^* A_3.$$

We denote by $\widehat{\mathcal{H}}$ the Hilbert space $\mathcal{H}$ endowed with the norm

$$\|h\|_{\widehat{\mathcal{H}}} = \|Dh\|_{\mathcal{H}}.$$

Furthermore, we set

$$\widetilde{\mathcal{H}} = \ell_2(\mathbb{N}^3, \omega, \widehat{\mathcal{H}});$$

this is similar to $\ell_2(\mathbb{N}^3, \omega, \mathbb{H})$, but the terms of the sequences are members of $\widehat{\mathcal{H}}$, and the norm of $\widehat{\mathcal{H}}$ replaces the $|\cdot|$ of $\mathbb{H}$. Theorem 3.1 is used to prove the next auxiliary result, which of course is well-known in the complex case.

Proposition 4.1 *The map*

$$\tau(h \otimes \xi) = (h\xi)$$

*is well defined and extends to a right linear unitary map between the right quaternionic Hilbert spaces* $\widehat{\mathcal{H}} \otimes \ell_2(\mathbb{N}^3, \omega, \mathbb{H})$ *and* $\widetilde{\mathcal{H}}$.

*Proof* Let $h \otimes \xi$ be an elementary tensor and let $q \in \mathbb{H}$. Then

$$\tau((h \otimes \xi)q) = \tau(h \otimes (\xi q)) = h\xi q = (h\xi)q = (\tau(h \otimes \xi))q,$$

and the right linearity of $\tau$ follows.

For $k = 1, 2$, let $h_k \in \widehat{\mathcal{H}}$ and $\xi^k = (\xi^k_\alpha)$, where the sequences have only a finite number of nonzero entries. Then

$$\begin{aligned}\langle \tau(h_1 \otimes \xi^1), \tau(h_2 \otimes \xi^2)\rangle_{\widehat{\mathcal{H}} \otimes \ell_2(\mathbb{N}^3, \omega, \mathbb{H})} &= \langle h_1\xi^1, h_2\xi^2\rangle_{\ell_2(\mathbb{N}^3, \omega, \widehat{\mathcal{H}})}\\ &= \sum_\alpha \omega(\alpha)\langle h_1\xi^1_\alpha, h_2\xi^2_\alpha\rangle_{\widehat{\mathcal{H}}}\\ &= \sum_\alpha \omega(\alpha)\overline{\xi^2_\alpha}\langle h_1, h_2\rangle_{\widehat{\mathcal{H}}}\xi^1_\alpha\\ &= \langle \langle h_1, h_2\rangle_{\widehat{\mathcal{H}}}\xi^1, \xi^2\rangle_{\ell_2(\mathbb{N}^3, \omega, \mathbb{H})}\\ &= \langle h_1 \otimes \xi^1, h_2 \otimes \xi^2\rangle_{\widehat{\mathcal{H}} \otimes \ell_2(\mathbb{N}^3, \omega, \mathbb{H})},\end{aligned}$$

and hence $\tau$ is an isometry. We now compute the adjoint of $\tau$. We denote by $(\widehat{h_i})$ an orthonormal basis of $\widehat{\mathcal{H}}$. Let $H = (H_\alpha) \in \widetilde{\mathcal{H}}$. We prove that

$$\tau^*(H) = \sum_i \widehat{h_i} \otimes (\langle H_\alpha, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}})_\alpha.$$

We have

$$\begin{aligned}\langle \tau^*(H), h \otimes \xi\rangle_{\widehat{\mathcal{H}} \otimes \ell_2(\mathbb{N}^3, \omega, \mathbb{H})} &= \langle H, h\xi\rangle_{\ell_2(\mathbb{N}^3, \omega, \widehat{\mathcal{H}})} = \sum_\alpha \omega(\alpha)\langle H_\alpha, h\xi_\alpha\rangle_{\widehat{\mathcal{H}}}\\ &= \sum_\alpha \Big\langle (\langle H_\alpha, h\rangle_{\widehat{\mathcal{H}}})_\alpha, \xi\Big\rangle_{\ell_2(\mathbb{N}^3, \omega, \mathbb{H})}\end{aligned}$$

on the one hand, and

$$\begin{aligned}\left\langle \sum_i \widehat{h_i} \otimes (\langle H_\alpha, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}})_\alpha, h \otimes \xi\right\rangle_{\widehat{\mathcal{H}} \otimes \ell_2(\mathbb{N}^3, \omega, \mathbb{H})} &= \sum_i \Big\langle (\langle H_\alpha, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}}\langle \widehat{h_i}, h\rangle_{\widehat{\mathcal{H}}})_\alpha, \xi\Big\rangle_{\ell_2(\mathbb{N}^3, \omega, \mathbb{H})}\\ &= \Big\langle (\langle H_\alpha, h\rangle_{\widehat{\mathcal{H}}})_\alpha, \xi\Big\rangle_{\ell_2(\mathbb{N}^3, \omega, \mathbb{H})}\end{aligned}$$

on the other hand, and the claim on the adjoint follows.

We can now prove that $\tau$ is unitary.

$$\tau\tau^*(H) = \tau\left(\sum_i \widehat{h_i} \otimes (\langle H_\alpha, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}})_\alpha\right) = \sum_i \widehat{h_i}\big(\langle H_\alpha, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}}\big)_\alpha = (H_\alpha)$$

and

$$\begin{aligned}\tau^*\tau(h \otimes \xi) = \tau^*(h\xi) &= \sum_i \widehat{h_i} \otimes \left(\big\langle (h\xi)_\alpha, \widehat{h_i}\big\rangle_{\widehat{\mathcal{H}}}\right)\\ &= \sum_i \widehat{h_i} \otimes \langle h, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}}\xi = \left(\sum_i \widehat{h_i}\langle h, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}}\right) \otimes \xi = h \otimes \xi,\end{aligned}$$

since $(h \otimes p\xi) = (hp) \otimes \xi$. ■

We now conclude the proof of the theorem in a number of steps. The proof of Step 1 is the same as in the complex case and will be omitted.

**Step 1:** *The map* $\Theta$ *defined by*

$$(\Theta(h))_\alpha = A^\alpha h, \quad h \in \mathcal{H},$$

*is an isometry from* $\mathcal{H}$ *into* $\ell_2(\mathbb{N}^3, \omega, \widehat{\mathcal{H}})$.

We define two other kinds of shifts. The map $S^\mu$ carries a sequence $(h_\alpha) \in \widehat{\mathcal{H}}$ to $(h_{\alpha+\mu})$. The map $\widetilde{S}^\mu$ has the same action on a sequence $(H_\alpha) \in \widetilde{\mathcal{H}}$.

**Step 2:** *For* $\mu \in \mathbb{N}^3$, *it holds that* $\widetilde{S}^\mu \tau = \tau(I_{\widehat{\mathcal{H}}} \otimes S^\mu)$.

Indeed, let $h \otimes (h_\alpha)$ be an elementary tensor in $\widehat{\mathcal{H}} \otimes \ell_2(\mathbb{N}^3, \omega, \mathbb{H})$. Then

$$\begin{aligned}\widetilde{S}^\mu \tau(h \otimes (h_\alpha)) &= \widetilde{S}^\mu (h h_\alpha) = (h h_{\alpha+\mu}) = \tau((h \otimes h_{\alpha+\mu})) = \tau(h \otimes S^\mu(h_\alpha)) \\ &= \big(\tau(I_{\widehat{\mathcal{H}}} \otimes S^\mu)\big)(h \otimes (h_\alpha)).\end{aligned}$$

**Step 3:** *For* $\mu \in \mathbb{N}^3$, *it holds that* $\widetilde{S}^\mu \Theta = \Theta A^\mu$, *and hence*

$$(I_{\widehat{\mathcal{H}}} \otimes S^\mu)(\tau^* \Theta) = (\tau^* \Theta) A^\mu \quad \text{and} \quad (I_{\widehat{\mathcal{H}}} \otimes Q(S))(\tau^* \Theta) = (\tau^* \Theta) Q(A), \tag{5}$$

*where* $Q(x_1, x_2, x_3) = \sum_\mu x^\mu q_\mu$ *is a polynomial with real coefficients.*

Indeed, for $h \in \widehat{\mathcal{H}}$, we have

$$\begin{aligned}\tau\Big(I_{\widehat{\mathcal{H}}} \otimes S^\mu\Big)\tau^* \Theta h &= \tau\Big(I_{\widehat{\mathcal{H}}} \otimes S^\mu\Big)\tau^*(A^\alpha h) \\ &= \tau\Big(I_{\widehat{\mathcal{H}}} \otimes S^\mu\Big)\bigg(\sum_i \widehat{h_i} \otimes \Big(\langle A^\alpha h, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}}\Big)_\alpha\bigg) \\ &= \tau\bigg(\sum_i \widehat{h_i} \otimes \Big(\langle A^{\alpha+\mu} h, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}}\Big)_\alpha\bigg) \\ &= \sum_i \widehat{h_i}\Big(\langle A^{\alpha+\mu} h, \widehat{h_i}\rangle_{\widehat{\mathcal{H}}}\Big)_\alpha = (A^{\alpha+\mu} h).\end{aligned}$$

Then the result also holds for any finite linear combination $\sum_\mu x^\mu q_\mu$ with real $q_\mu$.

The proof of the next step is a direct consequence of (5).

**Step 4:** *von Neumann inequality holds with the operators* $S_1, S_2, S_3$.

**Step 5:** *von Neumann inequality holds with the operators* $M_1, M_2, M_3$.

Indeed the map $T$ defined by

$$T(f_\alpha) = \sum_{\alpha \in \mathbb{N}^3} \zeta^\alpha(x) f_\alpha \omega(\alpha)$$

is an isomorphism of right quaternionic Hilbert spaces from $\ell_2(\mathbb{N}^3, \omega, \mathbb{H})$ onto $\mathcal{A}_{\mathbb{H}}$, and its adjoint is given by

$$T^* f = \left(\frac{f_\alpha}{\omega_\alpha}\right)_\alpha \quad \text{with} \quad f(x) = \sum_\alpha \zeta^\alpha f_\alpha.$$

Further,

$$\begin{aligned}(T S_\ell T^*)(f) &= T S_\ell \left(\frac{f_\alpha}{\omega_\alpha}\right) = T\left(\left(\frac{f_{\alpha+e^\ell}}{w_{\alpha+e^\ell}}\right)\right) = \sum_\alpha \zeta^\alpha \frac{f_{\alpha+e^\ell}}{\omega_{\alpha+e^\ell}} \omega_\alpha \\ &= \sum_\alpha \zeta^\alpha f_{\alpha+e^\ell} \frac{\alpha_\ell + 1}{|\alpha| + 1} = \sum_{\alpha \geq e^\ell} \zeta^{\alpha - e^\ell}(x) f_\alpha \frac{\alpha_\ell}{|\alpha|}\end{aligned}$$

and

$$(TS_\ell^* T^*)(f) = TS_\ell^*\left(\frac{f_\alpha}{\omega_\alpha}\right) = T\left(\frac{f_{\alpha-e^\ell}}{\omega_{\alpha-e^\ell}}\frac{\alpha_\ell}{|\alpha|}\right) = \sum_{\alpha\geq e^\ell}\zeta^\alpha(x)\frac{f_{\alpha-e^\ell}}{\omega_{\alpha-e^\ell}}\frac{\alpha_\ell}{|\alpha|}$$

$$= \sum_{\alpha\geq e^\ell}\zeta^\alpha(x) f_{\alpha-e^\ell}\frac{\alpha_\ell}{|\alpha|}\frac{(\alpha-e^\ell)!}{(|\alpha|-1)!} = \zeta_\ell \odot f = M_\ell f. \tag{6}$$

Thus

$$\|Q(S_1,S_2,S_3)\| = \|Q(T(S_1,S_2,S_3)T^*)\| = \|Q(T(S_1,S_2,S_3)T^*)^*\|$$
$$= \|Q(T(S_1^*,S_2^*,S_3^*)T^*)\| = \|Q(M_1,M_2,M_3)\|,$$

where the next to last equality uses the fact that $Q$ has real coefficients, and the last equality uses (6).

It remains to let $r \to 1$ to conclude the proof.



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