



Analysis and applications of replenishment problems under stepwise transportation costs and generalized wholesale prices

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ABSTRACT

In this study, we analyze the replenishment decision of a buyer with the objective of maximizing total expected profits. The buyer faces stepwise freight costs in inbound transportation and a hybrid wholesale price schedule given by a combination of all-units discounts with economies and diseconomies of scale. This general cost structure enables the model and the proposed solution to be also used for the supplier selection of a buyer under the single sourcing assumption. We show that the buyer's replenishment problem reduces to finding and comparing the solutions of the following two subproblems: (i) a replenishment problem involving wholesale prices given by an all-units discount schedule with economies of scale and a lower bound on the replenishment quantity, and (ii) a replenishment problem involving wholesale prices given by an all-units discount schedule with diseconomies of scale and an upper bound on the replenishment quantity. We propose solution methods for these two subproblems, each of which stands alone as practical problems, and utilize these methods to optimally solve the buyer's replenishment problem.

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1. Introduction and literature

Transportation costs are one of the main cost drivers observed in supply chain management. Research on integrated transportation and production/inventory decisions shows that companies may increase total profits by simultaneous planning of transportation and production/inventory decisions (see Aucamp, 1982; Hoque and Goyal, 2000; Lee, 1986; Tersine and Barman, 1994; Toptal et al., 2003). As shipment by trucks is one of the most common transportation modes, taking into account truck capacities and costs explicitly in solving replenishment problems may lead to competitive advantages for a company. In this study, we consider a buyer subject to *full truckload shipping* as a mode of inbound transportation and a *hybrid wholesale price schedule*. This price schedule involves all-units quantity discounts with diseconomies of scale up to a certain size of order quantity, followed by all-units quantity discounts with economies of scale for larger quantities.

In truckload (TL) transportation, each additional truck requires a fixed payment as opposed to less-than-truckload (LTL) transportation in which the related costs are in proportion to the shipment quantity. Aucamp (1982), Lee (1986), Toptal et al. (2003), Toptal and Çetinkaya (2006), and Toptal (2009) are some

examples of papers that model truck capacities and costs explicitly within the context of integrated replenishment and transportation decisions. Aucamp (1982) studies the classical economic order quantity (EOQ) problem assuming that the replenishment quantity is shipped via trucks having identical capacities and costs. Lee (1986) extends this study by modelling the availability of discounts on each additional truck used. Lee (1989) and Toptal et al. (2003) study the dynamic lot sizing problem and the single-warehouse, single-retailer replenishment problem, respectively, under the same transportation cost structure as in Aucamp (1982).

In comparison to the studies that consider deterministic demand (i.e., Aucamp, 1982; Lee, 1986, 1989; Toptal et al., 2003), there are also papers modelling TL shipments for inventory systems with stochastic demand (e.g., Toptal and Çetinkaya, 2006; Toptal, 2009; Ülkü and Bookbinder, 2012). Toptal and Çetinkaya (2006) study the problem of coordinating the replenishment decisions between a buyer and a vendor under transportation costs and capacities. Toptal (2009) proposes a solution for finding the order quantity that maximizes the single period expected profits of a company with stepwise freight costs and procurement costs given by an all-units discount schedule. Ülkü and Bookbinder (2012) study the shipment consolidation and pricing decisions of a manufacturer with multiple buyers who are sensitive to price and delivery time.

In this study, we consider a setting where the buyer is subject to the same freight cost structure as in Aucamp (1982). Moreover, we model a wholesale price schedule which exhibits a combination of

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economies of scale and diseconomies of scale over varying quantity intervals. There are different types of wholesale price schedules applied in practice and studied in the literature (see Benton and Park, 1996; Munson and Rosenblatt, 1998). Discount schedules with economies of scale, simply referred to as quantity discounts, are the commonly prevailing ones. Typically, in these price schedules (e.g., all-units, incremental), the unit price of an item is less for larger orders. On the other hand, in a quantity discount schedule with diseconomies of scale, the unit price of an item is more for larger orders. The changes in prices are defined by breakpoints in both of these schedules. Munson and Rosenblatt (1998) report that all-units quantity discounts, in which the discount is applied to all units in an order, is the most commonly practiced price schedule in the industry.

Quantity discounts with diseconomies of scale are also referred to as quantity premiums or quantity surcharges in the literature. Quantity premiums are common for energy products such as electricity usage and water consumption. Widrick (1985) notes that a supplier may use quantity premiums as a demarketing tool to discourage excessive consumption of a scarce resource such as water and fuel. Das (1984) discusses that this form of price schedule is also justifiable in case of limited supply, specifically, in developing economies. Quantity premiums may also be an efficient instrument for supply chain coordination when a supplier observes diseconomies of scale in replenishment costs (see Tomlin, 2003; Toptal and Çetinkaya, 2006). Tomlin (2003) studies a two-party, a manufacturer and a supplier, capacity procurement game and shows that a quantity premium cost schedule may be optimal for the manufacturer. Toptal and Çetinkaya (2006) consider a buyer–vendor system and show that when the vendor has transportation costs and capacities defined by a stepwise cost structure, quantity premiums may optimally coordinate the supply chain.

Quantity discounts are studied from the perspectives of both the buyers and the suppliers. While the former body of research focuses on the replenishment decisions under a quantity discount schedule (e.g., Abad, 1988; Arcelus and Srinivasan, 1995; Das, 1984; Hwang et al., 1990; Tersine and Barman, 1994, 1995; Toptal, 2009), the latter group investigates how a supplier should construct such price schedules (e.g., Banerjee, 1986; Lal and Staelin, 1984; Lee and Rosenblatt, 1986; Li and Liu, 2006; Monahan, 1984; Rubin and Benton, 2003; Weng, 1995). Since our focus is solving a buyer's replenishment problem under a given wholesale price schedule, our study falls into the first body of research. The wholesale price schedule considered herein is significantly different from those in earlier studies in its following feature: while the unit price of an item is more for larger orders up to a certain size, subsequent unit prices decrease with increasing order size. That is, the price schedule exhibits either diseconomies of scale or economies of scale over different quantity intervals. In order to emphasize this distinctive characteristic, we use the term “hybrid” in classifying the wholesale price schedule. It is important to note that quantity discounts with economies/diseconomies of scale are special cases of this general price schedule. There may exist several practical circumstances for a buyer to have procurement costs given by a hybrid price schedule. We show that an immediate context that a hybrid wholesale price schedule prevails is in case of supplier selection. More specifically, a profit maximizing buyer's supplier selection problem under the single sourcing assumption in a multiple suppliers setting can be modeled as a replenishment problem with a hybrid price schedule, if each supplier offers either an all-units quantity discount or an all-units quantity premium.

Two main types of supplier selection problems are identified in the literature, single-sourcing and multiple-sourcing. In single-sourcing, the purchaser is restricted to replenish from a single supplier whereas multiple-sourcing allows the replenishment quantity to be fulfilled through more than one supplier. Chaudhry et al.

(1993) study a supplier selection problem allowing multiple sourcing in a setting where a supplier offers either quantity discounts or quantity premiums. Xia and Wu (2007) also consider a multiple sourcing scenario, and assume that the vendors have supply limitations and they offer quantity discounts. Swift (1995) discusses reasons why single sourcing may be preferred in practice, among which, is developing long-term cooperative relations with a supplier. We cite Aissaoui et al. (2007) for a review of studies on supplier selection.

When single sourcing is assumed in a multiple supplier setting with each supplier offering either quantity discounts or quantity premiums, a buyer's replenishment problem can be solved using one of the two methods: (i) a replenishment problem can be solved for each supplier separately and the supplier leading to the maximum expected profits can be chosen; (ii) a single schedule for wholesale prices can be constructed and a replenishment problem can be solved under this new schedule. For the problem of interest in this paper, we provide a complete analysis using both methods. The first method requires solving two types of replenishment problems; one with stepwise freight costs and quantity discounts, and one with stepwise freight costs and quantity premiums. In the setting that is of concern, the replenishment problem has a nonrecurring nature and the buyer has strictly concave production/inventory related profits for a fixed purchasing price. It is important to note that, under these considerations, the replenishment problem involving stepwise freight costs and all-units quantity discounts has been solved by Toptal (2009). To the best of our knowledge, the problem with stepwise freight costs and all-units quantity premiums has not been studied. The second method, on the other hand, requires solving a replenishment problem with stepwise freight costs and a hybrid wholesale price schedule. Again, our review of the literature suggests that this problem has not been examined.

The contributions of this paper are as follows:

- (C1) As part of the first method outlined above, we solve the replenishment problem with quantity premiums and stepwise freight costs.
- (C2) As part of the second method, we extend the analysis in Toptal (2009) to consider a lower bound on the order quantity. Similarly, we extend our analysis in part (C1) to consider an upper bound on the order quantity.
- (C3) We combine the above results to work out the solution to the replenishment problem with stepwise freight costs and a hybrid wholesale price schedule.
- (C4) We show that the supplier selection problem for a profit maximizing purchaser under the single sourcing assumption, reduces to the problem of interest in this paper. We also describe how to construct a single hybrid price schedule out of several price menus each of which is either an all-units quantity discount or an all-units quantity premium.

It is important to note that, although our main objective is to arrive at (C3), the solutions to the subproblems described in (C1)–(C2) can be used on their own for practical purposes.

The rest of the paper is organized as follows. Section 2 presents the notation used in the paper and provides a generic mathematical formulation, which captures a wider class of problems than the specific one under consideration. In Section 3, we provide our analysis within the context of (C1)–(C3). Section 4 follows with a detailed discussion of how a supplier selection problem under the single sourcing assumption reduces to the problem studied in this paper. In Section 5, the proposed solution methodology is illustrated over an application to the Newsvendor Model setting. Section 6 concludes the paper with a discussion on possible future research directions.

2. Notation and problem formulation

We study the replenishment decision of a buyer who is subject to an all-units discount schedule with economies and diseconomies of scale in addition to stepwise freight costs. In particular, the unit wholesale price, denoted by $c(Q)$, is given by the following expression:

$$c(Q) = \begin{cases} c_0 & q_0 \leq Q < q_1 \\ c_1 & q_1 \leq Q < q_2 \\ \vdots & \vdots \\ c_h & q_h \leq Q < q_{h+1}, \\ \vdots & \vdots \\ c_{n-1} & q_{n-1} \leq Q < q_n \\ c_n & Q \geq q_n \end{cases} \quad (1)$$

where $q_0 = 0$ and c_h represents the wholesale price when the buyer's order quantity is in $[q_h, q_{h+1})$. The expression for $c(Q)$ pertains to a hybrid wholesale price schedule, when an index h , $0 < h < n$, exists such that $c_0 < c_1 < \dots < c_{h-1} < c_h$ and $c_h > c_{h+1} > \dots > c_{n-1} > c_n$. Note that if $h=0$, $c(Q)$ simply refers to an all-units discount schedule with economies of scale, whereas the case of $h=n$ leads to an all-units quantity discount schedule with diseconomies of scale. Fig. 1 illustrates the three possible forms that $c(Q)$ may assume.

In this setting, the buyer pays for the transportation costs. Specifically, $\lceil Q/P \rceil R$ is incurred for shipping an order quantity of Q , where P and R are the per truck capacity and the per truck cost, respectively. The production/inventory related expected profits of the buyer as a function of order quantity Q and wholesale price c_i are given by $G(Q, c_i)$. Here, $G(Q, c_i)$ is a strictly concave function of Q , for fixed value of c_i . Accordingly, the buyer's total expected profits are given by

$$\mathcal{H}(Q) = G(Q, c(Q)) - \left\lceil \frac{Q}{P} \right\rceil R. \quad (2)$$

The terms of $G(Q, c(Q))$ may include the buyer's expected revenue from sales and salvage, expected lost sales cost, fixed costs of replenishment, procurement costs, etc. In Section 5, we provide specific examples of this function.

Let $Q^{(i)}$ denote the unique maximizer of $G(Q, c_i)$ over $Q \geq 0$. $Q^{(i)}$ is classified as *realizable* if $q_i \leq Q^{(i)} < q_{i+1}$. Note that the buyer's expected profit function consists of $(n+1)$ pieces and its value on the $(i+1)^{st}$ piece is determined by $H^i(Q)$, where

$$H^i(Q) = G(Q, c_i) - \left\lceil \frac{Q}{P} \right\rceil R. \quad (3)$$

Suppose that $\tilde{Q}^{(i)}$ is a maximizer of $H^i(Q)$ over $Q \geq 0$. Similarly, $\tilde{Q}^{(i)}$ is classified as *realizable* if $\tilde{Q}^{(i)}$ falls into interval $[q_i, q_{i+1})$ (i.e., $q_i \leq \tilde{Q}^{(i)} < q_{i+1}$). We assume that $G(Q, c(Q))$ has the following characteristics:

(A1) If $c_i > (<)c_{i+1}$ then $Q^{(i)} < (>)Q^{(i+1)}$. That is, the maximizer of $G(Q, c_i)$ increases (decreases) as c_i decreases (increases).

(A2) If $c_i > (<)c_{i+1}$ then $G(Q, c_i) < (>)G(Q, c_{i+1})$. That is, for fixed value of Q , $G(Q, c_i)$ decreases (increases) as c_i increases (decreases).

(A3) If $c_i > c_{i+1}$ then $G(Q_2, c_{i+1}) - G(Q_2, c_i) > G(Q_1, c_{i+1}) - G(Q_1, c_i)$ for $Q_1 < Q_2 \leq Q^{(i)}$. If $c_i < c_{i+1}$, then $G(Q_2, c_i) - G(Q_2, c_{i+1}) > G(Q_1, c_i) - G(Q_1, c_{i+1})$ for $Q_1 < Q_2 \leq Q^{(i)}$. That is, the change in $G(Q, c_i)$ when c_i is decreased (increased), increases (decreases) with respect to Q .

The buyer's replenishment problem, which we refer to as *Problem HPTC* (Hybrid Price Transportation Cost), is then given by the following general formulation.

$$(\mathcal{HPTC}) \quad \begin{aligned} & \max \quad \mathcal{H}(Q), \\ & \text{s.t.} \quad Q \geq 0. \end{aligned}$$

As discussed earlier, an all-units hybrid price schedule consists of two main parts: an all-units premiums up to the h^{th} price interval followed by an all-units discounts after the h^{th} price interval, where h is the price interval such that $c_{h-1} < c_h$ and $c_h > c_{h+1}$. The solution method we provide in the next section for *Problem HPTC*, utilizes this characteristic of a hybrid price schedule. That is, we consider the following two subproblems and we compare their optimal solutions to arrive at a maximizer for *Problem HPTC*: a replenishment problem involving stepwise freight costs and an all-units discount schedule with a lower bound constraint on the order quantity, that is *Problem DPPTC_{LB}* (Discounted Price Transportation Cost with Lower Bound), and, a replenishment problem involving stepwise freight costs and an all-units premium schedule with an upper bound constraint on the order quantity, that is *Problem PPPTC_{UB}* (Premium Price Transportation Cost with Upper Bound).

Our solution for *Problem DPPTC_{LB}* relies on an analysis for its version with no positive lower bound on the order quantity, that is *Problem DPPTC* (Discounted Price Transportation Cost), discussed in an earlier paper by Toptal (2009). In the current paper, we also study the replenishment problem involving stepwise freight costs and an all-units premium schedule, *Problem PPPTC* (Premium Price Transportation Cost). We then extend this analysis to consider an upper bound on the order quantity, which we refer to as *Problem PPPTC_{UB}*. Next, the notation used throughout the paper is summarized. Additional notation will be defined as needed.

- Q : Replenishment quantity of the buyer.
- n : Number of price break points.
- q_i : Quantity where the i th breakpoint appears, $0 \leq i \leq n$.
- $c(Q)$: Unit wholesale price as a function of order quantity.
- P : Per truck capacity.
- R : Per truck cost.
- $\mathcal{H}(Q)$: Profit function of the buyer.
- $H^i(Q)$: Profit function of the buyer at wholesale price level c_i , defined over $Q \geq 0$.
- $G(Q, c_i)$: Buyer's profit component defined over $Q \geq 0$ for price level c_i , without transportation costs.
- Q^* : Maximizer of $\mathcal{H}(Q)$.
- $\tilde{Q}^{(i)}$: Maximizer of $H^i(Q)$.
- $Q^{(i)}$: Maximizer of $G(Q, c_i)$.

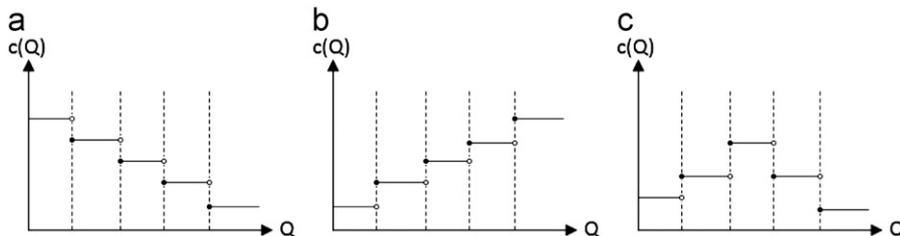


Fig. 1. Illustration of the possible forms of $c(Q)$. (a) All-units discounts. (b) All-units premiums. (c) All-units hybrid.

3. Analysis of the problem

In this section, we propose a solution for *Problem \mathcal{HPTC}* and describe the analysis we follow to arrive at this solution (Q^*). Based on the fact that $c(Q)$ is a hybrid price schedule, our solution approach relies on finding and comparing the profits at the maximizers over the two quantity intervals for which the wholesale prices exhibit either diseconomies of scale or economies of scale.

Our solution approach can be described in more detail as follows: If $Q^* < q_h$, then Q^* coincides with the optimizer Q^P (i.e., $Q^* = Q^P$) of the following problem: maximizing the buyer's expected profits under the all-units premium schedule $\bar{c}^P(Q)$ and stepwise freight costs with the upper bound constraint $Q < q_h$. Here, we define $\bar{c}^P(Q)$ as follows: $\bar{c}^P(Q) = c(Q)$ for $Q \in [0, q_h]$ and $\bar{c}^P(Q) = c_h$ for $Q \in [q_h, \infty)$. Similarly, if $Q^* \geq q_h$, then Q^* coincides with the optimizer Q^D (i.e., $Q^* = Q^D$) of the following problem: maximizing the buyer's expected profits under the all-units discount schedule $\bar{c}^D(Q)$ and stepwise freight costs with the lower bound constraint $Q \geq q_h$. Here, we define $\bar{c}^D(Q)$ as follows: $\bar{c}^D(Q) = c_h$ for $Q \in [0, q_{h+1})$ and $\bar{c}^D(Q) = c(Q)$ for $Q \in [q_{h+1}, \infty)$. In order to find Q^P , we consider a type of replenishment problem which has been introduced in Section 2 as *Problem \mathcal{PPTC}_{UB}* . Similarly, in order to reach a value for Q^D , we consider a type of replenishment problem which has been introduced in the same section as *Problem \mathcal{DPTC}_{LB}* . Finally, we set

$$Q^* = \arg \max\{\mathcal{H}(Q^D), \mathcal{H}(Q^P)\}.$$

In mathematical terms, *Problem \mathcal{PPTC}_{UB}* and *Problem \mathcal{DPTC}_{LB}* can be described as follows:

$$(\mathcal{PPTC}_{UB}) \quad \begin{array}{ll} \max & \mathcal{H}(Q) \\ \text{s.t.} & 0 \leq Q < UB, \end{array}$$

and

$$(\mathcal{DPTC}_{LB}) \quad \begin{array}{ll} \max & \mathcal{H}(Q) \\ \text{s.t.} & Q \geq LB, \end{array}$$

where LB and UB are nonnegative real numbers (note that in the analysis of *Problem \mathcal{HPTC}* , we take $LB = UB = q_h$). In *Problem \mathcal{PPTC}_{UB}* , $c(Q)$ is given by an all-units quantity premium schedule. In *Problem \mathcal{DPTC}_{LB}* , $c(Q)$ is given by an all-units quantity discount schedule. In this section, we first begin with an analysis of *Problem \mathcal{PPTC}_{UB}* by setting $UB = \infty$, that is *Problem \mathcal{PPTC}* (Premium Price Transportation Cost). We then proceed with the case of $0 < UB < \infty$. This is followed by an analysis of *Problem \mathcal{DPTC}_{LB}* . An important property that is common to all these problems, which also is an underlying factor in our solution approach, is that their objective functions have a piecewise structure and the function value on the $(i+1)^{\text{st}}$ piece is determined by $H^i(Q)$ as given in Expression (3). Therefore, some structural properties of $H^i(Q)$ function and the solution to the following problem, i.e., *Problem \mathcal{UPTC}* (Uniform Price Transportation Cost), will be relevant to the upcoming analysis.

$$(\mathcal{UPTC}) \quad \begin{array}{ll} \max & H^i(Q) \\ \text{s.t.} & Q \geq 0. \end{array}$$

We report the following result from Toptal (2009), which provides the solution to the above problem.

Result 1. The solution to *Problem \mathcal{UPTC}* is given by

$$\tilde{Q}^{(i)} = \begin{cases} \arg \max\{H^i(mP), H^i((m+1)P)\} & \text{if } \mathcal{F} \neq \emptyset \\ \arg \max\{H^i(Q^{(i)}), H^i((l-1)P)\} & \text{if } \mathcal{F} = \emptyset \end{cases}$$

where

$$\mathcal{F} = \{k \in \{0, 1, 2, \dots\} : G((k+1)P, c_i) - G(kP, c_i) \leq R, (k+1)P \leq Q^{(i)}\},$$

$$l = \lceil Q^{(i)} / P \rceil \text{ and } m = \min\{k \text{ s.t. } k \in \mathcal{F}\} \text{ when } \mathcal{F} \neq \emptyset.$$

Result 1 indicates that $\tilde{Q}^{(i)}$ is either equal to $Q^{(i)}$ or an integer multiple of a full truck load less than that. Note also that, in both cases of the result, multiple solutions may exist. In the first case, if $G((m+1)P, c_i) - G(mP, c_i) = R$, then both mP and $(m+1)P$ maximize $H^i(Q)$. Similarly, in the second case, if $G(Q^{(i)}, c_i) - G((l-1)P, c_i) = R$, then both $Q^{(i)}$ and $(l-1)P$ maximize $H^i(Q)$.

3.1. Analysis of *Problem \mathcal{PPTC}_{UB}* : the case of $UB = \infty$

In this section, we analyze *Problem \mathcal{PPTC}_{UB}* by setting $h=n$ in Expression (1) and $UB = \infty$. This problem has been referred to as *Problem \mathcal{PPTC}* earlier in the text. The solution relies on Result 1 which we cite from Toptal (2009) and the following structural properties of $\mathcal{H}(Q)$. The proofs of Properties 1 and 2 will be omitted as they are very similar to those of Properties 9 and 10 in Toptal (2009). The proofs of all other results are presented in Appendix.

Property 1. $H^i(\tilde{Q}^{(i)}) > H^{i+1}(\tilde{Q}^{(i+1)})$, $\forall i$ s.t. $0 \leq i \leq n-1$. That is, the optimal function values at consecutive $H^i(Q)$'s are decreasing.

Note that Property 1 implies $H^0(\tilde{Q}^{(0)}) > H^1(\tilde{Q}^{(1)}) > \dots > H^n(\tilde{Q}^{(n)})$. The next property presents an ordinal relationship between the maximizers of $H^i(Q)$ functions at consecutive values of price index i .

Property 2. We have $\tilde{Q}^{(i)} \geq \tilde{Q}^{(i+1)}$, $\forall i$ s.t. $0 \leq i \leq n-1$. In other words, the maximizers of consecutive $H^i(Q)$ functions are nonincreasing.

An implication of Property 1 is that, if a maximizer of $H^i(Q)$ is realizable, then $\mathcal{H}(\tilde{Q}^{(i)}) \geq \mathcal{H}(Q), \forall Q > \tilde{Q}^{(i)}$. Combining this with Property 2 further leads to the fact that, in finding a maximizer of $\mathcal{H}(Q)$, we do not need to consider quantities larger than the largest realizable $\tilde{Q}^{(i)}$, if there exists any.

Property 3. If there exists $k \in [0, n-1]$ such that some maximizer of $H^k(Q)$ is less than q_{k+1} (i.e., $\tilde{Q}^{(k)} < q_{k+1}$), then all maximizers of $H^i(Q)$ are less than q_j (i.e., $\tilde{Q}^{(i)} < q_j$) $\forall j > k$.

Let $\mathcal{F}_2 = \{k \in \{0, 1, \dots, n\} \text{ s.t. all maximizers of } H^k(Q) \text{ are greater than or equal to } q_{k+1}\}$. If $\mathcal{F}_2 \neq \emptyset$, define $n_2 = \max\{k \text{ s.t. } k \in \mathcal{F}_2\}$. It then follows from Property 3 that the maximizers of $H^k(Q)$ functions for consecutive price indices k , such that $0 \leq k \leq n_2$, are all greater than or equal to q_{k+1} .

Property 4. If $\mathcal{F}_2 = \emptyset$, then there exists a maximizer of $H^0(Q)$, say $\tilde{Q}^{(0)}$, which is realizable and optimally solves *Problem \mathcal{PPTC}* .

It should be noted that the solution to the classical economic order quantity model with all-units discount schedule builds on the fact that there exists at least one realizable EOQ (see Hadley and Whitin, 1963, pp. 62–66). It is shown in Toptal (2009) that the same result holds for the Newsvendor Model and the generalization of the Newsvendor Model including stepwise freight costs with all-units discounts. On the other hand, while solving the classical economic order quantity model with all-units premium schedule, existence of a realizable EOQ is not guaranteed (see Das, 1984). The same result can be easily shown for the Newsvendor Model with all-units quantity premium. Property 5 proves this result for the generalization of Newsvendor Model with stepwise freight costs and quantity premiums.

Property 5. If $\mathcal{F}_2 \neq \emptyset$, there exists at most one price index i such that some maximizer of $H^i(Q)$ is realizable and that can only be $n_2 + 1$.

Property 6. If $\mathcal{F}_2 \neq \emptyset$ and

- If $H^{n_2+1}(Q)$ has a realizable maximizer, say $\tilde{Q}^{(n_2+1)}$, then in finding a solution for *Problem \mathcal{PPTC}* , we can focus on $Q \leq \tilde{Q}^{(n_2+1)}$.
- If $H^{n_2+1}(Q)$ has no realizable maximizer, then in finding a solution for *Problem \mathcal{PPTC}* , we can focus on $Q < q_{n_2+1}$.

Property 6 implies that an optimal solution to Problem \mathcal{PPTC} lies in $[0, \bar{Q}^{(n_2+1)}]$ or in $[0, q_{n_2+1}]$ depending on whether $H^{n_2+1}(Q)$ has a realizable maximizer or not. It further leads to the fact that, we should consider at most n_2+1 subproblems where each subproblem is in the form of Problem \mathcal{UPTC} with the additional constraint that $q_i \leq Q < q_{i+1}$, $i \leq n_2$. Next proposition characterizes the solution of Problem \mathcal{UPTC} with this additional constraint.

Proposition 1. Let i be the index of a price interval such that $i \leq n_2$ and let \bar{Q}_i^* be defined as follows:

$$\bar{Q}_i^* = \begin{cases} q_{i+1} - \epsilon & \text{if } \lceil \frac{q_i}{P} \rceil = \lceil \frac{q_{i+1} - \epsilon}{P} \rceil, \\ \arg \max \left\{ H^i(q_{i+1} - \epsilon), H^i \left(\left\lfloor \frac{q_{i+1} - \epsilon}{P} \right\rfloor P \right) \right\} & \text{o.w.} \end{cases}$$

where ϵ is a very small, positive number. We have $H^i(\bar{Q}_i^*) \geq H^i(Q)$, $\forall Q \in [q_i, q_{i+1})$.

Note that Proposition 1 not only solves $\max\{H^i(Q) : Q \in [q_i, q_{i+1}), i \leq n_2\}$ but also $\max\{H^i(Q) : Q \in (0, q_{i+1}), \bar{Q}^{(i)} > q_{i+1}\}$. This fact is also utilized in our analysis for Problem \mathcal{PPTC}_{UB} . In the next corollary, which follows from Properties 4 and 6, Proposition 1 and Result 1, we introduce an algorithm for finding the smallest maximizer of Problem \mathcal{PPTC} , i.e., $Q^{(p)}$.

Corollary 1. The following algorithm gives an optimal solution for Problem \mathcal{PPTC} .

1. Form the set \mathcal{F}_2 . If $\mathcal{F}_2 = \emptyset$, set $Q^{(p)}$ to the smallest realizable maximizer of $H^0(Q)$ and stop. Otherwise go to Step 2.
2. Find n_2 and check if any maximizer of $H^{n_2+1}(Q)$ is realizable.
 - (a) If there exists any realizable maximizer of $H^{n_2+1}(Q)$, set $Q^{(p)}$ to the smallest and go to Step 3.
 - (b) If no maximizer of $H^{n_2+1}(Q)$ is realizable, set $Q^{(p)} = q_{n_2+1}$ and go to Step 3.
3. Starting from $i = n_2$ back to $i = 0$.
 - (a) Find \bar{Q}_i^* using Proposition 1 (if there are alternative values for \bar{Q}_i^* , choose the smallest).
 - (b) If $\mathcal{H}(\bar{Q}_i^*) \geq \mathcal{H}(Q^{(p)})$, set $Q^{(p)} = \bar{Q}_i^*$.
4. Return $Q^{(p)}$.

It should be emphasized that while constructing the set \mathcal{F}_2 , one should start from the first price index, and stop as soon as a price index j such that some maximizer of $H^j(Q)$ is less than q_{j+1} , is reached.

3.2. Analysis of Problem \mathcal{PPTC}_{UB} : the case of $UB < \infty$

In this section, we consider Problem \mathcal{PPTC}_{UB} for the case of $UB < \infty$ and when wholesale prices are given by Expression (1) under $h = n$. The next proposition presents an algorithm for finding a solution to this problem (i.e., Q^p).

Proposition 2. Suppose that $c(Q)$ represents an all-units premium schedule and the buyer is subject to $Q < UB$. The following algorithm gives an optimal solution for Problem \mathcal{PPTC}_{UB} with $UB < \infty$.

- Assume momentarily that $UB = \infty$ and use Corollary 1 to find the smallest maximizer $Q^{(p)}$.
- If $Q^{(p)} < UB$, then set $Q^p = Q^{(p)}$.
- If $Q^{(p)} \geq UB$, let $w = \max\{i : q_i < UB\}$ and redefine the $(w+1)^{st}$ interval to be (q_w, UB) . Then, $Q^p = \arg \max\{\mathcal{H}(\bar{Q}_i^*) : 0 \leq i \leq w\}$, where \bar{Q}_i^* is determined by Proposition 1.

We note that, Proposition 2 characterizes a solution for Problem \mathcal{PPTC}_{UB} for any given all-units premiums schedule and

upper bound. That is, UB does not have to coincide with a quantity where a price breakpoint occurs.

3.3. Analysis of problem \mathcal{DPPTC}_{LB} : the case of $LB > 0$

In this section, we consider Problem \mathcal{DPPTC}_{LB} for the case of $LB > 0$ and when wholesale prices are given by Expression (1) under $h = 0$. The next proposition presents an algorithm for finding a solution to this problem (i.e., Q^D).

Proposition 3. Suppose that $c(Q)$ represents an all-units discount schedule and the buyer is subject to $Q \geq LB$. The following algorithm gives an optimal solution for Problem \mathcal{DPPTC}_{LB} with $LB > 0$.

- Assume momentarily that $LB = 0$ and use a modified version of Corollary 4 in Toptal (2009) to find the largest maximizer $Q^{(d)}$.
- If $Q^{(d)} \geq LB$, then $Q^D = Q^{(d)}$.
- If $Q^{(d)} < LB$, let r_1 and r_2 be defined as in Toptal (2009). Moreover, let $w = \min\{i : q_{i+1} \geq LB\}$ and redefine the $(w+1)^{st}$ interval as $[LB, q_{w+1})$. Then $w \geq r_1$.
 - If $w > r_2$, then $Q^D = \arg \max\{\mathcal{H}(q_i) : w \leq i \leq n\}$.
 - If $w \leq r_2$, then set $Q^D = \arg \max\{\mathcal{H}(Q_i^*) : w \leq i \leq r_2\}$, where Q_i^* is determined by Proposition 2 in Toptal (2009). If $r_2 < n$, compute $q_{max} = \arg \max\{\mathcal{H}(q_{r_2+1}), \mathcal{H}(q_{r_2+1}), \dots, \mathcal{H}(q_n)\}$. If $\mathcal{H}(q_{max}) > \mathcal{H}(Q^D)$, let $Q^D = q_{max}$.

We note that Corollary 4 and Proposition 2 in Toptal (2009) can be easily modified to find the largest maximizer $Q^{(d)}$ by always choosing the larger quantity whenever alternative solutions exist within the optimization algorithm.

4. A supplier selection problem

In this section, we show that the replenishment decision of a buyer who orders from one of the $s \geq 2$ suppliers can be modeled as Problem \mathcal{HPTC} , under the single sourcing assumption. Suppose that each supplier offers either an all-units quantity discount schedule or an all-units quantity premium schedule. The buyer's wholesale price from supplier u ($u = 1, 2, \dots, s$), $c^u(Q)$, is given by

$$c^u(Q) = \begin{cases} c_0^u & q_0^u \leq Q < q_1^u \\ c_1^u & q_1^u \leq Q < q_2^u \\ c_2^u & q_2^u \leq Q < q_3^u \\ \vdots & \vdots \\ c_{n^u-1}^u & q_{n^u-1}^u \leq Q < q_{n^u}^u \\ c_{n^u}^u & Q \geq q_{n^u}^u, \end{cases} \quad (4)$$

where $q_0^u = 0 \forall u = 1, 2, \dots, s$. Here, c_i^u represents the wholesale price offered by supplier u when the buyer's order quantity is in $[q_i^u, q_{i+1}^u)$. Supplier u has n^u ($n^u \geq 1$) price breakpoints. Let \mathcal{D} and \mathcal{P} denote the set of suppliers who offer quantity discounts and quantity premiums, respectively. That is, if $u \in \mathcal{D}$ we have $c_0^u > c_1^u > \dots > c_{n^u}^u$. Similarly, if $u \in \mathcal{P}$ we have $c_0^u < c_1^u < \dots < c_{n^u}^u$. In this setting, the buyer has to decide jointly how much to order and from which supplier to order.

Due to the single sourcing assumption, the outcome of the supplier-selection problem is implied from the replenishment quantity of the buyer. More specifically, given the replenishment quantity of the buyer, the supplier who offers the minimum wholesale price is selected. As discussed in Section 1, a method for solving the buyer's joint replenishment and supplier selection problem is to find the optimal replenishment quantity for each supplier separately and then, choose the solution which leads to the maximum expected profits. The problem of the buyer, is then to $\max_{u=1,2,\dots,s} \{\mathcal{H}_u(Q_u^*)\}$, where $\mathcal{H}_u(Q)$ denotes the buyer's

expected profit under the price schedule offered by supplier u , i.e., when $c(Q) = c^u(Q)$, and $Q_{(u)}^*$ is the optimal replenishment quantity to be ordered from supplier u . Note that, this method utilizes the solution for Problem \mathcal{PPTC} provided in Section 3.1 and the solution for Problem \mathcal{DPITC} proposed in Toptal (2009).

Another solution approach is to construct a unified price schedule $c(Q)$, based on the individual price schedules of the suppliers, and then, to determine the optimal replenishment quantity. The unified price schedule that the buyer faces is given by $c(Q) = \min_{1 \leq u \leq s} \{c^u(Q) : Q \geq 0\}$. Once the buyer decides on the optimal replenishment quantity under this price schedule, the supplier who offers the minimum wholesale price for the optimal replenishment quantity is selected. In the rest of this section, we present some properties of $c(Q)$. The proofs of Properties 7, 8 and 9 are omitted as they are trivial.

Property 7. If $\mathcal{P} = \emptyset$ or $\mathcal{D} = \emptyset$, then $c(Q)$ corresponds to an all-units discount or an all-units premium schedule, respectively.

It follows from Property 7 that the supplier selection problem described above reduces to Problem \mathcal{DPITC} when $\mathcal{P} = \emptyset$ and, it reduces to Problem \mathcal{PPTC} when $\mathcal{D} = \emptyset$. We note that the cases highlighted in Property 7 are sufficient but not necessary for $c(Q)$ to be either an all-units discount or an all-units premium schedule. It is still possible that $c(Q)$ is in the form of either one of these price schedules when both $\mathcal{P} \neq \emptyset$ and $\mathcal{D} \neq \emptyset$. In Property 8, we analyze such cases. For simplicity, we utilize Property 7 in the following way. Let $c^D(Q)$ be the all-units discount schedule constructed by only considering those suppliers such that $u \in \mathcal{D}$, and let c_i^D be the price for interval $[q_i^D, q_{i+1}^D)$. Also let n^D be the number of breakpoints for $c^D(Q)$. Similarly, define $c^P(Q)$, c_i^P , $[q_i^P, q_{i+1}^P)$ and n^P for the all-units premium schedule constructed by only considering those suppliers such that $u \in \mathcal{P}$, $u = 1, 2, \dots, s$. In other words, we reduce the s -suppliers scenario to 2-suppliers scenario. One of these suppliers offers an all-units discount schedule given by $c^D(Q)$, and we refer to this supplier as supplier D . The other supplier offers an all-units premium schedule given by $c^P(Q)$, and we refer to this supplier as supplier P .

Property 8. Suppose that $c^D(Q)$ and $c^P(Q)$ are constructed for given sets $\mathcal{D} \neq \emptyset$ and $\mathcal{P} \neq \emptyset$. If $c_0^D \leq c_0^P$, then $c(Q) = c^D(Q)$, or, if $c_{n^D}^D \geq c_{n^P}^P$, then $c(Q) = c^P(Q)$.

Given that one of the cases in Property 8 holds, the supplier selection problem reduces to either Problem \mathcal{DPITC} or Problem \mathcal{PPTC} . However, similar to Property 7, the conditions stated are not necessary but only sufficient for the special cases to occur.

Property 9. Let $\hat{Q} = \inf_{Q \geq 0} \{c^D(Q) \leq c^P(Q)\}$. If $0 < \hat{Q} < \infty$, then $c(Q)$ corresponds to an all-units hybrid price schedule, where $c(Q) = c^D(Q)$ for $Q < \hat{Q}$ and, $c(Q) = c^P(Q)$ for $Q \geq \hat{Q}$.

When the condition in Property 9 holds, the buyer faces a hybrid price schedule and his/her replenishment problem can be formulated as in Problem \mathcal{HPITC} . In the next section, we present an example of a supplier selection problem for a buyer that operates under the conditions of the classical Newsvendor Model. We also illustrate the applications of the solution procedures that have been developed for the underlying subproblems.

5. Application to the Newsvendor setting

In this section, we study the supplier selection problem of a buyer under single sourcing assumption, i.e., the buyer has to choose one supplier among many to replenish from. For illustrative purposes, a two-suppliers case where Supplier 1 offers an all-units premium schedule and Supplier 2 offers an all-units

discount schedule, is considered. Note that, s -suppliers case for $s > 2$ can be reduced to 2-suppliers case as implied by Property 7. We assume that the company operates under the conditions of the classical Newsvendor setting, and faces transportation costs and capacities as in Expression (2). The buyer has a single replenishment opportunity at the beginning of a period during which he/she faces random demand. In case the ordered quantity exceeds the demand, excess items are salvaged at $\$v/unit$. On the other hand, if the demand exceeds the ordered quantity, the buyer incurs a loss of goodwill cost $\$b/unit$. The retail price is fixed at $\$r/unit$. Let X and $f(x)$ denote the random demand amount and its probability density function, respectively. Then, the expected profit of the buyer as a function of his/her order quantity Q , is given by

$$\mathcal{H}(Q) = (r-v)\mu - (c(Q)-v)Q + (r+b-v) \int_Q^\infty (Q-x)f(x) dx - \left[\frac{Q}{P}\right]R, \tag{5}$$

where μ is the expected value of demand. The summation of the first three terms in the above expression is the expected profits in the typical Newsvendor setting, except for the fact that the unit procurement cost (i.e., $c(Q)$) is a function of Q . We refer to Silver et al. (1998, pp. 404–406) for its derivation. The last term, which is the cost of shipment, is subtracted to find the expected profits in the setting of interest. Assume that the unified price schedule $c(Q)$ that the company faces as a result of individual price schedules $c^i(Q)$ ($i = 1, 2$) has a hybrid structure. When the price level is fixed at c_i , the expected profit without the stepwise freight costs is

$$G(Q, c_i) = (r-v)\mu - (c_i-v)Q + (r+b-v) \int_Q^\infty (Q-x)f(x) dx. \tag{6}$$

$G(Q, c_i)$ is strictly concave with respect to Q and the unique maximizer $Q^{(i)}$ satisfies

$$F(Q^{(i)}) = \frac{r+b-c_i}{r+b-v}, \tag{7}$$

where $F(\cdot)$ is the distribution function of demand.

It can be easily shown that Expressions (6) and (7) satisfy assumptions (A1), (A2) and (A3) described in Section 2. Therefore, the supplier selection problem of the buyer can be formulated and solved using the methods proposed in this paper. In the next example, for the purpose of illustration, we use Example 1 in Toptal (2009) and extend it to two-suppliers case, where one supplier offers a price given by an all-units discount schedule and the other offers a price given by an all-units premium schedule.

Example 1. Consider a buyer with the following parameters: $r = 35$, $b = 0$, $v = 15$, $R = 150$ and $P = 100$. The buyer may order from Supplier 1 or Supplier 2, who charge $c^1(Q)$ and $c^2(Q)$ as given by

$$c^1(Q) = \begin{cases} 18.9 & 0 \leq Q < 400 \\ 19.7 & 400 \leq Q < 675 \\ 20.5 & 675 \leq Q < 900 \\ 21.5 & Q \geq 900, \end{cases}$$

and

$$c^2(Q) = \begin{cases} 21 & 0 \leq Q < 650 \\ 20 & 650 \leq Q < 701 \\ 19.9 & 701 \leq Q < 1200 \\ 19 & Q \geq 1200. \end{cases}$$

Demand is exponentially distributed with rate $\lambda = 0.002$.

Solution: We will solve this example using both of the solution methods. Recall that the first method involves solving two replenishment problems, that is one for each supplier. The second method requires forming the hybrid price schedule out

of individual price menus, and solving a single replenishment problem with this price schedule. Let us start with the first method. The objective functions of the two replenishment problems can be obtained from Expression (5) by plugging in $c^1(Q)$ and $c^2(Q)$ separately. That is, the buyer maximizes the following two functions over $Q \geq 0$:

$$10\,000 - c^1(Q) \times Q + 15Q - 10\,000e^{-0.002Q} - \left\lceil \frac{Q}{100} \right\rceil 150, \tag{8}$$

and

$$10\,000 - c^2(Q) \times Q + 15Q - 10\,000e^{-0.002Q} - \left\lceil \frac{Q}{100} \right\rceil 150. \tag{9}$$

Since $c^1(Q)$ refers to an all-units premium schedule, we follow the steps of the algorithm provided in Corollary 1 for maximizing Expression (8). We start with forming the set \mathcal{F}_2 . As $\tilde{Q}^{(0)} = 700$ and it is greater than $q_1^1 = 400$, the index of the first price interval is included in set \mathcal{F}_2 . Continuing with the next lowest price, we find that $\tilde{Q}^{(1)} = 600$. Since $\tilde{Q}^{(1)} < q_2^1 = 675$, we conclude that $\mathcal{F} = \{0\}$. This implies $n_2 = 0$.

We next check if $\tilde{Q}^{(n_2+1)} = \tilde{Q}^{(1)}$ is realizable. Since $400 \leq \tilde{Q}^{(1)} = 600 < 675$, $\tilde{Q}^{(1)}$ is realizable. We set $Q^{(p)} = 600$ and proceed with Step 3 of the algorithm. Proposition 1 implies that \tilde{Q}_0^* is either $q_1^1 - \epsilon = 399.999$ or $\lfloor 399.99/100 \rfloor 100 = 300$. As 399.99 results in larger profits than 300 does, we conclude that $\tilde{Q}_0^* = 399.999$. The second part of Step 3 involves comparing the function values that 600 and 399.999 yield in Expression (8). As a result, we find that $Q^{(p)} = 399.99$. Therefore, if the buyer orders from Supplier 1, he/she will make a replenishment for 399.999 and expect to have 3346.705 money units of profit.

We refer to Example 1 in Toptal (2009) for the solution to the buyer's replenishment problem if the buyer orders from Supplier 2 (i.e., the solution to maximizing Expression (9)). It is reported that if the buyer in this example chooses to order from Supplier 2, he/she will make a replenishment for 693.147 units and expect a profit of 2984.264 money units. As part of the first method, we finally compare the buyer's maximum expected profits if he/she orders from Supplier 1 or Supplier 2. Since $3346.705 > 2984.264$, we conclude that the buyer should choose Supplier 1 and order 399.999 units.

We next illustrate the solution to this example using the second method. As a result of $c^1(Q)$ and $c^2(Q)$, the buyer practically faces the following hybrid price schedule:

$$c(Q) = \begin{cases} 18.9 & 0 \leq Q < 400 \\ 19.7 & 400 \leq Q < 675 \\ 20 & 675 \leq Q < 701 \\ 19.9 & 701 \leq Q < 1200 \\ 19 & Q \geq 1200. \end{cases}$$

We first form $\bar{c}^p(Q)$ and $\bar{c}^d(Q)$ using $c(Q)$. Since $c_1 < c_2 > c_3$, we have $h=2$ and

$$\bar{c}^p(Q) = \begin{cases} 18.9 & 0 \leq Q < 400 \\ 19.7 & 400 \leq Q < 675 \\ 20 & Q \geq 675, \end{cases} \quad \bar{c}^d(Q) = \begin{cases} 20 & 0 \leq Q < 701 \\ 19.9 & 701 \leq Q < 1200 \\ 19 & Q \geq 1200. \end{cases}$$

As part of the second method, we solve the following two subproblems: *Problem \mathcal{PPTC}_{UB}* with $c(Q) = \bar{c}^p(Q)$ and $UB=675$, and *Problem \mathcal{DPTC}_{LB}* with $c(Q) = \bar{c}^d(Q)$ and $LB=675$. Using Proposition 2, we find that the solution to *Problem \mathcal{PPTC}_{UB}* by setting $c(Q) = \bar{c}^p(Q)$ and $UB = \infty$ (i.e., 399.999) already satisfies the upper bound constraint. Therefore, we conclude that $Q^p = 399.999$. Using Proposition 3, we find that the solution to *Problem \mathcal{DPTC}_{LB}* by setting $c(Q) = \bar{c}^d(Q)$ and $LB=0$ (i.e., 693.147) is already greater than the lower bound 675. Therefore, we conclude that

$Q^d = 693.147$. Finally, comparing the expected profits at 399.99 and 693.147, we find that $Q^* = 399.999$. Furthermore, since $c^1(399.999) < c^2(399.999)$, the buyer chooses Supplier 1.

The above example can also be used to illustrate the impact of considering transportation costs and capacities explicitly on the replenishment and supplier selection decisions of a buyer. We next discuss the following three cases:

- The buyer does not take into account transportation costs and capacities in his/her supplier selection and replenishment decisions: In this case, if the buyer chooses Supplier 1, he/she will order 674.999 units with expected profits amounting to 4235.09 money units excluding truck costs. Similarly, if the buyer chooses Supplier 2, he/she will order 1200 units with expected profits amounting to 4292.82 money units excluding truck costs. Hence, the buyer will choose Supplier 2 and order 1200 units. The expected profits of the buyer will then be 2492.82 money units including truck costs.
- The buyer does not take into account transportation costs and capacities in his/her order replenishment decision, but in his/her supplier selection decision: In this case, if the buyer chooses Supplier 1, he/she will order again 674.999 units but he/she is aware that his/her profit will be 3185.09 money units including truck costs. Similarly, if the buyer chooses Supplier 2, he/she will order again 1200 units but he/she is aware that his/her profit will be 2492.82 money units including truck costs. Hence, the buyer will choose Supplier 1 and order 674.999 units. The expected profits of the buyer will be 3185.09 money units including truck costs.
- The buyer regards transportation costs and capacities explicitly in both of his/her decisions: In this case, we know from the solution of the above example that, the buyer will choose Supplier 1 and order 399.999 units. The expected profit of the buyer will be 3346.705 money units including truck costs.

As it can be seen from the above three cases, consideration of transportation costs and capacities explicitly in solving the joint replenishment and supplier selection problem of the buyer, has a significant impact on his/her expected profits. When this issue is considered only for the supplier selection, not for the replenishment decision, the buyer will achieve 27.77% $((3185.09 - 2492.82) / 2492.82 \times 100\%)$ savings compared to the case when it is not considered at all. When transportation costs and capacities are considered explicitly for both of the decisions, the buyer will achieve 34.25% $((3346.705 - 2492.82) / 2492.82 \times 100\%)$ savings over the case when they are not considered at all, and 5.07% $((3346.705 - 3185.09) / 3185.09 \times 100\%)$ savings over the case when they are considered only for supplier selection.

6. Conclusion and future research

In this paper, we studied the replenishment problem of a buyer by modeling transportation costs and capacities explicitly and considering a general whole price structure. In the setting of interest, the buyer pays for the inbound transportation of his/her one-time inventory replenishment. The wholesale price schedule exhibits economies and diseconomies of scale over varying quantity intervals (i.e., a hybrid price schedule). The production/inventory related expected profits of the buyer are modeled as a general function of the order quantity. Solving the buyer's replenishment problem to maximize his/her expected profits in view of these cost and profit structures exhibits certain challenges due to the piecewise form of the objective function. Therefore, the proposed solution is algorithmic and it relies on several structural properties of the objective function, which are proved in the paper.

In order to arrive at a solution for finding the replenishment quantity of the buyer in this setting, several subproblems are defined and analyzed. First, a replenishment problem with transportation considerations and wholesale prices given by quantity premiums, is solved. Secondly, the solution is extended to consider an upper bound on the order quantity. Thirdly, a previous study by Toptal (2009) is extended to consider a lower bound on the order quantity in a replenishment problem with transportation considerations and wholesale prices given by quantity discounts. It is important to emphasize that each of the solutions to these subproblems can be used alone for other practical problems.

In the paper, we also show within the context of a supplier selection problem that, the wholesale price schedule that a buyer practically faces under single sourcing assumption turns out to have a hybrid structure. Therefore, we discuss how the solutions to different subproblems studied in this paper can be utilized to solve the joint supplier selection and replenishment decision of a buyer. Based on some numerical instances, we also report our results about the impact of modeling transportation costs and capacities explicitly on these decisions.

In this study, inbound shipment costs are modeled assuming truckload transportation. Some recent papers consider cases where the replenishment quantity can be shipped using a combination of truckload and less-than-truckload transportation (e.g., Mendoza and Ventura, 2008; Rieksts and Ventura, 2010). Our study can also be extended to consider different transportation modes. We note that the types of wholesale price schedules considered in our study (all-units discounts, all-units premiums, all-units hybrid) can prevail as part of freight rate discounts in the context of less-than-truckload transportation (e.g., Tersine and Barman, 1994). Another direction for future research concerns solving the replenishment decisions in multi-stage inventory systems where different stages face transportation costs in the form of stepwise freight costs and/or unit freight rate discounts (see Glock, 2012 for a recent review of the literature on joint economic lot size models).

Appendix A

A.1. Proof of Property 3

Due to Property 2, we know that any maximizer $\tilde{Q}^{(k+1)}$ of $H^{k+1}(Q)$ satisfies $\tilde{Q}^{(k+1)} \leq \tilde{Q}^{(k)}$. Since $\tilde{Q}^{(k)} < q_{k+1}$, it follows that $\tilde{Q}^{(k+1)} < q_{k+1}$. Using the fact that $\tilde{Q}^{(k+2)} \leq \tilde{Q}^{(k+1)}$ and $q_{k+1} < q_{k+2}$, we have $\tilde{Q}^{(k+2)} < q_{k+2}$. Continuing in this fashion, it can be shown that $\tilde{Q}^{(j)} < q_j$ for $j = k+3, k+4, \dots, n$.

A.2. Proof of Property 4

$\mathcal{F}_2 = \emptyset$ implies that some maximizer of $H^0(Q)$ is less than q_1 and hence realizable. Let us refer to this maximizer as $\tilde{Q}^{(0)}$. Since $\tilde{Q}^{(0)}$ is realizable, we have $\mathcal{H}(\tilde{Q}^{(0)}) = H^0(\tilde{Q}^{(0)})$. It follows from Property 1 that $H^0(\tilde{Q}^{(0)}) > H^i(\tilde{Q}^{(0)})$, $\forall i \geq 1$. Therefore, $\mathcal{H}(\tilde{Q}^{(0)}) > \mathcal{H}(Q)$, $\forall Q \geq q_1$. Combining this with the fact that $\mathcal{H}(\tilde{Q}^{(0)}) \geq \mathcal{H}(Q)$ for all Q such that $q_0 \leq Q < q_1$, we conclude that $\tilde{Q}^{(0)}$ is a maximizer of $\mathcal{H}(Q)$.

A.3. Proof of Property 5

Using the definition of n_2 and Property 3, we have that if $\mathcal{F}_2 \neq \emptyset$, all the maximizers of $H^k(Q)$ are greater than or equal to q_{k+1} (i.e., $\tilde{Q}^{(k)} \geq q_{k+1}$) $\forall k < n_2 + 1$. Therefore, there exists no $k < n_2 + 1$ such that some maximizer of $H^k(Q)$ is realizable. Now,

consider the price index $n_2 + 1$. It follows from the definition of n_2 that there exists a maximizer of $H^{n_2+1}(Q)$ that is less than q_{n_2+1} . Due to Property 3, this further implies that all the maximizers of $H^k(Q)$ functions are less than q_k $\forall k > n_2 + 1$ (i.e., $\tilde{Q}^{(k)} < q_k$, $\forall k > n_2 + 1$). Hence, there exists no $k > n_2 + 1$ such that some maximizer of $H^k(Q)$ is realizable.

A.4. Proof of Property 6

Let us first prove the first part of the property. Since $\tilde{Q}^{(n_2+1)}$ is a realizable maximizer of $H^{n_2+1}(Q)$, we have $\mathcal{H}(\tilde{Q}^{(n_2+1)}) \geq \mathcal{H}(Q) \forall Q$ s.t. $\tilde{Q}^{(n_2+1)} < Q < q_{n_2+2}$. Furthermore, it follows from Property 1 that $H^{n_2+1}(\tilde{Q}^{(n_2+1)}) > H^k(\tilde{Q}^{(k)}) \forall k > n_2 + 1$, and hence, $\mathcal{H}(\tilde{Q}^{(n_2+1)}) > \mathcal{H}(Q) \forall Q \geq q_{n_2+2}$. Since $\mathcal{H}(\tilde{Q}^{(n_2+1)}) \geq \mathcal{H}(Q) \forall Q$ s.t. $\tilde{Q}^{(n_2+1)} < Q < q_{n_2+2}$ and $\mathcal{H}(\tilde{Q}^{(n_2+1)}) > \mathcal{H}(Q) \forall Q \geq q_{n_2+2}$, we conclude that in finding a solution for Problem \mathcal{PPTC} , we can focus on $Q \leq \tilde{Q}^{(n_2+1)}$.

The proof of the second part follows from the definition of n_2 . Specifically, there exists a maximizer of $H^{n_2+1}(Q)$ that is less than q_{n_2+2} , say $\tilde{Q}^{(n_2+1)}$ (i.e., $\tilde{Q}^{(n_2+1)} < q_{n_2+2}$). Since $H^{n_2+1}(Q)$ has no realizable maximizer, we must have $\tilde{Q}^{(n_2+1)} < q_{n_2+1}$. Utilizing the fact that $H^i(\tilde{Q}^{(n_2+1)}) > H^{n_2+1}(\tilde{Q}^{(n_2+1)}) \forall i < n_2 + 1$, we have $\mathcal{H}(\tilde{Q}^{(n_2+1)}) > H^{n_2+1}(\tilde{Q}^{(n_2+1)})$. Furthermore, it follows from Property 1 that we have $H^{n_2+1}(\tilde{Q}^{(n_2+1)}) > H^i(\tilde{Q}^{(i)}) \forall i > n_2 + 1$. Therefore, $\mathcal{H}(\tilde{Q}^{(n_2+1)}) > \mathcal{H}(Q)$, $\forall Q \geq q_{n_2+2}$. Since $\tilde{Q}^{(n_2+1)}$ is a maximizer for $H^{n_2+1}(Q)$, we also have $H^{n_2+1}(\tilde{Q}^{(n_2+1)}) \geq \mathcal{H}(Q)$, $\forall Q$ such that $q_{n_2+1} \leq Q < q_{n_2+2}$. This, in turn, implies that $\mathcal{H}(\tilde{Q}^{(n_2+1)}) > \mathcal{H}(Q)$, $\forall Q$ such that $q_{n_2+1} \leq Q < q_{n_2+2}$. Combining this with the result that $\mathcal{H}(\tilde{Q}^{(n_2+1)}) > \mathcal{H}(Q)$, $\forall Q \geq q_{n_2+2}$, we conclude that in finding a solution for Problem \mathcal{PPTC} , we can focus on $Q < q_{n_2+1}$.

A.5. Proof of Proposition 1

It follows from the definition of n_2 that all maximizers of $H^i(Q)$ are greater than or equal to q_{i+1} (i.e., $\tilde{Q}^{(i)} \geq q_{i+1}$) for $i \leq n_2$. We know from Result 1 cited from Toptal (2009) that $H^i(Q)$ is piecewise increasing with respect to Q in $(0, \tilde{Q}^{(i)}]$ and $H^i(kP) < H^i((k+1)P)$ for $k \in \mathbb{Z}^+$ s.t. $(k+1)P \leq \tilde{Q}^{(i)}$. Therefore, if $\lceil q_i/P \rceil = \lceil (q_{i+1}-\epsilon)/P \rceil$, we have $H^i(q_{i+1}-\epsilon) > H^i(Q)$ for all $Q \in [q_i, q_{i+1})$. If $\lceil q_i/P \rceil \neq \lceil (q_{i+1}-\epsilon)/P \rceil$, either $q_{i+1}-\epsilon$ or $\lfloor (q_{i+1}-\epsilon)/P \rfloor P$ or both maximize $H^i(Q)$ over $[q_i, q_{i+1})$.

A.6. Proof of Proposition 2

The proof will follow by considering the following two cases: $Q^{(p)} < UB$ and $Q^{(p)} \geq UB$. In the first case, the unconstrained maximizer satisfies the upper bound constraint, therefore, it is also an optimal solution to the constrained problem. In the second case, $Q^{(p)}$ is not feasible, therefore it is not optimal. In this case, letting $w = \max\{i : q_i < UB\}$ so that $q_w \leq UB < q_{w+1}$, we redefine the $(w+1)^{\text{st}}$ interval as $[q_w, UB)$. Now, any feasible solution to Problem \mathcal{PPTC}_{UB} lies within the first $(w+1)$ quantity intervals of the updated price schedule. Since $Q^{(p)} \geq UB$, we conclude, due to Corollary 1, that $w \leq n_2 + 1$. Proposition 1 and its proof imply that \tilde{Q}_i^* dominate all the other order quantities within $[q_i, q_{i+1})$ for all $0 \leq i \leq n_2$. Therefore, if $w \leq n_2$, the optimal solution to \mathcal{PPTC}_{UB} is given by the quantity among all \tilde{Q}_i^* s over $0 \leq i \leq w$, which gives the maximum objective function value. Furthermore, Corollary 1 implies that we have $w = n_2 + 1$ only if $q_w < UB \leq Q^{(p)} = \tilde{Q}^{(n_2+1)} < q_{w+1}$. Since $Q^{(p)} \geq UB$, Proposition 1 can again be used to find the

maximizer over $[q_w, UB)$, that is \bar{Q}_w^* . Thus, we have $Q^P = \arg \max\{\mathcal{H}(\bar{Q}_i^*) : 0 \leq i \leq w\}$, where \bar{Q}_i^* is determined by Proposition 1.

A.7. Proof of Proposition 3

The proof will follow by considering the following two cases: $Q^{(d)} \geq LB$ and $Q^{(d)} < LB$. In the first case, the nonnegative maximizer satisfies the positive lower bound constraint, therefore, it is an optimal solution. In the second case, $Q^{(d)}$ is not feasible, therefore it is not optimal. In this case, letting $w = \min\{i : q_{i+1} \geq LB\}$ we redefine the $(w+1)^{\text{st}}$ interval as $[LB, q_{w+1})$. Since $Q^{(d)} \geq \bar{Q}^{(r_1)}$, it turns out that $w \geq r_1$. If $w > r_2$, Property 7 in Toptal (2009) implies that we should only consider the breakpoints q_w, q_{w+1}, \dots, q_n . If $w \leq r_2$, we analyze the following two parts of the feasible region separately: $LB \leq Q < q_{r_2+1}$ and $Q \geq q_{r_2+1}$. Again, Property 7 in Toptal (2009) implies that we should only consider the breakpoints $q_{r_2+1}, q_{w+1}, \dots, q_n$ in the latter part of the feasible region. On the other hand, since $\bar{Q}^i < q_i$, for i s.t. $w \leq i \leq r_2$, the maximizer over each quantity interval $[q_i, q_{i+1})$ (i.e., Q_i^*) in the first part of the feasible region can be found using Proposition 2 in Toptal (2009). Reducing the feasible region to these finite number of solutions, the required value Q^D can be found by comparing and choosing the one which yields the maximum expected profits.

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