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## Sensitivity Reduction by Strongly Stabilizing Controllers for MIMO Distributed Parameter Systems

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**Abstract**—This note investigates a sensitivity reduction problem by stable stabilizing controllers for a linear time-invariant multi-input multioutput distributed parameter system. The plant we consider has finitely many unstable zeros, which are simple and blocking, but can possess infinitely many unstable poles. We obtain a necessary condition and a sufficient condition for the solvability of the problem, using the matrix Nevanlinna-Pick interpolation with boundary conditions. We also develop a necessary and sufficient condition for the solvability of the interpolation problem, and show an algorithm to obtain the solutions. Our method to solve the interpolation problem is based on the Schur-Nevanlinna algorithm.

**Index Terms**—Distributed parameter systems,  $\mathcal{H}^\infty$ -control, strong stabilization.

### I. INTRODUCTION

In this note, we study the problem of finding stable controllers that stabilize a multi-input multioutput distributed parameter system while reducing, simultaneously, the sensitivity of the system. That is, the problem of *strong stabilization with sensitivity reduction*.

A background motivation for seeking stable controllers is that unstable poles of the controllers are known to lead to performance degradation in feedback systems under various performance objectives [1]–[3]. Moreover, stable controllers are also robust to sensor failures [4] and to plant nonlinearities [5]. Stable controllers have other theoretical or practical advantages, see, e.g., [1], [6], and the references therein.

For finite dimensional systems, various approaches have been developed for finding stable stabilizing controllers that achieve a desired  $\mathcal{H}^\infty$  performance level, see, e.g., [6]–[12] and their references. For infinite dimensional systems, some works have also been reported recently [13]–[15]. For example, [14] has extended the technique used in [8] to find strongly stabilizing controllers that lead to optimal  $\mathcal{H}^\infty$  sensitivity levels for a class of single-input single-output systems with time delays. In [16], it was shown that every stabilizable linear multi-input multi-output plant is strongly stabilizable. However, strong stabilization with sensitivity reduction for multi-input multioutput distributed parameter systems is largely open at present.

We generalize the method of [9] to a class of multi-input multi-output distributed parameter systems. The plants we consider have only finitely many unstable zeros, all of which are simple and blocking, but they are allowed to have infinitely many unstable poles. We obtain stable controllers for the sensitivity reduction problem, using the matrix Nevanlinna-Pick interpolation problem with boundary conditions. We also prove that the interpolation problem is solvable if and only if

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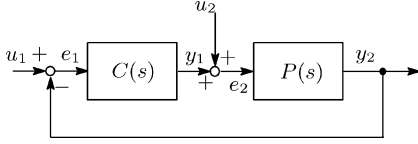


Fig. 1. Feedback system.

the Pick matrix consisting of the interior conditions is positive definite. To obtain solutions of this interpolation problem, we show an iterative algorithm similar to the well-known Schur-Nevanlinna algorithm [1].

The note is organized as follows: Section II gives the statement of the sensitivity reduction problem with stable controllers. In Section III, we reduce this problem to the interpolation with unimodular matrices in  $\mathcal{H}^\infty$  under some assumptions. We propose an algorithm for attaining low sensitivity by stable controllers in Section IV. The algorithm is based on the matrix Nevanlinna-Pick interpolation problem with boundary conditions, so we discuss the interpolation problem in Section V. We give a numerical example in Section VI, and conclusion in Section VII.

#### Notation

Let  $\mathbb{C}_+$  denote the open right half-plane  $\{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ , and let  $\bar{\mathbb{C}}_+$  be the closed right half-plane  $\{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$ .  $\mathcal{H}^\infty$  denotes the set of functions that are bounded and analytic in  $\mathbb{C}_+$ . We denote by  $\mathcal{F}^\infty$  the field of fractions of  $\mathcal{H}^\infty$ .  $\mathbf{M}(R)$  is used as a generic symbol to denote the set of matrices with elements in a commutative ring  $R$ , of whatever size. When it is necessary to show explicitly the size of a matrix, the notation  $G \in R^{p \times q}$  is used to indicate that  $G$  is a  $p \times q$  matrix with entries in  $R$ . For a complex matrix  $M$ , its conjugate transpose is denoted by  $M^*$ . For  $H \in \mathbf{M}(\mathcal{H}^\infty)$ , the  $\mathcal{H}^\infty$  norm is defined as  $\|H\|_\infty := \sup_{s \in \bar{\mathbb{C}}_+} \|H(s)\|$ , where  $\|M\|$  denotes the maximum singular value of the matrix  $M$ .

## II. PROBLEM STATEMENT

Consider the linear, continuous-time, time-invariant feedback system given in Fig. 1. Let plant  $P$  and controller  $C$  belong to  $\mathbf{M}(\mathcal{F}^\infty)$ . The feedback system in Fig. 1 is *internally stable* if the transfer matrix  $H(P, C)$  from  $u_1, u_2$  to  $e_1, e_2$

$$H(P, C) = \begin{bmatrix} (I + PC)^{-1} & -(I + PC)^{-1}P \\ C(I + PC)^{-1} & I - C(I + PC)^{-1}P \end{bmatrix} \in \mathbf{M}(\mathcal{H}^\infty). \quad (\text{II.1})$$

We say that  $C$  *stabilizes*  $P$ , and  $P$  is *stabilizable* if the feedback system is internally stable. Let  $\mathcal{C}(P)$  represent the set of all controllers that stabilize  $P$ .  $P$  is *strongly stabilizable* if  $\mathcal{C}(P)$  contains a stable controller, that is,  $\mathbf{M}(\mathcal{H}^\infty) \cap \mathcal{C}(P) \neq \emptyset$ .

Our problem is stated as follows.

**Problem II.1:** Given  $P \in \mathbf{M}(\mathcal{F}^\infty)$ ,  $W_1, W_2 \in \mathbf{M}(\mathcal{H}^\infty)$  and  $\rho > 0$ , determine whether there exists a controller  $C \in \mathbf{M}(\mathcal{H}^\infty) \cap \mathcal{C}(P)$  such that

$$\|W_1(I + PC)^{-1}W_2\|_\infty < \rho. \quad (\text{II.2})$$

Also, if one exists, find such a controller  $C$ .

Our aim is to give a sufficient condition for the solvability of Problem II.1 under some assumptions. We also propose a design method for such a controller.

## III. STRONG STABILIZATION AND SENSITIVITY REDUCTION

In this section, we reduce strong stabilization to interpolation by unimodular matrices in  $\mathcal{H}^\infty$ , and we formulate an interpolation problem with an  $\mathcal{H}^\infty$  norm condition equivalent to Problem II.1 under some assumptions. The interpolation problem is similar to the matrix Nevanlinna-Pick interpolation problem, but the solution needs to be unimodular in  $\mathbf{M}(\mathcal{H}^\infty)$ .

Let us first study strong stabilization only. The following lemma gives a necessary and sufficient condition for strong stabilization:

**Lemma III.1:** Let  $P \in \mathbf{M}(\mathcal{F}^\infty)$  be stabilizable. Suppose that  $P$  has the form  $P = D^{-1}N$ , where  $D, N \in \mathbf{M}(\mathcal{H}^\infty)$  are strongly left coprime in the sense of [17], i.e., there exist  $X, Y \in \mathbf{M}(\mathcal{H}^\infty)$  such that

$$NX + DY = I. \quad (\text{III.1})$$

Then  $P$  is strongly stabilizable if and only if there exists a  $C \in \mathbf{M}(\mathcal{H}^\infty)$  such that

$$(D + NC)^{-1} \in \mathbf{M}(\mathcal{H}^\infty). \quad (\text{III.2})$$

*Proof:*

( $\Leftarrow$ ): We have

$$\begin{aligned} (I + PC)^{-1} &= (I + D^{-1}NC)^{-1} \\ &= (D^{-1}(D + NC))^{-1} = (D + NC)^{-1}D. \end{aligned}$$

Moreover

$$\begin{aligned} (I + PC)^{-1}P &= (D + NC)^{-1}N, \\ C(I + PC)^{-1} &= C(D + NC)^{-1}D, \\ C(I + PC)^{-1}P &= C(D + NC)^{-1}N. \end{aligned}$$

Since  $C, D, N$ , and  $(D + NC)^{-1}$  are in  $\mathbf{M}(\mathcal{H}^\infty)$ , we obtain (II.1). Hence  $P$  is strongly stabilizable.

( $\Rightarrow$ ): Since  $P$  is stabilizable,  $P$  admits a strongly right coprime factorization [17]:

$$P = \tilde{N}\tilde{D}^{-1}, \quad \tilde{N}, \tilde{D} \in \mathbf{M}(\mathcal{H}^\infty).$$

Moreover, (III.1) is satisfied for some  $X, Y \in \mathbf{M}(\mathcal{H}^\infty)$ . Then all controllers are of the form  $(X + \tilde{D}Q)(Y - \tilde{N}Q)^{-1}$ , where  $Q \in \mathbf{M}(\mathcal{H}^\infty)$  [17]. Since  $P$  is strongly stabilizable, there exists a  $Q_0 \in \mathbf{M}(\mathcal{H}^\infty)$  such that  $C = (X + \tilde{D}Q_0)(Y - \tilde{N}Q_0)^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ . Additionally, we have from (III.1)

$$\begin{aligned} D + NC &= D + N(X + \tilde{D}Q_0)(Y - \tilde{N}Q_0)^{-1} \\ &= (D(Y - \tilde{N}Q_0) + N(X + \tilde{D}Q_0))(Y - \tilde{N}Q_0)^{-1} \\ &= (Y - \tilde{N}Q_0)^{-1}. \end{aligned}$$

Hence we obtain  $(D + NC)^{-1} = Y - \tilde{N}Q_0 \in \mathbf{M}(\mathcal{H}^\infty)$ .  $\blacksquare$

Lemma III.1 suggests the following problem to find stable stabilizing controllers.

**Problem III.2:** Given  $D, N \in \mathbf{M}(\mathcal{H}^\infty)$ , find a  $C \in \mathbf{M}(\mathcal{H}^\infty)$  satisfying (III.2).

Under the following assumption on  $D$  and  $N$ , we can reduce Problem III.2 to an interpolation problem with unimodular matrices.

**Assumption III.3:**  $D, N \in \mathbf{M}(\mathcal{H}^\infty)$  are strongly left coprime.  $N$  is square and  $N$  has the form  $N = \phi N_o$ , where  $\phi \in \mathcal{H}^\infty$  and  $N_o, N_o^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ , and  $\phi$  is a nonzero rational function satisfying

$\phi(\infty) \neq 0$ , and possesses distinct zeros  $z_1, \dots, z_n$  in  $\bar{\mathcal{C}}_+$ . All elements of  $N_o$ ,  $D$ ,  $X$ , and  $Y$  in (III.1) are meromorphic functions.

Under Assumption III.3, we prove that Problem III.2 is equivalent to the following problem.

**Problem III.4:** Given  $z_1, \dots, z_n \in \bar{\mathcal{C}}_+$  and complex square matrices  $A_1, \dots, A_n$ , find a  $U \in \mathbf{M}(\mathcal{H}^\infty)$  satisfying  $U^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$  and

$$U(z_i) = A_i, \quad i = 1, \dots, n. \quad (\text{III.3})$$

We start with the following lemma:

**Lemma III.5:** Consider Problem III.2 under Assumption III.3. We restrict the solutions to matrices whose elements are meromorphic functions. Define  $A_i := D(z_i)$  for  $i = 1, \dots, n$ . Then Problem III.2 is equivalent to Problem III.4 with interpolation data  $\{z_i\}_{i=1}^n$  and  $\{A_i\}_{i=1}^n$ .

*Proof:* Let  $C$  be a solution of Problem III.2. Define  $U := D + NC$ . Then by (III.2)  $U$  satisfies  $U$ ,  $U^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$  and

$$U(z_i) = D(z_i) + \phi(z_i)N_o(z_i)C(z_i) = D(z_i) = A_i.$$

Hence  $U$  is a solution to Problem III.4.

Conversely, suppose that  $U$  is a solution to Problem III.4 with  $\{z_i\}_{i=1}^n$  and  $\{A_i\}_{i=1}^n$ . Define

$$C := \frac{1}{\phi} N_o^{-1}(U - D).$$

Then  $C$  satisfies  $(D + NC)^{-1} = U^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$  and  $\phi C = N_o^{-1}(U - D) \in \mathbf{M}(\mathcal{H}^\infty)$ . If  $C \notin \mathbf{M}(\mathcal{H}^\infty)$ , then  $C$  has some poles in  $\bar{\mathcal{C}}_+$  that are canceled by the zeros of  $\phi$ . Let  $z_k$  be one of such poles. Then we have  $N_o^{-1}(z_k)(U(z_k) - A_k) = (\phi C)(z_k) \neq 0$ , which contradicts (III.3). ■

Before proceeding to sensitivity reduction by strongly stabilizing controllers, we need to recall the definitions of co-inner and co-outer matrix functions.  $F \in \mathbf{M}(\mathcal{H}^\infty)$  is said to be *co-inner* if  $F(\bar{s})^*$  is inner. Similarly,  $G \in \mathbf{M}(\mathcal{H}^\infty)$  is said to be *co-outer* if  $G(\bar{s})^*$  is outer.

The following theorem shows that every function in  $\mathbf{M}(\mathcal{H}^\infty)$  admits a unique co-inner-outer factorization.

**Theorem III.6 ([18]):** Let  $H$  be in  $(\mathcal{H}^\infty)^{p \times q}$ .  $H$  admits a co-inner-outer factorization of the form  $H = GF$ , where  $G \in (\mathcal{H}^\infty)^{p \times r}$  is co-outer and  $F \in (\mathcal{H}^\infty)^{r \times q}$  is co-inner for some  $r$ .  $F$  and  $G$  are unique to within multiplication by a constant unitary matrix.

Let us next consider Problem II.1. We place the following additional assumption on  $W_1$ ,  $W_2$ , and  $D$ :

**Assumption III.7:** All elements of  $W_1$  and  $W_2$  are meromorphic functions.  $W_1$  is unimodular in  $\mathbf{M}(\mathcal{H}^\infty)$ . If we factorize  $DW_2$  in the form  $DW_2 = (DW_2)_{co} \cdot (DW_2)_{ci}$ , where  $(DW_2)_{co}$  is co-outer and  $(DW_2)_{ci}$  is co-inner, then  $(DW_2)_{co}$  is also unimodular in  $\mathbf{M}(\mathcal{H}^\infty)$ .

We can obtain a solution for Problem II.1 under Assumption III.3 and III.7, using a solution of the following problem.

**Problem III.8:** Suppose that  $z_1, \dots, z_n \in \bar{\mathcal{C}}_+$  are distinct, and that  $B_1, \dots, B_n$  are complex square matrices. Suppose also that  $\rho > 0$ . Find an  $F \in \mathbf{M}(\mathcal{H}^\infty)$  satisfying  $F^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ ,  $\|F\|_\infty < \rho$ , and  $F(z_i) = B_i$  for  $i = 1, \dots, n$ .

**Theorem III.9:** Consider Problem II.1. We assume that there exist  $D, N \in \mathbf{M}(\mathcal{H}^\infty)$  such that  $P = D^{-1}N$ . Let Assumptions III.3 and III.7 hold. Define

$$B_i := W_1(z_i)D(z_i)^{-1}(DW_2)_{co}(z_i), \quad i = 1, \dots, n.$$

If there exists a solution  $F$  of Problem III.8 with  $\{z_i\}_{i=1}^n$ ,  $\{B_i\}_{i=1}^n$ , and  $\rho$ , then

$$C := N^{-1}(DW_2)_{co}F^{-1}W_1 - P^{-1} \quad (\text{III.4})$$

gives a solution of Problem II.1.

*Proof:* First of all, we prove that  $D(z_i)$  is invertible for  $i = 1, \dots, n$ . Since  $\phi(z_i) = 0$ ,  $D(z_i)Y(z_i) = I$  follows by (III.1). Hence  $D(z_i)^{-1}$  exists and  $D(z_i)^{-1} = Y(z_i)$ .

Since

$$\begin{aligned} W_1(I + PC)^{-1}W_2 &= W_1(D + NC)^{-1}DW_2 \\ &= W_1(D + NC)^{-1}(DW_2)_{co} \cdot (DW_2)_{ci} \end{aligned}$$

defining  $F := W_1(D + NC)^{-1}(DW_2)_{co}$ , we have

$$\|W_1(I + PC)^{-1}W_2\|_\infty = \|F(DW_2)_{ci}\|_\infty = \|F\|_\infty. \quad (\text{III.5})$$

Suppose that there exists a solution  $F$  to Problem III.8 with  $\{z_i\}_{i=1}^n$ ,  $\{B_i\}_{i=1}^n$  and  $\rho$ . Then  $C$  in (III.4) satisfies (II.2) by (III.5) and  $C \in \mathbf{M}(\mathcal{H}^\infty) \cap \mathcal{C}(P)$  by Lemma III.1 and III.5. Hence  $C$  in (III.4) is a solution to Problem II.1. ■

The following corollary gives a necessary condition for the solvability of Problem II.1.

**Corollary III.10:** Consider Problem II.1 whose solutions are restricted to meromorphic matrix functions. Under the same hypotheses of Theorem III.9, suppose that Problem II.1 is solvable. Then there exists an  $F \in \mathbf{M}(\mathcal{H}^\infty)$  such that  $\|F\|_\infty < \rho$  and  $F(z_i) = B_i$  for  $i = 1, \dots, n$ .

*Proof:* Obvious from the proof of Theorem III.9. ■

At the end of this section, we discuss the assumption of  $\phi$  in Assumption III.3.

**Remark III.11:** For simplicity, we assume that the unstable zeros of  $\phi$  are distinct in Assumption III.3. However, even when they are not distinct, we can develop the result similar to Lemma III.5.

**Remark III.12:** If  $D$  is a matrix whose elements are rational, then we can allow  $\phi$  to be strictly proper. However, if  $D$  is not rational and if  $\phi$  is strictly proper, in the same way as [14], we should replace  $\phi$  with  $\phi_\varepsilon(s) = \phi(s)(1 + \varepsilon s)^m$ , where  $\varepsilon > 0$  and  $m$  is the relative degree of  $\phi$ . This makes sure that we do not have to deal with interpolation conditions at infinity, but this leads to an improper term like PD controllers in the controller.

**Remark III.13:** We assume that  $\phi$  is scalar, and then we reduce strong stabilization with sensitivity reduction to the *matrix* Nevanlinna-Pick interpolation. However, this assumption of  $\phi$  could be weakened at the cost of going to the *tangential* Nevanlinna-Pick interpolation [19]. Details will be reported in a future work.

#### IV. DESIGN OF STABLE CONTROLLERS ATTAINING LOW SENSITIVITY

In this section, we develop a design method of strongly stabilizing controllers, extending the technique of [9] to multi-input multioutput systems with time delays.

The design method is based on the following lemma.

**Lemma IV.1:** Suppose that  $G \in \mathbf{M}(\mathcal{H}^\infty)$  is square and that  $\|G\|_\infty < 1$ . Then, for every complex number  $\lambda \neq 0$

$$F := \frac{\lambda}{2}(G + I) \quad (\text{IV.1})$$

satisfies  $F, F^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$  and  $\|F\|_\infty < |\lambda|$ .

*Sketch of Proof:* We can easily prove this lemma by the small gain theorem and the triangle inequality, so we omit the proof. ■

We obtain the following theorem from Lemma IV.1.

**Theorem IV.2:** Consider Problem III.8. Let  $\lambda$  be a complex number satisfying  $|\lambda| = \rho$ . If  $G \in \mathbf{M}(\mathcal{H}^\infty)$  satisfies  $\|G\|_\infty < 1$  and

$$G(z_i) = \frac{2}{\lambda}B_i - I, \quad i = 1, \dots, n$$

then  $F$  defined by (IV.1) is a solution of Problem III.8.

*Proof:* Obvious from Lemma IV.1. ■

The problem of finding  $G$  in Theorem IV.2 and that of finding  $F$  in Corollary III.10 are a matrix Nevanlinna-Pick interpolation problem with boundary conditions. The interpolation problem is solvable if and only if the Pick matrix consisting of the interior conditions is positive definite. Moreover, we can obtain a solution to the interpolation problem. The details are given in the next section.

We construct a solution of Problem II.1 by the following algorithm.

*A Solution to Problem II.1:*

Step 1: Let  $\lambda \in \mathbb{C}$  satisfy  $|\lambda| = \rho$ . Let  $G(z_i)$  be defined as follows:

$$G(z_i) = \frac{2}{\lambda} W_1(z_i) D(z_i)^{-1} (D W_2)_{co}(z_i) - I, \quad i = 1, \dots, n.$$

Step 2: Solve the matrix Nevanlinna-Pick interpolation problem with boundary conditions of  $G$ .

Step 3: Calculate a solution of Problem III.8 by (IV.1).

Step 4: Compute a solution of Problem II.1 by (III.4).

## V. MATRIX NEVANLINNA-PICK INTERPOLATION PROBLEM

The matrix Nevanlinna-Pick interpolation was studied well in [1], [20], and many works related to the interpolation have been reported over the last several years. For example, a theory of the interpolation with complexity constraints has been developed in [21].

Our objective in this section is to show that the matrix Nevanlinna-Pick interpolation problem with boundary conditions is solvable if and only if the Pick matrix consisting of the interior conditions is positive definite. Another aim is to show an algorithm similar to the Schur-Nevanlinna algorithm [1] for obtaining the solutions.

Since the results in [1], [20] are developed for the unit disk  $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ , it is convenient to map the open right half plane onto the unit disk via the bilinear transformation

$$s \mapsto z = \frac{s-1}{s+1}.$$

That is, in this section, we define  $\mathcal{H}^\infty$  as the set of functions that are bounded and analytic in  $\mathbb{D}$ , and the  $\mathcal{H}^\infty$  norm is defined as  $\|H\|_\infty := \sup_{z \in \mathbb{D}} \|H(z)\|$  for  $H \in \mathbf{M}(\mathcal{H}^\infty)$ .

### A. Interpolating Interior Conditions

Let us first introduce the interpolation problem with interior conditions only. The problem is solved in [1], [20]. We here extend the approach of [1], [20], when we consider the interpolation problem with both interior and boundary conditions.

We give the statement of *the matrix Nevanlinna-Pick interpolation problem* as follows:

*Problem V.1 ([1], [20]):* Given distinct complex numbers  $\lambda_1, \dots, \lambda_n \in \mathbb{D}$  and complex matrices  $F_1, \dots, F_n$  satisfying  $\|F_i\| < 1$  for every  $i$ , find a  $\Phi \in \mathbf{M}(\mathcal{H}^\infty)$  satisfying  $\|\Phi\|_\infty < 1$  and  $\Phi(\lambda_i) = F_i$  for  $i = 1, \dots, n$ .

In what follows, we use the notation of the form  $(\lambda_1, \dots, \lambda_n; F_1, \dots, F_n)$  to indicate the interpolation data as above, i.e., associating values  $F_i$  at  $\lambda_i$ .

It is well known that Problem V.1 is solvable if and only if the associated Pick matrix is positive definite.

*Theorem V.2 ([1], [20]):* Consider Problem V.1. Define the block matrix

$$P := \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix} \quad (\text{V.1})$$

where

$$P_{kl} := \frac{1}{1 - \bar{\lambda}_k \lambda_l} (I - F_k^* F_l), \quad k, l = 1, \dots, n.$$

Then Problem V.1 is solvable if and only if  $P > 0$ .

Let  $\mathcal{B} := \{M \in \mathbb{C}^{p \times q} \mid \|M\| < 1\}$ . We need the following lemma when we construct an algorithm for obtaining solutions of the interpolation problem, and when we consider the problem with boundary conditions.

*Lemma V.3 ([1], [20]):* Let  $E \in \mathcal{B}$ . Define

$$A := (I - EE^*)^{-1/2}, \quad B := -(I - EE^*)^{-1/2} E \\ C := -(I - E^* E)^{-1/2} E^*, \quad D := (I - E^* E)^{-1/2}$$

where  $M^{1/2}$  denotes the Hermitian square root of  $M$ . Then the mapping

$$T_E : \mathcal{B} \rightarrow \mathcal{B} : X \mapsto (AX + B)(CX + D)^{-1} \quad (\text{V.2})$$

is well defined and bijective.

We obtain a solution of Problem V.1 with  $T_E$  in (V.2) by the following corollary.

*Corollary V.4 ([1], [20]):* Consider Problem V.1. Define

$$y(z) := \frac{|\lambda_1|(z - \lambda_1)}{\lambda_1(1 - \bar{\lambda}_1 z)}, \quad (\text{V.3})$$

$$F'_i := \frac{1}{y(\lambda_i)} T_{F_1}(F_i), \quad i = 2, \dots, n. \quad (\text{V.4})$$

Then the original problem is solvable if and only if the Nevanlinna-Pick problem with  $n-1$  interpolation conditions  $(\lambda_2, \dots, \lambda_n; F'_2, \dots, F'_n)$  is solvable. Moreover, there exist a solution  $\Phi_n$  of the original problem with  $n$  conditions and a solution  $\Phi_{n-1}$  of the problem with  $n-1$  conditions such that  $\Phi_n(z) = T_{F_1}^{-1}(y(z)\Phi_{n-1}(z))$ .

For computing solutions of Problem V.1, Corollary V.4 suggests an iterative algorithm called the Schur-Nevanlinna algorithm. In addition, it follows from Corollary V.4 that there exist solutions whose entries are rational, whenever the problem is solvable.

### B. Interpolating Interior and Boundary Conditions

In this subsection, we consider the matrix Nevanlinna-Pick interpolation problem with boundary conditions. To solve this problem, we reduce it to the interpolation problem with boundary conditions only, which is always solvable.

We denote by  $\mathcal{RH}^\infty$  the subset of  $\mathcal{H}^\infty$  consisting of rational functions. Let  $\partial\mathbb{D}$  be the boundary of the unit disc  $\mathbb{D}$ . *The matrix Nevanlinna-Pick interpolation problem with boundary conditions* is stated as follows:

*Problem V.5:* Given distinct complex numbers  $\lambda_1, \dots, \lambda_n \in \mathbb{D}$ ,  $r_1, \dots, r_m \in \partial\mathbb{D}$  and complex matrices  $F_1, \dots, F_n, G_1, \dots, G_m$  such that  $\|F_i\| < 1$ ,  $\|G_j\| < 1$  for every  $i, j$ . Find a  $\Phi \in \mathbf{M}(\mathcal{RH}^\infty)$  satisfying  $\|\Phi\|_\infty < 1$  and

$$\Phi(\lambda_i) = F_i, \quad \Phi(r_j) = G_j, \quad i = 1, \dots, n, \quad j = 1, \dots, m.$$

The scalar version of Problem V.5 is studied in [22, Ch. 2] and [23]. The tangential one is also developed in [19, Ch. 21]. The approach of [22, Ch. 2] and [19, Ch. 21] is based on the corresponding Pick matrix. On the other hand, the method of [23] is based on the Schur-Nevanlinna algorithm. We here extend the method of [23] to the matrix case.

Our aim of this subsection is to prove the following theorem.

**Theorem V.6:** Problem V.5 is solvable if and only if Problem V.1 with the interpolation data  $(\lambda_1, \dots, \lambda_n; F_1, \dots, F_n)$  is solvable.

To prove Theorem V.6, we need to reduce Problem V.5 to the following problem.

**Problem V.7:** Given distinct complex numbers  $r_1, \dots, r_m \in \partial\mathbb{D}$  and complex matrices  $G_1, \dots, G_m$  satisfying  $\|G_j\| < 1$  for every  $j$ . Find a  $\Psi \in \mathbf{M}(\mathcal{RH}^\infty)$  satisfying  $\|\Psi\|_\infty < 1$  and  $\Psi(r_j) = G_j$  for  $j = 1, \dots, m$ .

This problem is called *the boundary Nevanlinna-Pick interpolation problem*.

**Lemma V.8 ([24]):** Problem V.7 is always solvable.

We can prove Lemma V.8 in the same way as in [24]. However, by the Schur-Nevanlinna algorithm, we here prove Lemma V.8 in a more straightforward way than that given in [24].

**Proof of Lemma V.8:** It suffices to show that there exists a boundary Nevanlinna-Pick interpolation problem with  $m - 1$  interpolation conditions in such a way that if the problem with  $m - 1$  conditions is solvable, then the original problem with  $m$  conditions is also solvable.

Let  $\epsilon > 0$ . We define

$$y_\epsilon(z) := \frac{1}{r_1} \frac{z - r_1}{(1 + \epsilon) - \bar{r}_1 z},$$

$$G'_j := \frac{1}{y_\epsilon(r_j)} T_{G_1}(G_j), \quad j = 2, \dots, m.$$

First we show that there exists  $\epsilon > 0$  such that  $\|G'_j\| < 1$  for every  $j$ . Since  $G_j$  is in  $\mathcal{B}$ ,  $T_{G_1}(G_j)$  is also in  $\mathcal{B}$  by Lemma V.3. Hence there exists  $\epsilon$  such that

$$0 < \epsilon < \min_{j=2, \dots, m} \left( |r_j - r_1| \cdot \left( \frac{1}{\|T_{G_1}(G_j)\|} - 1 \right) \right). \quad (\text{V.5})$$

For every  $\epsilon$  in (V.5),  $G'_j$  satisfies

$$\begin{aligned} \|G'_j\| &= \left\| \frac{1}{y_\epsilon(r_j)} T_{G_1}(G_j) \right\| \\ &= \left| 1 - \frac{\epsilon r_1}{r_j - r_1} \right| \cdot \|T_{G_1}(G_j)\| \\ &\leq \left( 1 + \frac{\epsilon}{|r_j - r_1|} \right) \cdot \|T_{G_1}(G_j)\| < 1. \end{aligned}$$

Next suppose that there exists a solution  $\Psi_{m-1} \in \mathbf{M}(\mathcal{RH}^\infty)$  of a boundary Nevanlinna-Pick problem with  $m - 1$  conditions  $(r_2, \dots, r_m; G'_2, \dots, G'_m)$ . Then  $\Psi_m(z) := T_{G_1}^{-1}(y_\epsilon(z)\Psi_{m-1}(z))$  is a solution of the original problem with  $m$  conditions. In fact,  $\|y_\epsilon\Psi_{m-1}\|_\infty < 1$ , because  $\|y_\epsilon\|_\infty < 1$ . Therefore,  $\Psi_m$  is in  $\mathbf{M}(\mathcal{RH}^\infty)$  and  $\|\Psi_m\|_\infty < 1$  by Lemma V.3. Next we confirm that  $\Psi_m$  satisfies the interpolation conditions. For  $j = 2, \dots, m$ , we have

$$\begin{aligned} \Psi_m(r_j) &= T_{G_1}^{-1}(y_\epsilon(r_j)\Psi_{m-1}(r_j)) \\ &= T_{G_1}^{-1}(y_\epsilon(r_j)G'_j) = T_{G_1}^{-1}(T_{G_1}(G_j)) = G_j. \end{aligned}$$

Furthermore, for  $j = 1$

$$\Psi_m(r_1) = T_{G_1}^{-1}(y_\epsilon(r_1)\Psi_{m-1}(r_1)) = T_{G_1}^{-1}(0) = G_1.$$

Hence  $\Psi_m$  is a solution of the original problem with  $m$  conditions.

It has been proved that we can reduce every Problem V.7 to another problem V.7 that has one interpolation condition less. Continuing this way, we arrive at Problem V.7 with only one condition, which always admits a solution. Therefore, Problem V.7 is always solvable. ■

Finally, we prove Theorem V.6 by Corollary V.4 and Lemma V.8.

**Proof of Theorem V.6:** The necessity is straightforward.

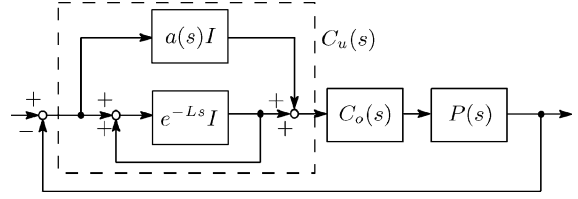


Fig. 2. Repetitive control system.

We show the sufficiency as follows. Suppose that Problem V.1 with the interpolation data  $(\lambda_1, \dots, \lambda_n; F_1, \dots, F_n)$  is solvable. Using Corollary V.4, we can show the existence of a function satisfying  $n - 1$  interior conditions and  $m$  boundary conditions derived by (V.4). Since  $y$  defined by (V.3) is an inner function, the new interpolating value on the boundary

$$\bar{G}_j := \frac{1}{y(r_j)} T_{F_1}(G_j)$$

satisfies  $\|\bar{G}_j\| < 1$  by Lemma V.3. Continuing this way, we can finally reduce Problem V.5 to Problem V.7. Moreover, Problem V.7 is always solvable by Lemma V.8. Therefore, Problem V.5 is solvable if Problem V.1 with conditions  $(\lambda_1, \dots, \lambda_n; F_1, \dots, F_n)$  is solvable. ■

Theorem V.2 and V.6 show that the solvability of Problem V.5 is also equivalent to the positive definiteness of the Pick matrix in (V.1). In addition, the proof of Lemma V.8 and that of Theorem V.6 suggest that we can compute a solution of Problem V.5 by an iterative algorithm similar to the Schur-Nevanlinna algorithm.

## VI. EXAMPLE

Consider the repetitive control system [25], [26] given in Fig. 2, where  $L := 3$ ,  $a(s) := s/(s + 1)$

$$P(s) := \begin{bmatrix} \frac{s+1}{s+2} & \frac{e^{-2s}}{s+1} \\ 0 & \frac{s+2}{s-\frac{1}{15}} \end{bmatrix}, \quad \text{and}$$

$$C_u(s) := \left( \frac{e^{-Ls}}{1 - e^{-Ls}} + a(s) \right) I = \frac{s + e^{-3s}}{(s + 1)(1 - e^{-3s})} I.$$

The internal model principle for the class of pseudorational impulse response matrices [26] shows that under the hypothesis of exponential stability of the closed-loop system, exponential decay of the error signal for any reference signal with a fixed period  $L$  is equivalent to the existence of the internal model  $e^{-Ls}/(1 - e^{-Ls})$ . The principle is a precise generalization of the well-known finite-dimensional counterpart [27].

It follows from this principle that the controllers we consider can be separated into two part  $C = C_u C_o$ , where  $C_u$  is the part of the internal model and has infinitely many poles on the imaginary axis, and  $C_o$  is the stable part to be designed. For the design of  $C_o$ , we can consider the product  $C_u P =: P_o$  to be the new plant to be controlled.

To guarantee exponential stability, it is desirable that  $H(P, C)$  in (II.1) has no poles in the region  $\mathbb{C}_{-\epsilon} := \{s \in \mathbb{C} \mid \text{Re } s \geq -\epsilon\}$ , where  $\epsilon > 0$  is fixed [28]. Therefore, we study sensitivity reduction with stable controllers for the following plant and weighting functions.

$$\begin{aligned} \tilde{P}(s) &:= P_o(s - \epsilon) = C_u(s - \epsilon)P(s - \epsilon), \\ W_1(s) &:= \frac{s + 1}{10s + 1} \begin{bmatrix} 1 & \frac{1}{10} \\ 0 & 1 \end{bmatrix}, \quad W_2(s) := I. \end{aligned}$$

Once we find the solution  $\tilde{C}$  of the problem, we determine the stable part  $C_o(s) := \tilde{C}(s + \epsilon)$ . Since  $\tilde{C}$  is in  $\mathbf{M}(\mathcal{H}^\infty)$ ,  $C_o$  does not have poles in  $\mathbb{C}_{-\epsilon}$ .

We take  $\varepsilon = 0.01$ , so  $\tilde{P}$  has infinitely many unstable poles. However it has only two zeros in  $\tilde{C}_+$ :  $\alpha \approx (0.156 + \varepsilon) + 0.607j$ ,  $\beta \approx (0.156 + \varepsilon) - 0.607j$ , which come from  $C_u(s - \varepsilon)$  and are blocking. Using the factorization method of [14], we can factor  $\tilde{P}$  as  $\tilde{P}(s) = \phi D^{-1} N_o$ , where

$$\phi(s) := \frac{(s - \alpha)(s - \beta)}{(s - \varepsilon + 1)^2}, \quad D(s) := \frac{1 - e^{3\varepsilon} e^{-3s}}{e^{-3s} - e^{3\varepsilon}} \begin{bmatrix} 1 & 0 \\ 0 & \frac{s - \varepsilon - \frac{1}{15}}{s + \varepsilon + \frac{1}{15}} \end{bmatrix},$$

$$N_o(s) := \frac{(s - \varepsilon + 1)(s - \varepsilon + e^{-3(s - \varepsilon)})}{(e^{-3s} - e^{3\varepsilon})(s - \alpha)(s - \beta)} \begin{bmatrix} \frac{s - \varepsilon + 1}{s - \varepsilon + 2} & \frac{e^{-2(s - \varepsilon)}}{s + \varepsilon + \frac{1}{15}} \\ 0 & \frac{s - \varepsilon + 1}{s - \varepsilon + 2} \end{bmatrix}.$$

$N_o$  given above satisfies  $N_o^{-1} \in \mathbf{M}(\mathcal{H}^\infty)$ . We can easily check whether  $D$  and  $N := \phi N_o$  are strongly left coprime by the matrix Nevanlinna-Pick interpolation problem in the same way as the scalar case [22, Ch. 3].

The minimum of  $\rho$  obtained by the proposed method is  $\rho_{\min} := 0.578$ , and the stable controller  $\tilde{C}$  is given as

$$\tilde{C} = \frac{2}{\rho_{\min}} \cdot \frac{1}{\phi} N_o^{-1} (G + I)^{-1} W_1 - \tilde{P}^{-1},$$

where

$$G(s) \approx \begin{bmatrix} \frac{-0.79(s+0.28)(s-0.073)(s^2+0.46s+0.056)}{(s^2+0.57s+0.081)(s^2+0.51s+0.18)} & \\ \frac{-0.057(s^2+0.49s+0.060)(s^2-0.33s+0.40)}{(s^2+0.57s+0.081)(s^2+0.51s+0.18)} & \\ \frac{0.031(s+1.37)(s+0.29)(s^2+0.56s+0.37)}{(s^2+0.57s+0.081)(s^2+0.51s+0.18)} & \\ \frac{-1.00(s-0.27)(s+0.29)(s^2+0.51s+0.18)}{(s^2+0.57s+0.081)(s^2+0.51s+0.18)} & \end{bmatrix}.$$

On the other hand, by Corollary III.10, we obtain a lower bound of  $\rho$  achieved by a stable controller, 0.272.

The controller we construct for  $P$  is distributed. To obtain an implementable finite dimensional controller, we have to approximate the controller; see, e.g., [29]–[31] and references therein.

## VII. CONCLUSION

In this note, the sensitivity reduction problem with stable controllers has been studied for a linear time-invariant multi-input multioutput distributed parameter system. It is still open to obtain a necessary and sufficient condition for the solvability of the problem. However, we have shown that a necessary condition and a sufficient condition can be reduced to the matrix Nevanlinna-Pick interpolation with boundary conditions, if the system has finitely many unstable zeros and if all of them are simple and blocking. The interpolation problem is solvable if and only if the Pick matrix consisting of the interior conditions is positive definite. We can obtain the solutions of the interpolation problem, extending the well-known Schur-Nevanlinna algorithm.

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